Multipole Expansion for Radiation;Vector Spherical Harmonics

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We seek a more systematic treatment of the multipole expansion for radiation. The strategy will be to consider three regions:

1. Radiation zone: $r \gg \lambda \gg d$. This is the region we have already considered for the dipole radiation. But we will see that there is a deeper connection between the usual multipole moments and the radiation at large distances (which in all cases falls as $1/r$).
2. Intermediate zone (static zone): $\lambda \gg r \gg d$. Note that time derivatives are of order $1/\lambda$ ($c = 1$), while derivatives with respect to r are of order $1/r$, so in this region time derivatives are negligible, and the fields appear static. Here we can do a conventional multipole expansion.
3. Near zone: $d \gg r$. Here is is more difficult to find simple approximations for the fields.

Our goal is to match the solutions in the intermediate and radiation zones. We will see that in the intermediate zone, because of the static nature of the field, there is a multipole expansion identical to that of electrostatics (where moments are evaluated at each instant). This solution will match onto outgoing spherical waves, all falling as $\frac{e^{ikr-i\omega t}}{r}$, but with a sequence of terms suppressed by powers of d/λ. So we have two kinds of expansion going on, a different one in each region.

Let's first review some of the special functions we will need for this analysis. The basic equation which will interest is the Helmholtz equation (similar to the Schrodinger equation):

$$(\vec{\nabla}^2 + \omega^2)\psi(\vec{x}, \omega) = 0. \tag{1}$$

One can expand the solutions for fixed r, in spherical coordinates:

$$\psi(\vec{x}, \omega) = \sum_{\ell,m} f_{\ell,m}(r) Y_{\ell,m}(\theta, \phi). \tag{2}$$

$f_{\ell,m}$ obeys:

$$\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + k^2 - \frac{\ell(\ell+1)}{r^2}\right] f_\ell(r) = 0. \tag{3}$$

The $f_\ell(r)$'s are "spherical Bessel functions," $f_\ell(r) = g_\ell(kr)$. The g_ℓ's are rather nice functions (you will encounter them in quantum mechanics in studying scattering theory). Typically they are presented as four types, the j_ℓ's, n_ℓ's, and h_ℓ's, spherical Bessel, Neumann, and Hankel functions. They are distinguished by their behaviors at the origin and at ∞. The j_ℓ's are regular at the origin, n_ℓ's are singular; both behave like sines at ∞. There are actually two types of h's,

$$h_\ell^{(1)} = j_\ell + in_\ell; \quad h_\ell^{(2)} = j_\ell - in_\ell. \tag{4}$$

These behave as $e^{\pm ix}$ at ∞, corresponding to outgoing and incoming spherical waves. They are both singular at the origin (a suitable linear combination is not).

$$j_0(x) = \frac{\sin x}{x} \quad n_0(x) = -\frac{\cos(x)}{x} \quad h_0^{(1)}(x) = \frac{e^{ix}}{ix} \tag{5}$$

with relatively simple asymptotics:

$$x \to 0 : j_\ell(x) \to \frac{x^\ell}{(2\ell+1)!!} \quad n_\ell(x) \to -\frac{(2\ell-1)!!}{x^{\ell+1}}. \tag{6}$$

$$x \to \infty : j_\ell(x) \to \frac{1}{x}\sin(x - \frac{\ell\pi}{2}) \quad n_\ell(x) \to -\frac{1}{x}\cos(x - \frac{\ell\pi}{2}). \tag{7}$$

$$h_\ell^{(1)} \to (-i)^{\ell+1}\frac{e^{ix}}{x} \quad h_\ell^{(2)} \to (i)^{\ell+1} e^{\frac{-ix}{x}} \tag{8}$$

With $e^{-i\omega t}$ time dependence, $h_\ell^{(1)} \sim e^{-i\omega t + ikr}$ is an outgoing spherical wave.

Green's function for the Helmholtz equation:

$$(\vec{\nabla}^2 + k^2)G(\vec{x}, \vec{x}') = -\delta(\vec{x} - \vec{x}') \tag{9}$$

$$G(\vec{x}, \vec{x}') = \frac{e^{ik|\vec{x}-\vec{x}'|}}{4\pi|\vec{x} - \vec{x}'|} \tag{10}$$

$$= \sum_{\ell,m} g_\ell(r, r') Y^*_{\ell,m}(\theta'\phi') Y_{\ell m}(\theta\phi)$$

where $g_\ell(r, r') = ikj_\ell(kr_<)h^{(1)}_\ell(kr_>)$.

Jackson proves this by looking at the equations for the coefficients of each $Y_{\ell m}$. But we can see why this must be correct rather simply. First, for $r < r'$, say, for each ℓ the equation for g_ℓ is the same equation we studied above, in both r and r'. As $r \to 0$, the result should be non-singular, so we must take the j_ℓ's. For r', as $r' \to \infty$, we would like outgoing spherical wave behavior, which fixes $h_\ell^{(1)}$. The argument is symmetric in r and r', which explains the $r_<, r_>$.

Now we can fix the coefficients by requiring that as $k \to 0$, this should go over to

$$\sum_{\ell,m} \frac{1}{2\ell+1} \frac{1}{r_>} \left(\frac{r_<}{r_>} \right)^{\ell} Y^*_{\ell m}(\theta', \phi') Y_{\ell m}(\theta, \phi). \tag{11}$$

Examining the expansions of h,j for small argument, yields the coefficient above:

$$ik \frac{(kr_<)^{\ell}}{(2\ell+1)!!} \left(-\frac{(2\ell-1)!!}{(kr_>)^{\ell+1}} \right) \tag{12}$$

$$= \frac{1}{2\ell+1} \frac{1}{r_>} \left(\frac{r_<}{r_>} \right)^{\ell}$$

So we are ready to attack the problem. We seek a generalization of the static multipole expansion. We will follow a slight modification of Jackson's strategy. We start by considering the intermediate zone. It turns out to be simpler to work with $\vec{E}$ and $\vec{H}$ (I will follow Jackson in using $\vec{H}$ rather than $\vec{B}$).

$\vec{E}$ and $\vec{H}$ are vector quantities. It is helpful to start by considering the scalar quantities $\vec{r} \cdot \vec{E}$ and $\vec{r} \cdot \vec{H}$. The first, in electrostatics, can be completely determined from the charge distribution; this is the same problem we will encounter in the intermediate zone. $\vec{r} \cdot \vec{H}$ is similar. We will focus mostly on the first.

Since

$$\vec{\nabla}^2 \vec{r} \cdot \vec{E} = \vec{r} \cdot \vec{\nabla}^2 \vec{E} + 2\vec{\nabla} \cdot \vec{E} = \vec{r} \cdot \vec{\nabla}^2 \vec{E} \tag{13}$$

and similarly for $\vec{H}$, it follows that:

$$(\vec{\nabla}^2 + \omega^2) \begin{Bmatrix} \vec{r} \cdot \vec{H} \\ \vec{r} \cdot \vec{E} \end{Bmatrix} = 0. \tag{14}$$

These are scalar quantities, so they can readily be expanded in spherical harmonics:

$$\vec{r} \cdot \vec{E} = \sum_{\ell,m} k\sqrt{\ell(\ell+1)} a_E(\ell, m) g_{\ell,m}(kr) Y_{\ell,m}(\Omega). \tag{15}$$

and similarly for $\vec{r} \cdot \vec{H}$.

Now for the problems which typically interest us, in which we have a localized source of radiation, of size much smaller than a wavelength, we can rather easily figure out the coefficients a_E. Consider, first, the “intermediate zone", $\lambda \gg r \gg d$. In this regime, the fields are essentially static. You can see this by comparing derivatives:

$$\frac{\partial}{\partial r} \sim \frac{1}{r} \quad \frac{1}{c}\frac{\partial}{\partial t} \sim \frac{1}{\lambda}. \tag{16}$$

So we can neglect time derivatives in this regime. But this means that

$$\vec{E} \approx -\vec{\nabla}\phi \tag{17}$$

In this static regime,

$$\phi \approx - \int d^3x' \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|}. \tag{18}$$

(this also follows from the form of the Green's function of the Helmholtz equation in the limit that we can neglect k). Then

$$\vec{r} \cdot \vec{E} = -\vec{r} \cdot \vec{\nabla}\phi = -\frac{\partial \phi}{\partial r} \tag{19}$$

But we know the form of ϕ far from a localized charge distribution:

$$\phi(\vec{x}) = \sum_{\ell,m} \frac{4\pi}{2\ell+1} \frac{q_{\ell,m} Y_{\ell,m}(\Omega)}{r^{\ell+1}}. \tag{20}$$

We can construct $\frac{\partial\phi}{\partial r}$, so for small r,

$$a_{\ell,m}\sqrt{\ell(\ell+1)}g_\ell(kr) \approx 4\pi \frac{\ell+1}{2\ell+1} \frac{1}{r^{\ell+1}} q_{\ell,m}. \tag{21}$$

Now we want to match to the general solution. Away from $r = 0$, we have

$$g_\ell(kr) = h_\ell^{(1)}(kr) \tag{22}$$

due to the requirement of outgoing spherical wave boundary conditions. Now we match, noting that (eqns. 9.85,9.88 in Jackson)

$$h_\ell^{(1)} \approx -i\frac{(2\ell-1)!!}{x^{\ell+1}} \tag{23}$$

so

$$a_E(\ell, m) = \frac{ck^{\ell+2}}{i(2\ell+1)!!}\left(\frac{\ell+1}{\ell}\right)^{1/2} q_{\ell m}. \tag{24}$$

Knowledge of $\vec{r} \cdot \vec{E}$ determines $\vec{H}$ and $\vec{E}$, through the usual equations for plane waves in free space:

$$\vec{H} = -\frac{i}{k}\vec{\nabla} \times \vec{E} \qquad \vec{E} = \frac{i}{k}\vec{\nabla} \times \vec{H} \tag{25}$$

So, e.g., for the"TM" ($\vec{r} \cdot \vec{H} = 0$) case, let's solve the full set of equations. Take (a guess)

$$\vec{H}_{\ell m} = A_{\ell m}(r)\vec{L}Y_{\ell,m} \tag{26}$$

Note first that $\vec{L}A(r) = 0$. Using the language of quantum mechanics, this is because $A(r)$ is rotationally invariant, and $\vec{L}$ is the generator of infinitesimal rotations. But it follows from:

$$-i\epsilon_{ijk}x_j\partial_k A(r) = -i\epsilon_{ijk}x_j\frac{x_k}{r}\frac{\partial A(r)}{\partial r} = 0. \tag{27}$$

Then it is easy to see that:

$$\vec{r} \cdot \vec{E} = \frac{i}{k} \vec{r} \cdot (\vec{\nabla} \times \vec{H}) = \frac{i}{k} r_i \epsilon_{ijk} A_{\ell m} \partial_j L_k Y_{\ell m} \tag{28}$$

$$\frac{1}{k} \vec{L}^2 A_{\ell m}(r) Y_{\ell, m} = \frac{1}{k} A_{\ell m} \ell(\ell + 1) Y_{\ell m}.$$

so that the connection of $A(r)$ and the expansion coefficients of $\vec{r} \cdot \vec{E}$ can be read off immediately (Jackson's 9.122). You can check that this configuration (with the appropriate $\vec{H}$, as above) solves the full set of Maxwell's equations; similarly for the "TE" modes.

It is natural to define “vector spherical harmonics":

$$\vec{X}_{\ell m}(\theta, \phi) = \frac{1}{\sqrt{\ell(\ell+1)}} \vec{L} Y_{\ell m}(\theta, \phi). \tag{29}$$

From the properties of the spherical harmonics, it is easy to see that:

$$\int \vec{X}^*_{\ell m} \cdot \vec{X}_{\ell' m'} d\Omega = \delta_{\ell,\ell'} \delta_{m,m'} \tag{30}$$

$$\int \vec{X}^*_{\ell m} \cdot (\vec{r} \times \vec{X}_{\ell' m'}) d\Omega = 0 \tag{31}$$

(the last follows from an integration by parts and use of the commutation relations of $\vec{L}$ and $\vec{r}$).

$$\vec{H} = \sum_{\ell m} A_{\ell m}(r) \sqrt{\ell(\ell+1)} \vec{X}_{\ell m}. \tag{32}$$

Following your text, we can derive this more directly. Rather than work with $\vec{A}$ and ϕ, it is convenient to work with $\vec{r} \cdot \vec{E}$ and $\vec{r} \cdot \vec{H}$ directly. We focus on the latter. Defining a field, $\vec{E}'$ which is divergence free,

$$\vec{E}' = \vec{E} + \frac{i}{\omega \epsilon_0} \vec{J} \tag{33}$$

we have the Maxwell equations (in Jacksons 9.160, set $Z_0 = 1$, and $\mathcal{M} = 0$)

$$\vec{\nabla} \cdot \vec{H} = 0 \quad \vec{\nabla} \cdot \vec{E}' = 0 \tag{34}$$

$$\vec{\nabla} \times \vec{E}' - ik\vec{H}' = \frac{i}{\omega \epsilon_0} \vec{\nabla} \times \vec{J} \quad \vec{\nabla} \times \vec{H} + ik\vec{E}' = 0.$$

Now take the curl of the third equation, substitute for $\vec{\nabla} \times \vec{H}$ from the fourth, to give:

$$(\nabla^2 + k^2)\vec{E}' = -i/k\vec{\nabla} \times (\vec{\nabla} \times \vec{J}). \tag{35}$$

(There is a similar equation for $\vec{H}$). So

$$(\nabla^2 + k^2)\vec{r} \cdot \vec{E}' = -i/k\,(\vec{L} \cdot \vec{\nabla} \times \vec{J}). \tag{36}$$

So now we can solve for $\vec{r} \cdot \vec{E}'$ using the Green's function for the Helmholtz equation:

$$\vec{r} \cdot \vec{E}'(x) = -\frac{1}{4\pi k} \int \frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x} - \vec{x}'|} \vec{L}' \cdot \vec{\nabla}' \times \vec{J}(\vec{x}') d^3x'. \qquad (37)$$

Now

$$\vec{L} \cdot \vec{\nabla} \times \vec{J} = -i(\vec{r} \times \vec{\nabla}) \cdot (\vec{\nabla} \times \vec{J}) \qquad (38)$$

can be simplified using our ϵ identity trick.

We need to be careful about the ordering of the derivatives. The expression above is:

$$
\begin{aligned}
&-i\epsilon_{ijk}\epsilon_{klm}(r_i\partial_j\partial_\ell J_m) \\
&= -i(r_i\partial_j\partial_i J_j - r_i\partial_j\partial_j J_i) \\
&= ir_i\nabla^2 J_i - i\vec{r}\cdot\vec{\nabla}(\vec{\nabla}\cdot\vec{J}) \\
&= i\nabla^2(r_iJ_i) - 2i\vec{\nabla}\cdot\vec{J} - i\vec{r}\cdot\vec{\nabla}(\vec{\nabla}\cdot\vec{J})
\end{aligned}
\tag{39}
$$

This can be rewritten:

$$\vec{L} \cdot \vec{\nabla} \times \vec{J} = i\nabla^2(\vec{r} \cdot \vec{J}) - \frac{i}{r}\frac{\partial}{\partial r}(r^2\vec{\nabla} \cdot \vec{A}) \tag{40}$$

So using this, we have (using the expansion of the Green's function in spherical Bessel functions, eqns. 9.95-9.97 in Jackson:

$$a_E(\ell, m) = \frac{k^2}{\sqrt{\ell(\ell+1)}} \int j_\ell(kr) Y^*_{\ell,m} \left[\frac{1}{k^2} \nabla^2 (\vec{r} \cdot \vec{J}) - \frac{ic}{k} \frac{1}{r} \frac{\partial}{\partial r}(r^2 \rho) \right] d^3x. \tag{41}$$

Integrating by parts on the ∇^2 term twice, and using the wave equation (without sources) gives a factor of k^2; but we can actually drop this term in the case of a localized source, leaving our earlier expression for a_E (using the small argument limit of the Bessel functions).

Exercise: Verify the matching for the electric dipole and quadrupole moments, i.e. check that these results reproduce our calculation of the dipole and quadrupole radiation, checking the connection between the large and small distance expressions.

Applications

Jackson considers two applications of this method. First are cases where low moments vanish. This happens in some atoms and nuclei. Alternatively, it is interesting, for example, for antennae when the wavelength is not too different than the size of the antenna. (Think of FM, 100 MHz).
Jackson also (in 9.4) discusses the problem of determining the current in an antenna, and shows that this is actually a boundary value problem, where one must solve simultaneously for the current and the field.