Waveguides and Resonant Cavities: Energy Loss

With applications to the problem of axion detection

Physics 214 2011, Electricity and Magnetism

Michael Dine
Department of Physics
University of California, Santa Cruz

January 2011

The structure of the modes in a waveguide depends on the geometry of the guide (assuming perfectly conducting walls. For definiteness we will write formulas for TM modes; for TE modes, see your text and homework.
[In these notes, I revert to SI units, as in your text]
With $\psi = E_z$, basic equation is

$$(\nabla_t^2 + \gamma_\lambda^2)\psi_\lambda = 0. \tag{1}$$

One has then

$$\vec{E}_t = \pm \frac{ik}{\gamma^2} \vec{\nabla}_t \psi; \quad \vec{B}_t = \frac{1}{i\omega} \vec{\nabla}_t \vec{E}_t. \tag{2}$$

In addition

$$\gamma^2 = \frac{\omega^2 \mu\epsilon}{c^2} - k^2. \tag{3}$$

from which we read off:

1. Cutoff frequency:

$$\omega_\lambda = \frac{c\gamma_\lambda}{\sqrt{\mu\epsilon}} \tag{4}$$

   (restoring μ)

2. Phase veclocity:

$$v_p = \frac{\omega}{k_\lambda} = \frac{c}{\sqrt{\mu\epsilon}} \frac{1}{\sqrt{1 - (\frac{\omega_\lambda}{\omega})^2}} > \frac{c}{\sqrt{\mu\epsilon}}. \tag{5}$$

   The group velocity is

$$v_g = \frac{d\omega}{dk} = \frac{c^2}{\mu\epsilon} \frac{1}{v_p}. \tag{6}$$

Energy flow: Want to determine Poynting vector (determines energy flowing per unit area per unit time along the guide); integrate over cross section per unit time (energy flowing per unit time) Fully determined in terms of ψ. [We will do this analysis one mode at a time] Can also calculate the energy density per unit length, and see that ratio is the group velocity. Energy attenuated in the walls: Can calculate the energy flow into the walls; integrating over the perimeter gives energy attenuated per unit length per unit time, $\frac{dP}{dz}$. Result will be proportional to skin depth times geometrical and frequency dependent factors, which can be worked out, again, mode by mode. Exponential decay of signal with distance.

Poynting vector, Energy Flow

$$\hat{z} \cdot \vec{S} = \frac{1}{2} \hat{z} \cdot (\vec{E} \times \vec{H}^*). \tag{7}$$

Substituting the explicit forms:

$$\hat{z} \cdot \vec{S} = \frac{\omega k}{2\gamma^4} \epsilon |\vec{\nabla}_t \psi|^2. \tag{8}$$

In the integral of the energy flux over the area, we can integrate by parts:

$$P = \frac{\omega k}{2\gamma^4} \epsilon \int_A \psi^* \nabla_t^2 \psi d^2 a \tag{9}$$

so

$$P = \frac{1}{2}\sqrt{\mu\epsilon} \left(\frac{\omega}{\omega_\lambda}\right)^2 \left(1 - \frac{\omega_\lambda^2}{\omega^2}\right)^{1/2} \epsilon \int_A \psi^* \psi d^2 a. \tag{10}$$

Energy Loss

$$\frac{dP}{dz} = -\frac{1}{2\sigma\delta} \oint_C |\hat{n} \times \vec{H}|^2 d\ell \tag{11}$$

$$= -\frac{1}{2\sigma\delta} \left(\frac{\omega}{\omega_\lambda}\right)^2 \oint \frac{1}{\mu^2 \omega_\lambda^2} |\frac{\partial \psi}{\partial n}|^2 d\ell.$$

On has, from the equation for ψ that the integral is of order

$$\oint \frac{1}{\mu^2 \omega_\lambda^2} |\frac{\partial \psi}{\partial n}|^2 d\ell \sim \xi_\lambda \mu \epsilon \frac{C}{A} \int_A |\psi|^2 d^2 a. \tag{12}$$

This allows an expression for the attenuation constant,

$$P = e^{-\beta_\lambda z} \tag{13}$$

in terms of geometrical factors and the skin depth.

Energy Loss in Cavities

Define quality factor, Q by:

$$E(t) = E_0 e^{-\frac{\omega_0}{2Q}} e^{-i\omega_0 t} \tag{14}$$

which exhibits Breit-Wigner shape.
Q can be calculated like β_λ. One calculated the energy density, and compares with the power loss, where one has to integrate over all of the sides. One finds

$$Q = \frac{d}{\delta}\frac{1}{2}\left(1 + \xi_\lambda \frac{Cd}{4A}\right). \tag{15}$$

Small $\delta \to$ large Q.

Application: Axion Search Experiment

If axions constitute dark matter, equations of EM modified, with new terms, such as

$$\vec{\nabla} \times \vec{B} = -\frac{\partial \vec{E}}{\partial t} + \frac{\dot{a}(\vec{x}, t)}{f_a} \vec{B}. \tag{16}$$

Last term, if large $\vec{B}$ and if the universe if filled with axion field: source for $\vec{A}$, oscillating at frequency of $a(\vec{x}, t)$; this frequency is $m_a c^2/\hbar$. In practice, microwave range.