Waveguides and Resonant Cavities: Energy Loss
With applications to the problem of axion detection

Physics 214 2011, Electricity and Magnetism

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The structure of the modes in a waveguide depends on the geometry of the guide (assuming perfectly conducting walls. For definiteness we will write formulas for TM modes; for TE modes, see your text and homework. [In these notes, I revert to SI units, as in your text]

With $\psi = E_z$, basic equation is

$$\left(\nabla_t^2 + \gamma_\lambda^2\right)\psi_\lambda = 0. \quad (1)$$

One has then

$$\vec{E}_t = \pm \frac{ik}{\gamma^2} \vec{\nabla}_t \psi; \quad \vec{B}_t = \frac{1}{i\omega} \vec{\nabla}_t \vec{E}_t. \quad (2)$$
In addition

\[ \gamma^2 = \frac{\omega^2 \mu \varepsilon}{c^2} - k^2. \]  \hspace{1cm} (3)

from which we read off:

1. Cutoff frequency:

\[ \omega_\lambda = \frac{c \gamma_\lambda}{\sqrt{\mu \varepsilon}} \]  \hspace{1cm} (4)

(restoring \( \mu \))

2. Phase velocity:

\[ v_p = \frac{\omega}{k_\lambda} = \frac{c}{\sqrt{\mu \varepsilon}} \frac{1}{\sqrt{1 - \left(\frac{\omega_\lambda}{\omega}\right)^2}} > \frac{c}{\sqrt{\mu \varepsilon}}. \]  \hspace{1cm} (5)

The group velocity is

\[ v_g = \frac{d\omega}{dk} = \frac{c^2}{\mu \varepsilon} \frac{1}{v_p}. \]  \hspace{1cm} (6)
Energy flow: Want to determine Poynting vector (determines energy flowing per unit area per unit time along the guide); integrate over cross section per unit time (energy flowing per unit time) Fully determined in terms of $\psi$. [We will do this analysis one mode at a time] Can also calculate the energy density per unit length, and see that ratio is the group velocity. Energy attenuated in the walls: Can calculate the energy flow into the walls; integrating over the perimeter gives energy attenuated per unit length per unit time, $\frac{dP}{dz}$. Result will be proportional to skin depth times geometrical and frequency dependent factors, which can be worked out, again, mode by mode. Exponential decay of signal with distance.
\[ \hat{z} \cdot \vec{S} = \frac{1}{2} \hat{z} \cdot (\vec{E} \times \vec{H}^*) . \]  

(7)

Substituting the explicit forms:

\[ \hat{z} \cdot \vec{S} = \frac{\omega k}{2\gamma^4} \epsilon |\nabla_t \psi|^2 . \]  

(8)

In the integral of the energy flux over the area, we can integrate by parts:

\[ P = \frac{\omega k}{2\gamma^4} \epsilon \int_A \psi^* \nabla^2_t \psi \, d^2a . \]  

(9)

so

\[ P = \frac{1}{2} \sqrt{\mu \epsilon} \left( \frac{\omega}{\omega_\lambda} \right)^2 \left( 1 - \frac{\omega^2}{\omega_\lambda^2} \right)^{1/2} \epsilon \int_A \psi^* \psi \, d^2a . \]  

(10)
\[
\frac{dP}{dz} = -\frac{1}{2\sigma\delta} \int_C |\hat{n} \times \vec{H}|^2 d\ell \\
= -\frac{1}{2\sigma\delta} \left( \frac{\omega}{\omega_\lambda} \right)^2 \int \frac{1}{\mu^2 \omega_\lambda^2} \left| \frac{\partial \psi}{\partial n} \right|^2 d\ell.
\]

On has, from the equation for \(\psi\) that the integral is of order
\[
\int \frac{1}{\mu^2 \omega_\lambda^2} \left| \frac{\partial \psi}{\partial n} \right|^2 d\ell \sim \xi_\lambda \mu \epsilon \frac{C}{A} \int_A |\psi|^2 d^2a.
\]

This allows an expression for the attenuation constant,
\[
P = e^{-\beta_\lambda z}
\]
in terms of geometrical factors and the skin depth.
Define quality factor, $Q$ by:

$$E(t) = E_0 e^{-\frac{\omega_0}{2Q}} e^{-i\omega_0 t}$$

which exhibits Breit-Wigner shape.

$Q$ can be calculated like $\beta_\lambda$. One calculated the energy density, and compares with the power loss, where one has to integrate over all of the sides. One finds

$$Q = \frac{d}{\delta} \frac{1}{2} \left( 1 + \xi \frac{Cd}{4A} \right).$$

Small $\delta \rightarrow$ large $Q$. 

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If axions constitute dark matter, equations of EM modified, with new terms, such as

\[ \nabla \times \vec{B} = -\frac{\partial \vec{E}}{\partial t} + \frac{\dot{a}(\vec{x}, t)}{f_a} \vec{B}. \]  (16)

Last term, if large \( \vec{B} \) and if the universe is filled with axion field: source for \( \vec{A} \), oscillating at frequency of \( a(\vec{x}, t) \); this frequency is \( m_a c^2 / \hbar \). In practice, microwave range.