Multipole Expansion of the Electromagnetic Fields

The derivation of E1, M1 and E2 was quite awkward.
We will now be a little more general and derive general expressions for the electromagnetic fields directly.

In preparation, first we consider harmonic fields

$$\vec{E}(\vec{r},t) = \vec{E}(\vec{r})e^{-i\omega t}$$

$$\vec{B}(\vec{r},t) = \vec{B}(\vec{r})e^{-i\omega t}$$

(as always, take the real part to obtain physical fields).

At a long distance from sources, we are in a region of space where $\vec{J} = \rho = 0$.
Then, Maxwell's equations are

$$\vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = ik\vec{B} \qquad\qquad k = \frac{\omega}{c}.$$

$$\vec{\nabla}\times\vec{B} = -ik\vec{E}$$

If we eliminate $\vec{E}$,

$$\vec{\nabla}\times\frac{i}{k}(\vec{\nabla}\times\vec{B}) = ik\vec{B}$$

or

$$\boxed{(\vec{\nabla}^2 + k^2)\vec{B}(\vec{r}) = 0}$$

where we have used $\vec{\nabla}\cdot\vec{B} = 0$.

Alternatively, we can eliminate $\vec{B}$ to obtain

$$\boxed{(\vec{\nabla}^2 + k^2)\vec{E}(\vec{r}) = 0}$$

These are the homogeneous Helmholtz equations

It is a little inconvenient to examine vector solutions to this equation. Instead, we will employ a trick.

<u>Theorem</u>: Let $\vec{F}$ be any vector field satisfying $\vec{\nabla}\cdot\vec{F}=0$.
Then, there exist scalar functions ψ and χ such that

$$\vec{F} = \vec{L}\psi + (\vec{\nabla}\times\vec{L})\chi$$

where $\vec{L} \equiv -i\vec{r}\times\vec{\nabla}$. The functions ψ, χ are not unique, but are unique up to

$$\psi(\vec{r}) \rightarrow \psi(\vec{r}) + f(r)$$
$$\chi(\vec{r}) \rightarrow \chi(\vec{r}) + g(r)$$

where f, g are radial functions. The functions ψ, χ are called <u>Debye potentials</u>.

<u>Proof</u>:
First we show that if $\vec{F} = \vec{L}\psi + (\vec{\nabla}\times\vec{L})\chi$, then ψ and χ can be found.
To do this, note the operator identities:

Note: $\vec{\nabla}\cdot\vec{F}=0$ automatically since $\vec{\nabla}\cdot\vec{L}=0$, $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{L})=0$.

$$\vec{L}\cdot(\vec{\nabla}\times\vec{L}) = 0$$
$$\vec{r}\cdot\vec{L} = 0$$
$$\vec{r}\cdot(\vec{\nabla}\times\vec{L}) = i\vec{L}^2 \quad \text{since } \vec{r}\cdot(\vec{\nabla}\times\vec{L}) = (\vec{r}\times\vec{\nabla})\cdot\vec{L}.$$

Then,

$$\vec{L}\cdot\vec{F} = \vec{L}^2\psi \qquad \text{note: } \vec{L}\cdot\vec{F} = -i\vec{r}\cdot(\vec{\nabla}\times\vec{F})$$

$$\vec{r}\cdot\vec{F} = i\vec{L}^2\chi$$

These equations can be solved by expanding in spherical harmonics, and noting that

$$\vec{L}^2 Y_{\ell m}(\theta,\phi) = \ell(\ell+1)\, Y_{\ell m}(\theta,\phi)$$

Thus, one can simply match coefficients of $Y_{\ell m}$ on both sides of the equation.
The $\ell=0$ piece of ψ, χ is purely radial and can be "gauged" away. $\vec{L}^2$ annihilates it, so it does not contribute to the field $\vec{F}$.
We see that F is determined by the radial components of $\vec{F}$ and $\vec{\nabla}\times\vec{F}$, if the expansion in Debye potentials is complete.

Theorem: Let $\vec{F}$ be any vector field that satisfies $\vec{\nabla}\cdot\vec{F}=0$. Then there exists scalar functions ψ and χ such that

$$\vec{F} = \vec{L}\psi + (\vec{\nabla}\times\vec{L})\chi$$

[If $\vec{\nabla}\cdot\vec{F}\neq 0$, need to add $\vec{\nabla}\phi$]

where $\vec{L} = -i\vec{r}\times\vec{\nabla}$. The functions ψ, χ are not unique, but are unique up to

$$\psi(\vec{r}) \to \psi(\vec{r}) + f(r)$$
$$\chi(\vec{r}) \to \chi(\vec{r}) + g(r)$$

If in addition, $(\vec{\nabla}^2 + k^2)\vec{F} = 0$, then ~~[illegible]~~ it follows that $(\vec{\nabla}^2+k^2)\psi(\vec{r})$ is a purely radial function, in which case one may choose $f(r)$ above such that

$$(\vec{\nabla}^2 + k^2)\psi(\vec{r}) = 0$$

Similarly, one can choose $g(\vec{r})$ such that

$$(\vec{\nabla}^2 + k^2)\chi(\vec{r}) = 0$$

Let us address the last statement first. Note that

$$\vec{\nabla}^2(\vec{r}\cdot\vec{F}) = \vec{r}\cdot(\vec{\nabla}^2\vec{F}) + 2\vec{\nabla}\cdot\vec{F}$$
$$= \vec{r}\cdot(\vec{\nabla}^2\vec{F})$$
$$= -k^2\,\vec{r}\cdot\vec{F}$$

Hence, $(\vec{\nabla}^2 + k^2)\,\vec{r}\cdot\vec{F} = 0$. But $\vec{r}\cdot\vec{F} = r\cdot(\vec{\nabla}\times\vec{L})\chi = i\vec{L}^2\chi$.

So, $(\vec{\nabla}^2+k^2)\vec{L}^2\chi = \vec{L}^2(\vec{\nabla}^2+k^2)\chi = 0$.

That is $(\vec{\nabla}^2+k^2)\chi$ is a radial function. Call it $h(r)$.
But, if we choose $g(r)$ such that $(\vec{\nabla}^2+k^2)g(r) = -h(r)$, then

$$(\vec{\nabla}^2+k^2)\left[\chi(\vec{r}) + g(r)\right] = h(r) - h(r) = 0.$$

Likewise

$$\vec{\nabla}^2(\vec{r}\cdot\vec{\nabla}\times\vec{F}) = \vec{r}\cdot\vec{\nabla}^2(\vec{\nabla}\times\vec{F}) + 2\,\vec{\nabla}\cdot(\vec{\nabla}\times\vec{F})$$

$$= \vec{r}\cdot\vec{\nabla}\times\vec{\nabla}^2\vec{F}$$

$$= -k^2(\vec{r}\cdot\vec{\nabla}\times\vec{F})$$

so, $(\vec{\nabla}^2 + k^2)\,\vec{r}\cdot\vec{\nabla}\times\vec{F} = 0$

But $\vec{r}\cdot\vec{\nabla}\times\vec{F} = \vec{r}\cdot\vec{\nabla}\times\left(\vec{L}\psi + (\vec{\nabla}\times\vec{L})\chi\right)$

$$= r\cdot\vec{\nabla}\times\vec{L}\psi + \vec{r}\cdot\vec{\nabla}\times(\vec{\nabla}\times\vec{L})\chi$$

Using $\vec{r}\cdot(\vec{\nabla}\times\vec{L}) = i\vec{L}^2$
$\vec{\nabla}\times(\vec{\nabla}\times\vec{L}) = -\vec{L}\cdot\vec{\nabla}^2$ $\Rightarrow$ $\vec{r}\cdot\vec{\nabla}\times(\vec{\nabla}\times\vec{L})\chi = -\vec{r}\cdot\vec{L}\,\vec{\nabla}^2 = 0.$

it follows that

$$\vec{r}\cdot(\vec{\nabla}\times\vec{F}) = i\vec{L}^2\psi$$

Hence $\vec{L}^2(\vec{\nabla}^2 + k^2)\psi = 0$

so $(\nabla^2 + k^2)\psi$ is a purely radial function. The same arguments then apply.

The above arguments show that

$$\vec{r}\cdot\vec{F} = i\vec{L}^2\chi$$
$$\vec{r}\cdot(\vec{\nabla}\times\vec{F}) = i\vec{L}^2\psi$$

So that χ, ψ are determined by $\vec{r}\cdot\vec{F}$ and $\vec{r}\cdot(\vec{\nabla}\times\vec{F})$

To complete the proof, we show that if:

$$\vec\nabla\cdot\vec F=0$$
$$\vec r\cdot\vec F=0$$
$$\vec r\cdot(\vec\nabla\times\vec F)=0$$

then $\vec F=0$. We do this by contradiction. Consider a sphere of radius R. On this sphere, lines of force $\vec F$ are tangential and do not cross.* Moreover they do not end on a source since $\vec\nabla\cdot\vec F=0$. Such a field clearly has a non-zero radial curl, unless $\vec F=0$. (* since $\vec F$ is uniquely defined)

Finally, we note that $\vec L f(r)=0$ for any radial function $f(r)$. Hence, I can shift either ψ or χ by an arbitrary radial function.

<u>Note</u>: explicit forms:

$$L_z=-i\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right)=-i\frac{\partial}{\partial\phi}$$

$$L_\pm\equiv L_x\pm iL_y = e^{\pm i\phi}\left(\pm\frac{\partial}{\partial\theta}+i\cot\theta\frac{\partial}{\partial\phi}\right)$$

$$\vec L^2=L_x^2+L_y^2+L_z^2$$

$$=-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)-\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

In spherical co-ordinates

$$\vec\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}-\frac{\vec L^2}{r^2}$$

$$\vec L=-i(\vec r\times\vec\nabla)=i\left(\hat\theta\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}-\hat\phi\frac{\partial}{\partial\theta}\right)$$

<u>Note</u>: $\vec\nabla\cdot\vec L=0$.

$$[L_i,L_j]=i\epsilon_{ijk}L_k$$

$$[\vec L^2,L_i]=L_j[L_j,L_i]+[L_j,L_i]L_j$$
$$=-i\epsilon_{ijk}(L_jL_k+L_kL_j)$$
$$=0$$

symmetric under $(j\leftrightarrow k)$

Since $\vec{E}, \vec{B}$ are divergenceless in a source-free region, I can write

$$\vec{B} = \vec{L}\psi^E - \frac{i}{k}(\vec{\nabla}\times\vec{L})\psi^M \qquad \text{factor } \frac{-i}{k} \text{ is conventional}$$

$$\vec{E} = \frac{i}{k}\vec{\nabla}\times\vec{B}$$

$$= \vec{L}\psi^M + \frac{i}{k}(\vec{\nabla}\times\vec{L})\psi^E$$

after using the identities.

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{L}) = -\vec{L}\vec{\nabla}^2 \qquad \vec{L}^2 = \frac{-1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

$$\vec{\nabla}^2\vec{L} = \vec{L}\vec{\nabla}^2 \qquad \text{since } \vec{\nabla}^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{\vec{L}^2}{r^2}$$

In particular, we can insist that $\psi^{E,M}$ satisfy:

$$(\vec{\nabla}^2 + k^2)\psi^E = 0$$
$$(\vec{\nabla}^2 + k^2)\psi^M = 0 \qquad **$$

So, e.g.

$$\frac{i}{k}\vec{\nabla}\times\vec{B} = \frac{i}{k}(\vec{\nabla}\times\vec{L})\psi^E - \frac{1}{k^2}\vec{L}\,\vec{\nabla}^2\psi^M$$

But $\vec{\nabla}^2\psi^M = -k^2\psi^M$.

** This is not a unique choice, but one can impose the two conditions $(\vec{\nabla}^2 + k^2)\psi^{E,M} = 0$ separately using the gauge freedom.

$$\vec{B} = \vec{L}\psi^E - \frac{i}{k}(\vec{\nabla}\times\vec{L})\psi^M$$

$\Rightarrow \quad \vec{r}\cdot\vec{B} = \frac{1}{k}\vec{L}^2\psi^M$ $\qquad\qquad$ since $\vec{r}\cdot(\vec{\nabla}\times\vec{L}) = i\vec{L}^2$

Thus since $\vec{\nabla}^2$ commutes with $\vec{L}^2$,

$$(\vec{\nabla}^2 + k^2)\,\vec{r}\cdot\vec{B} = \frac{1}{k}\vec{L}^2(\vec{\nabla}^2 + k^2)\psi^M$$

Now $\quad \vec{\nabla}^2\vec{r}\cdot\vec{B} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(xB_x + yB_y + zB_z)$

$$\frac{\partial^2}{\partial x^2}(xB_x) = \frac{\partial}{\partial x}\left(B_x + x\frac{\partial B_x}{\partial x}\right) = x\frac{\partial^2 B_x}{\partial x^2} + 2\frac{\partial B_x}{\partial x}$$

Thus,

$$\begin{aligned}
\vec{\nabla}^2(\vec{r}\cdot\vec{B}) &= x\frac{\partial^2 B_x}{\partial x^2} + y\frac{\partial^2 B_y}{\partial x^2} + z\frac{\partial^2 B_z}{\partial x^2} + 2\frac{\partial B_x}{\partial x} \\
&\quad + x\frac{\partial^2 B_x}{\partial y^2} + y\frac{\partial^2 B_y}{\partial y^2} + z\frac{\partial^2 B_z}{\partial y^2} + 2\frac{\partial B_y}{\partial y} \\
&\quad + x\frac{\partial^2 B_x}{\partial^2 z^2} + y\frac{\partial^2 B_y}{\partial z^2} + z\frac{\partial^2 B_z}{\partial z^2} + 2\frac{\partial B_z}{\partial z} \\
&= x\vec{\nabla}^2 B_x + y\vec{\nabla}^2 B_y + z\vec{\nabla}^2 B_z + 2\vec{\nabla}\cdot\vec{B} \\
&= \vec{r}\cdot\vec{\nabla}^2\vec{B} + 2\vec{\nabla}\cdot\vec{B} \\
&= \vec{r}\cdot\vec{\nabla}^2\vec{B} \qquad\qquad \text{since } \vec{\nabla}\cdot\vec{B} = 0 \\
&= -k^2\vec{r}\cdot\vec{B} \qquad\qquad \text{since } (\vec{\nabla}^2 + k^2)\vec{B} = 0
\end{aligned}$$

Therefore, $(\vec{\nabla}^2 + k^2)\,\vec{r}\cdot\vec{B} = 0$ and we conclude that

$$\vec{L}^2(\vec{\nabla}^2 + k^2)\psi^M = 0$$

This implies that

$$(\vec{\nabla}^2 + k^2)\,\psi^M = f(r)$$

where $f(r)$ is a radial function since $L^2 f(r) = 0$.

If we shift $\psi^M \to \psi^M + g(r)$ where $(\vec{\nabla}^2 + k^2)\,g(r) = f(r)$, then

$$(\vec{\nabla}^2 + k^2)\,\psi^M = 0$$

There is always a solution to $(\vec{\nabla}^2 + k^2)\,g(r) = f(r)$, namely

$$g(r) = \frac{-1}{4\pi} \int \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}\, f(r')\, d^3r'$$

Since $$(\vec{\nabla}^2 + k^2)\,\frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} = -4\pi\,\delta^3(\vec{r}-\vec{r}')$$

is the relevant Greens function corresponding to outgoing spherical waves at large r.

Similarly, $\vec{L}\cdot\vec{B} = \vec{L}^2\psi^E$ which after similar manipulations yields

$$(\vec{\nabla}^2 + k^2)\,\psi^E = h(r)$$

and again we can shift ψ^E so that $(\vec{\nabla}^2 + k^2)\,\psi^E = 0$.

Alternatively, from

$$\vec{E} = \frac{i}{k}(\vec{\nabla}\times\vec{L})\,\psi^E - \frac{1}{k^2}\,\vec{L}\,\vec{\nabla}^2\psi^M$$

so

$$\vec{r}\cdot\vec{E} = -\frac{1}{k}\,\vec{L}^2\psi^E \qquad \text{since } \vec{r}\cdot\vec{L} = 0$$

and we can simply repeat the previous analysis of ψ^M

Note that

$$(\vec{\nabla}^2+k^2)\vec{B}=0$$

$$(\vec{\nabla}^2+k^2)\vec{E}=0$$

necessarily are satisfied. (Alternatively, these equations are used to prove that one is free to choose $\psi^{E,M}$ to satisfy $(\vec{\nabla}^2+k^2)\psi^{E,M}=0$.)

As before, we can relate $\psi^{E,M}$ to the radial components:

$$\vec{r}\cdot\vec{B}=\frac{1}{k}\vec{L}^2\psi^M$$

$$\vec{r}\cdot\vec{E}=-\frac{1}{k}\vec{L}^2\psi^E$$

To proceed, we study the solutions to the homogeneous Helmholtz eq.

$$(\vec{\nabla}^2+k^2)\psi(\vec{r})=0$$

This is most easily solved in spherical co-ordinates. Since

$$\vec{\nabla}^2+k^2=\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}-\frac{\vec{L}^2}{r^2}+k^2$$

and $\vec{L}^2 Y_{\ell m}(\theta,\phi)=\ell(\ell+1)Y_{\ell m}(\theta,\phi)$

We can write in general

$$\psi(\vec{r})=\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}f_{\ell m}(r)Y_{\ell m}(\theta,\phi)$$

where

$$\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}+k^2-\frac{\ell(\ell+1)}{r^2}\right)f_{\ell m}(r)=0$$

This is an equation for spherical Bessel functions

$$f_{\ell m}(r) = a_{\ell m}\, j_\ell(kr) + b_{\ell m}\, n_\ell(kr)$$

where

$$j_\ell(x) \equiv \sqrt{\frac{\pi}{2x}}\, J_{\ell+1/2}(x)$$

$$n_\ell(x) \equiv \sqrt{\frac{\pi}{2x}}\, N_{\ell+1/2}(x)$$

It is convenient to define

$$h_\ell^{(1)}(x) \equiv \sqrt{\frac{\pi}{2x}}\left[J_{\ell+1/2}(x) + i N_{\ell+1/2}(x)\right]$$

$$h_\ell^{(2)}(x) \equiv \sqrt{\frac{\pi}{2x}}\left[J_{\ell+1/2}(x) - i N_{\ell+1/2}(x)\right]$$

Explicitly,

$$j_\ell(x) = (-x)^\ell \left(\frac{1}{x}\frac{d}{dx}\right)^\ell \left(\frac{\sin x}{x}\right)$$

$$n_\ell(x) = -(-x)^\ell \left(\frac{1}{x}\frac{d}{dx}\right)^\ell \left(\frac{\cos x}{x}\right)$$

so, e.g.

$$j_0(x) = \frac{\sin x}{x} \qquad n_0(x) = -\frac{\cos x}{x}$$

$$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x} \qquad n_1(x) = \frac{-\cos x}{x^2} - \frac{\sin x}{x}$$

$$h_0^{(1)}(x) = \frac{e^{ix}}{ix}$$

$$h_1^{(1)}(x) = -\frac{e^{ix}}{x}\left(1 + \frac{i}{x}\right) \qquad \text{etc.}$$

Large argument limits

$$j_\ell(x) \rightarrow \frac{1}{x}\sin\left(x - \frac{\ell\pi}{2}\right)$$

$$n_\ell(x) \rightarrow \frac{-1}{x}\cos\left(x - \frac{\ell\pi}{2}\right)$$

$$h_\ell^{(1)}(x) \rightarrow (-i)^{\ell+1}\frac{e^{ix}}{x}$$

Small argument limits

$$j_\ell(x) \longrightarrow \frac{x^\ell}{(2\ell+1)!!} \qquad (2\ell+1)!! = (2\ell+1)(2\ell-1)\cdots 5\cdot 3\cdot 1.$$

$$n_\ell(x) \longrightarrow \frac{-(2\ell-1)!!}{x^{\ell+1}}$$

Thus, we conclude that

$$\psi^E(\vec{r}) = \sum_{\ell m} \psi^E_{\ell m}\, h_\ell^{(1)}(kr)\, Y_{\ell m}(\theta,\phi)$$

$$\psi^M(\vec{r}) = \sum_{\ell m} \psi^M_{\ell m}\, h_\ell^{(1)}(kr)\, Y_{\ell m}(\theta,\phi)$$

Comments: at large $kr \gg 1$, $h_\ell(kr) \sim \frac{e^{ikr}}{kr}$

which is an outgoing spherical wave. This corresponds to outgoing radiation produced by sources.

The coefficients $\psi^E_{\ell m}$ and $\psi^M_{\ell m}$ are determined by matching these solutions on to the exact solutions in the presence of sources.

Note: the sum over ℓ starts at $\ell=1$, since the $\ell=0$ piece yields a purely radial function which has no effect on the physical fields.

Thus,

$$\vec{B}(\vec{r}) = \sum_{\ell m} \psi^{E}_{\ell m} \vec{L}\, h^{(1)}_\ell(kr)\, Y_{\ell m}(\Omega) + \sum_{\ell m} \psi^{M}_{\ell m} \left(\frac{-i}{k}\right) \vec{\nabla}\times \vec{L}\, h^{(1)}_\ell(kr)\, Y_{\ell m}(\Omega)$$

$$\vec{E}(\vec{r}) = \sum_{\ell m} \psi^{E}_{\ell m} \frac{i}{k} \vec{\nabla}\times \vec{L}\, h^{(1)}_\ell(kr)\, Y_{\ell m}(\Omega) + \sum_{\ell m} \psi^{M}_{\ell m} \vec{L}\, h^{(1)}_\ell(kr)\, Y_{\ell m}(\Omega)$$

In the radiation zone, $kr \gg 1$,

$$h^{(1)}_\ell(kr) = (-i)^{\ell+1} \frac{e^{ikr}}{r}$$

<u>Electric (ℓ, m) multipole</u>

$$\vec{B}^{(E)}_{\ell m} = (-i)^{\ell+1} \frac{e^{ikr}}{kr} \vec{L}\, Y_{\ell m}(\Omega)$$

$$\vec{E}^{(E)}_{\ell m} = \frac{i}{k} \vec{\nabla} \times \vec{B}^{(E)}_{\ell m}$$

<u>Note</u>:
$\vec{r}\cdot B^{(E)}_{\ell m} = 0$
so this is sometimes called TM

<u>Magnetic (ℓ, m) multipole</u>

$$\vec{E}^{(M)}_{\ell m} = (-i)^{\ell+1} \frac{e^{ikr}}{kr} \vec{L}\, Y_{\ell m}(\Omega)$$

$$\vec{B}^{(M)}_{\ell m} = -\frac{i}{k} \vec{\nabla} \times \vec{E}^{(M)}_{\ell m}$$

<u>Note</u>:

$\vec{r}\cdot E^{(M)}_{\ell m} = 0$
so this is sometimes called TE

The corresponding expansions of the vector potential $\vec{A}(\vec{r})$ and scalar potential $\phi(\vec{r})$ in the Lorentz gauge are given by H.J. Schnitzer, arXiv: physics/0509122.

Hill's [Am J. Phys. 22, 211 (1954)] defines the vector spherical harmonics

$$\vec{X}_{\ell m} = \frac{1}{\sqrt{\ell(\ell+1)}} \vec{L}\, Y_{\ell m}$$

$$\vec{V}_{\ell m} = -\sqrt{\frac{\ell+1}{2\ell+1}}\, \hat{r}\, Y_{\ell m} + \frac{1}{\sqrt{(\ell+1)(2\ell+1)}}\, r \vec{\nabla} Y_{\ell m}$$

$$\vec{W}_{\ell m} = \sqrt{\frac{\ell}{2\ell+1}}\, \hat{r}\, Y_{\ell m} + \frac{1}{\sqrt{\ell(2\ell+1)}}\, r \vec{\nabla} Y_{\ell m}$$

which satisfy

$$\int d\Omega\, \vec{X}^*_{\ell' m'} \cdot \vec{X}_{\ell m} = \int d\Omega\, \vec{V}^*_{\ell' m'} \cdot \vec{V}_{\ell m} = \int d\Omega\, \vec{W}^*_{\ell' m'} \cdot \vec{W}_{\ell m}$$

$$= \delta_{\ell \ell'} \delta_{m m'} \qquad \ell, \ell' \geq 1.$$

Note that the case of $\ell = 0$ is not defined for $\vec{X}_{\ell m}$ so we simply define $\vec{X}_{00} = 0$. Similarly the case of $\ell = 0$ is not well-defined ~~for~~ for $\vec{W}_{\ell m}$, so we also define $\vec{W}_{00} = 0$. Finally, $\vec{V}_{00} = \frac{-\hat{r}}{\sqrt{4\pi}}$ is well-defined.

Moreover,

$$\int d\Omega\, \vec{X}^*_{\ell' m'} \cdot \vec{V}_{\ell m} = \int d\Omega\, \vec{X}^*_{\ell' m'} \cdot \vec{W}_{\ell m} = \int d\Omega\, \vec{V}^*_{\ell' m'} \cdot \vec{W}_{\ell m} = 0$$

Thus, we can expand an arbitrary vector field as

$$\vec{A}(\vec{r}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} x_{lm}(r)\,\vec{X}_{lm} + v_{lm}(r)\,\vec{V}_{lm} + w_{lm}(r)\,\vec{W}_{lm}$$

Using the orthogonality relations,

$$x_{lm}(r) = \int d\Omega\, \vec{X}^{*}_{lm} \cdot \vec{A}$$

$$v_{lm}(r) = \int d\Omega\, \vec{V}^{*}_{lm} \cdot \vec{A}$$

$$w_{lm}(r) = \int d\Omega\, \vec{W}^{*}_{lm} \cdot \vec{A}$$

In quantum mechanics, the vector spherical harmonics correspond to the addition of orbital angular momentum l and spin angular momentum $s=1$. The result is $j = l+1, l-1, l$ which correspond to V, W and X respectively.

~~If $\vec{\nabla}\cdot\vec{A} = 0$, then only $\vec{X}_{lm}$ and $\hat{r}Y_{lm}$ are needed in the expansion of $\vec{A}$. Note that $\hat{r}Y_{lm}$ is a linear combination of $\vec{V}_{lm}$ and $\vec{W}_{lm}$~~

~~For example~~

$$\cancel{r\vec{\nabla}\times\vec{L}Y_{lm} = r^2\hat{r}\vec{\nabla}^2 Y_{lm} - r\vec{\nabla}Y_{lm}}$$

$$\cancel{= -l(l+1)\hat{r}Y_{lm} - r\vec{\nabla}Y_{lm}}$$

Vector spherical harmonics

$$\vec{L} = i\left[\hat{\theta}\,\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} - \hat{\phi}\frac{\partial}{\partial\theta}\right]$$

$$\vec{X}_{\ell m}(\theta,\phi) = \frac{1}{\sqrt{\ell(\ell+1)}}\,\vec{L}\,Y_{\ell m}(\theta,\phi) \qquad (\ell \neq 0)$$

(Define $\vec{X}_{00} = 0$.)

$$\int d\Omega\, \vec{X}^{*}_{\ell' m'}(\Omega)\cdot\vec{X}_{\ell m}(\Omega) = \delta_{\ell\ell'}\delta_{mm'}$$

[$\ell=0$ does not contribute to the multipoles except in the static limit of $k\to 0$.]

Radiated power

$$P = \oint \vec{S}\cdot d\vec{a}$$

$$\vec{S} = \frac{c}{8\pi}\,\mathrm{Re}\,\vec{E}\times\vec{B}^{*}$$

$$d\vec{a} = \hat{n}\,r^2 d\Omega \qquad \hat{n} \equiv \frac{\vec{r}}{r}.$$

(i) Electric multipole

$$\vec{B} = \psi^{E}_{\ell m}\,\vec{B}^{(E)}_{\ell m}$$

$$\vec{E} = \frac{i}{k}\vec{\nabla}\times\vec{B}$$

We need to work out $\dfrac{i\vec{\nabla}}{k}\times\left[\dfrac{e^{ikr}}{r}\vec{L}\,Y_{\ell m}(\Omega)\right]$

$$\frac{i}{k}\vec{\nabla}\times\left[\frac{e^{ikr}}{r}\vec{L}\,Y_{\ell m}(\Omega)\right] = \frac{i}{k}\vec{\nabla}\left(\frac{e^{ikr}}{r}\right)\times\vec{L}\,Y_{\ell m} + \frac{e^{ikr}}{kr}i\vec{\nabla}\times\vec{L}\,Y_{\ell m}$$

A useful operator identity:

$$i\vec{\nabla}\times\vec{L} = \vec{r}\,\vec{\nabla}^2 - \vec{\nabla}\left(1+r\frac{\partial}{\partial r}\right)$$

So we get

$$i\vec{\nabla}\left(\frac{e^{ikr}}{kr}\right)\times\vec{L}\,Y_{\ell m} + \frac{e^{ikr}}{kr}\left[\vec{r}(\vec{\nabla}^2 Y_{\ell m}) - \vec{\nabla}\left(1+r\frac{\partial}{\partial r}\right)Y_{\ell m}\right]$$

$$= -\frac{e^{ikr}}{kr}\left[k\,\hat{n}\times\vec{L}\,Y_{\ell m} - (\vec{r}\,\vec{\nabla}^2 - \vec{\nabla})Y_{\ell m}\right] + O\left(\frac{1}{r^2}\right)$$

$\hat{n} = \frac{\vec{r}}{r}$ since $\frac{\partial Y_{\ell m}}{\partial r} = 0$.

$$= -\frac{e^{ikr}}{r}\left[\hat{n}\times\vec{L}\,Y_{\ell m} - \frac{1}{kr}(\text{dimensionless function of } \theta,\phi)\right]$$

using $\vec{\nabla}^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{\vec{L}^2}{r^2}$

$\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial\theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial\phi}$

So for $kr \gg 1$

$$= \frac{-e^{ikr}}{r}\hat{n}\times\vec{L}\,Y_{\ell m}$$

That is,

$$\vec{E} = -\hat{n}\times\vec{B}$$

$$\vec{E}\times\vec{B}^* = -(\hat{n}\times\vec{B})\times\vec{B}^*$$

$$= \vec{B}^*\cdot(\hat{n}\times\vec{B})$$

$$= \hat{n}|\vec{B}|^2 - \vec{B}(\hat{n}\cdot\vec{B}^*)$$

But $\hat{n}\cdot\vec{B}=0$ (since $\vec{r}\cdot\vec{L}=0$).

Hence,

$$\vec{S} = \frac{c}{8\pi}\hat{n}|\vec{B}|^2 = \frac{c}{8\pi}\hat{n}\,|\psi^E_{\ell m}|^2\frac{1}{k^2 r^2}|\vec{L}Y_{\ell m}|^2$$

$$\vec{S}\cdot\hat{n}r^2 d\Omega = \frac{c}{8\pi k^2}|\psi^E_{\ell m}||\vec{L}Y_{\ell m}|^2 d\Omega$$

or

$$\frac{dP}{d\Omega} = \frac{c}{8\pi k^2}|\psi^E_{\ell m}|^2\,|\vec{L}Y_{\ell m}|^2$$

$$\boxed{\frac{dP}{d\Omega} = \frac{c}{8\pi k^2}|\psi^E_{em}|^2\,\ell(\ell+1)|\vec{X}_{em}|^2}$$

Integrating over angles yields

$$P = \frac{c\,\ell(\ell+1)}{8\pi k^2}|\psi^E_{\ell m}|^2$$

(ii) Magnetic Multipole

$$\vec{B} \to \vec{E}, \quad \vec{E} \to -\vec{B}$$

So,

$$\frac{dP}{d\Omega} = \frac{c}{8\pi k^2} |\Psi^M_{\ell m}|^2 \ell(\ell+1) |\vec{X}_{\ell m}|^2$$

Note that all the angular dependence is in $|\vec{X}_{\ell m}|^2$.

To work out $|\vec{X}_{\ell m}|^2$, we note that

$$L_+ Y_{\ell m} = \sqrt{(\ell-m)(\ell+m+1)}\, Y_{\ell, m+1} \qquad L_\pm = L_x \pm i L_y$$

$$L_- Y_{\ell m} = \sqrt{(\ell+m)(\ell-m+1)}\, Y_{\ell, m-1}$$

$$L_z Y_{\ell m} = m Y_{\ell m}$$

$$\begin{aligned}\vec{L}\cdot\vec{L} &= L_x^2 + L_y^2 + L_z^2 \\ &= \tfrac{1}{4}(L_+ + L_-)\cdot(L_+ + L_-) - \tfrac{1}{4}(L_+ - L_-)(L_+ - L_-) + L_z^2 \\ &= \tfrac{1}{2}(L_+L_- + L_-L_+) + L_z^2 \\ &= \tfrac{1}{2}(L_+L_+^* + L_-L_-^*) + L_z^2.\end{aligned}$$

So,

$$\ell(\ell+1)|\vec{X}_{\ell m}|^2 = |\vec{L} Y_{\ell m}|^2 = \tfrac{1}{2}\left[|L_+Y_{\ell m}|^2 + |L_-Y_{\ell m}|^2 + |L_z Y_{\ell m}|^2\right]$$

$$\begin{aligned} &= \tfrac{1}{2}(\ell-m)(\ell+m+1)|Y_{\ell,m+1}|^2 \\ &\quad + \tfrac{1}{2}(\ell+m)(\ell-m+1)|Y_{\ell,m-1}|^2 \\ &\quad + m^2|Y_{\ell m}|^2 \end{aligned}$$

Thus,

$$\frac{dP}{d\Omega} = \frac{c}{8\pi k^2}|\Psi_{\ell m}|^2\left[\tfrac{1}{2}(\ell-m)(\ell+m+1)|Y_{\ell,m+1}|^2 + \tfrac{1}{2}(\ell+m)(\ell-m+1)|Y_{\ell,m-1}|^2 + m^2|Y_{\ell m}|^2\right]$$

example: $\ell = 1$ $\qquad\qquad$ $Y_{\ell,-m} = (-1)^m Y_{\ell,m}^*$

$m=0$: $\quad 2|Y_{11}|^2 = \frac{3}{4\pi}\sin^2\theta$

$m=1$ $\quad |Y_{10}|^2 + |Y_{11}|^2 = \frac{3}{8\pi}(\sin^2\theta + 2\cos^2\theta)$

$$= \frac{3}{8\pi}(1+\cos^2\theta)$$

$m=-1$ $\quad |Y_{10}|^2 + |Y_{1,-1}|^2 = \frac{3}{8\pi}(1+\cos^2\theta)$

We note, e.g. that the $m=0$ case would correspond to a dipole oscillating parallel to the z axis.

To distinguish $E\ell$ and $M\ell$ radiation, one must measure the polarization of the radiation since the corresponding polarizations are at right angles to each other.

Radiated angular momentum

C. G. Gray Am J Phys 46 (1978) 169.

The time averaged rate of radiation of angular momentum is given by

Sign error in Gray since $\overleftrightarrow{T}\times\vec{r}$ is the angular momentum flux density.

$$\frac{d\vec{L}_{ang}}{dt} = \vec{\tau} = -\oint \vec{r}\times\overleftrightarrow{T}\cdot d\vec{a}$$

where $\overleftrightarrow{T} = \frac{1}{8\pi}\mathrm{Re}\left[\vec{E}\vec{E}^* + \vec{B}\vec{B}^* - \frac{1}{2}(|\vec{E}|^2 + |\vec{B}|^2)\mathbb{1}\right]$

Let us compute this for $E\ell$ radiation.

$(\hat{n} = \vec{r}/r)$

$$[(\vec{r}\times\overleftrightarrow{T})\cdot d\vec{a}]_k = \varepsilon_{ijk} r_i T_{j\ell} n_\ell \, da$$

$d\vec{a} = \hat{n} r^2 d\Omega, \qquad \hat{n} = \frac{\vec{r}}{r}, \quad$ so $\quad (\vec{r}\times \mathbb{1})\cdot d\vec{a} = 0.$

Also, $\vec{B}\cdot\hat{n} = 0$. Hence,

[$\vec{B}\cdot\hat{n} = 0$ at $O(\frac{1}{r})$. Can one justify dropping terms proportional to $\vec{B}$ entirely?]

$$\vec{\tau} = \frac{-1}{8\pi} \operatorname{Re} \oint (\vec{r}\times\vec{E})(\vec{E}^*\cdot\hat{n})\, r^2 d\Omega$$

$$\vec{E} = -\hat{n}\times\vec{B} + O\left(\frac{1}{k^2r^2}\right)$$

$$\vec{B} = \psi^E_{\ell m}(-i)^{\ell+1}\frac{e^{ikr}}{kr}\vec{L}Y_{\ell m}$$

Unfortunately, the leading term in $\vec{E}^*\cdot\hat{n} = -\hat{n}\cdot(\hat{n}\times\vec{B}^*) = 0.$
So, we will need the $O(\frac{1}{k^2r^2})$ term. We must go back to

$$\vec{E} = \frac{i}{k}\vec{\nabla}\times\vec{B}$$

Using $\vec{r}\cdot\vec{\nabla}\times\vec{L} = i\vec{L}^2$ and $\vec{L}^2 Y_{\ell m} = \ell(\ell+1)Y_{\ell m}$,

$$\vec{E}\cdot\hat{n} = -\frac{\ell(\ell+1)}{kr}\psi^E_{\ell m}(-i)^{\ell+1}\frac{e^{ikr}}{kr}Y_{\ell m}(\Omega)$$

Using $\vec{r}\times(\vec{\nabla}\times\vec{L}) = -\vec{L}(1+\vec{r}\cdot\vec{\nabla})$,

$$\vec{r}\times\vec{E} = -\psi^E_{\ell m}\frac{i}{k}\left(1 + r\frac{\partial}{\partial r}\right)(-i)^{\ell+1}\frac{e^{ikr}}{kr}\vec{L}Y_{\ell m}(\Omega)$$

Easier: use $\vec{E} = -\hat{n}\times\vec{B}$. Then $\vec{r}\times\vec{E} = -r\hat{n}\times(\hat{n}\times\vec{B}) = \cancel{r}\vec{B} = (-i)^{\ell+1}\frac{e^{ikr}}{k}\vec{L}Y_{\ell m}(\Omega)$

The end result is

$$\vec{\tau} = \frac{\ell(\ell+1)}{8\pi k^3}|\psi^E_{\ell m}|^2 \operatorname{Re}\int (\vec{L}Y_{\ell m})Y^*_{\ell m}\, d\Omega$$

Since $L_z Y_{\ell m} = mY_{\ell m}$ and $\int |Y_{\ell m}|^2 d\Omega = 1$,

$$(\vec{\tau})_i = \frac{\ell(\ell+1)m}{8\pi k^3}|\psi^E_{\ell m}|^2\delta_{i3}$$

$\tau_1 = \tau_2 = 0$ for a fixed (ℓm)-mod
The z-direction is special since
$Y_{\ell m}$ is an eigenstate of L_3. More
generally, one has linear combinations

Thus,

$$\frac{t_3}{P} = \frac{m}{\omega} \qquad \omega = ck$$

which has a nice interpretation in terms of photons with angular momentum $m\hbar$ and energy $\hbar\omega$.

Multipole moments

Our final task is to relate $\psi_{\ell m}$ to the sources. Jackson considers the case of harmonic charge density, current density and magnetization. To simplify the discussion, I will set the magnetization to zero. Then, Maxwell's equations for the harmonic fields, writing

$$\rho(\vec{x},t) = \rho(\vec{x}) e^{-i\omega t}$$

$$\vec{J}(\vec{x},t) = \vec{J}(\vec{x}) e^{-i\omega t}$$

are

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{E} = ik\vec{B}$$

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho \qquad \vec{\nabla}\times\vec{B} = -ik\vec{E} + \frac{4\pi\vec{J}}{c}$$

Eliminating $\vec{E}$ yields

$$(\vec{\nabla}^2 + k^2)\vec{B} = -\frac{4\pi}{c}\vec{\nabla}\times\vec{J}$$

Eliminating $\vec{B}$ yields

$$(\vec{\nabla}^2 + k^2)\vec{E} = -\frac{4\pi}{c} ik\vec{J} + 4\pi\vec{\nabla}\rho$$

Recall:
$\vec{r}\cdot\vec{B} = \frac{1}{k}\vec{L}^2\psi^M(\vec{x})$
$r\cdot\vec{E} = -\frac{1}{k}\vec{L}^2\psi^E(\vec{x})$

Since we need only match $\vec{r}\cdot\vec{E}$ and $\vec{r}\cdot\vec{B}$ with the fields at large distance, we consider

$$(\vec{\nabla}^2 + k^2)\vec{r}\cdot\vec{B} = -\frac{4\pi}{c}\vec{r}\cdot(\vec{\nabla}\times\vec{J})$$

$$(\vec{\nabla}^2 + k^2)\vec{r}\cdot\vec{E} = -\frac{4\pi}{c} ik\vec{r}\cdot\vec{J} + 4\pi(2\rho + \vec{r}\cdot\vec{\nabla}\rho)$$

where we have used the identity

$$\vec{r}\cdot(\vec{\nabla}^2\vec{E}) = \vec{\nabla}^2(\vec{r}\cdot\vec{E}) - 2\vec{\nabla}\cdot\vec{E}.$$

Thus, we must solve the inhomogeneous Helmholtz equation. To do this, we consider the general equation

$$(\vec{\nabla}^2 + k^2)\, G(\vec{r},\vec{r}\,') = -4\pi\delta^3(\vec{r}-\vec{r}\,')$$

Writing

$$G(\vec{r},\vec{r}\,') = G(\vec{r}-\vec{r}\,')$$

by translational invariance and putting

$$G(\vec{r}) = \frac{1}{(2\pi)^3}\int d^3q\, \tilde{G}(\vec{q})\, e^{i\vec{q}\cdot\vec{r}}$$

$$\delta^3(\vec{r}) = \frac{1}{(2\pi)^3}\int d^3q\, e^{i\vec{q}\cdot\vec{r}}$$

Thus,

$$(k^2 - q^2)\,\tilde{G}(q) = -4\pi$$

$$\tilde{G}(q) = \frac{4\pi}{q^2-k^2}$$

$$G(\vec{r}) = \frac{4\pi}{(2\pi)^3}\int d^3q\, \frac{e^{i\vec{q}\cdot\vec{r}}}{q^2-k^2}$$

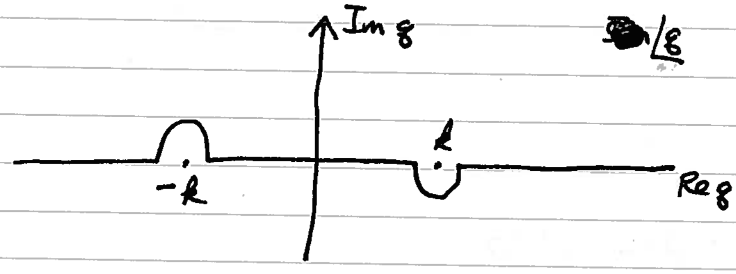
$$= \frac{4\pi}{(2\pi)^3}\, 2\pi\int_0^\infty \frac{q^2dq}{q^2-k^2}\int_{-1}^{1} d\cos\theta\, e^{iqr\cos\theta}$$

$$= \frac{1}{\pi}\int_0^\infty \frac{q^2dq}{q^2-k^2}\,\frac{1}{iqr}\left(e^{iqr}-e^{-iqr}\right)$$

$$= \frac{2}{\pi r}\int_0^\infty \frac{q\sin qr\,dq}{q^2-k^2}$$

$$= \frac{1}{\pi r}\int_{-\infty}^{\infty} \frac{q\sin qr\,dq}{q^2-k^2}$$

We have to regulate the singularity. We choose

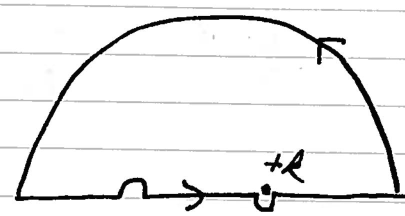


$$q^2 = k^2 + i\varepsilon \Rightarrow q = \pm\left(k + \frac{i\varepsilon}{2}\right).$$

in order to get outgoing spherical waves

First look at

$$\frac{1}{\pi r}\int_{-\infty}^{\infty} \frac{q e^{iqr}\,dq}{q^2-k^2-i\varepsilon}$$



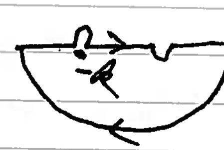
Since $r>0$, we must close the contour in the upper half plane. Then we get

$$= \frac{1}{\pi r}\,2\pi i\,\frac{1}{2k}\,k\,e^{ikr}.$$

$$= \frac{i}{r}e^{ikr}$$

Next, look at

clockwise contour in lower half-plane



$$\frac{1}{\pi r}\int_{-\infty}^{\infty} \frac{q e^{-iqr}\,dq}{q^2-k^2-i\varepsilon} = \frac{1}{\pi r}(-2\pi i)\frac{1}{-2k}(-k)e^{ikr} = \frac{-i}{r}e^{ikr}$$

Thus,

$$\frac{1}{\pi r}\int_{-\infty}^{\infty} \frac{q \sin qr}{q^2 - k^2 - i\epsilon} dq = \frac{1}{r} e^{ikr}$$

That is,

$$\boxed{G(\vec{r},\vec{r}') = \frac{1}{|\vec{r}-\vec{r}'|} e^{ik|\vec{r}-\vec{r}'|}}$$

Then, the solution to

$$(\vec{\nabla}^2 + k^2)\psi(\vec{r}) = f(\vec{r})$$

is

$$\psi(\vec{r}) = -\frac{1}{4\pi}\int d^3r'\, G(\vec{r},\vec{r}') f(\vec{r}')$$

Thus,

$$\vec{r}\cdot\vec{B} = \frac{1}{c}\int d^3r' \frac{1}{|\vec{r}-\vec{r}'|} e^{ik|\vec{r}-\vec{r}'|}\, \vec{r}'\cdot\vec{\nabla}'\times J(\vec{r}')$$

$$r\cdot\vec{E} = \frac{1}{c}\int d^3r' \frac{1}{|\vec{r}-\vec{r}'|} e^{ik|\vec{r}-\vec{r}'|}\left[ik\,\vec{r}'\cdot\vec{J}(\vec{r}') - c(2+\vec{r}'\cdot\vec{\nabla}')\rho(\vec{r}')\right]$$

Our strategy is to expand in spherical harmonics and match on to the large distance expressions.

To do this, we need to expand

$$\frac{1}{|\vec{r}-\vec{r}'|} e^{ik|\vec{r}-\vec{r}'|} = \sum_{\ell m} g_\ell(r, r') Y^*_{\ell m}(\Omega') Y_{\ell m}(\Omega)$$

Take $\vec{\nabla}^2 + k^2$ of this equation.

$$(\vec{\nabla}^2 + k^2) \frac{1}{|\vec{r}-\vec{r}'|} e^{ik|\vec{r}-\vec{r}'|} = -4\pi \delta^3(\vec{r}-\vec{r}')$$

$$(\vec{\nabla}^2 + k^2)\, g(r, r') Y_{\ell m}(\Omega)$$

$$= \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + k^2 - \frac{\vec{L}^2}{r^2} \right) g_\ell(r, r') Y_{\ell m}(\Omega)$$

$$= \left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + k^2 - \frac{\ell(\ell+1)}{r^2} \right) g_\ell(r, r') Y_{\ell m}(\Omega)$$

Thus, if we choose

$$\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + k^2 - \frac{\ell(\ell+1)}{r^2} \right) g_\ell(r, r') = -\frac{4\pi}{r^2} \delta(r-r')$$

then,

$$(\vec{\nabla}^2 + k^2) \sum_{\ell m} g_\ell(r, r') Y^*_{\ell m}(\Omega') Y_{\ell m}(\Omega)$$

$$= -\frac{4\pi}{r^2} \delta(r-r') \sum_{\ell m} Y^*_{\ell m}(\Omega') Y_{\ell m}(\Omega)$$

$$= -\frac{4\pi}{r^2} \delta(r-r') \delta(\Omega-\Omega')$$

$$= -4\pi \delta^3(\vec{r}-\vec{r}').$$

$$f(\Omega) = \sum_{\ell m} h_{\ell m} Y_{\ell m}(\Omega)$$

$$\int d\Omega' f(\Omega') Y^*_{\ell m}(\Omega') = h_{\ell m}$$

Hence, $f(\Omega) = \sum_{\ell m} \int d\Omega' f(\Omega') Y_{\ell m}(\Omega) Y^*_{\ell m}(\Omega')$

$$= \int d\Omega' f(\Omega') \delta(\Omega-\Omega')$$

$g_\ell(r,r')$ is the radial Green function. It can be solved using the standard methods.

$$g_\ell(r,r') = 4\pi i k\, j_\ell(kr_<)\, h_\ell^{(1)}(kr_>) \qquad \begin{matrix} r_> = \max\{r,r'\} \\ r_< = \min\{r,r'\} \end{matrix}$$

This guarantees finiteness as $r \to 0$ and outgoing spherical waves for $r \to \infty$.

method: for $r \neq r'$, the solution is a linear combination of Bessel functions.
At $r=r'$, integrate to get the discontinuity of the first derivative.

Thus,

$$\frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} = 4\pi i k \sum_{\ell=0}^{\infty} j_\ell(kr_<)\, h_\ell^{(1)}(kr_>) \sum_{m=-\ell}^{\ell} Y_{\ell m}^*(\Omega') Y_{\ell m}(\Omega)$$

Thus,

$$\vec{r}\cdot\vec{B} = \frac{4\pi i k}{c} \sum_{\ell m} \int d^3r'\, j_\ell(kr_<)\, h_\ell^{(1)}(kr_>)\, Y_{\ell m}^*(\Omega') Y_{\ell m}(\Omega)\, \vec{r}'\cdot\vec{\nabla}'\times J(\vec{r}')$$

Now, let us look at large $|\vec{r}|$. Then, $r_> = r$, $r_< = r'$

$$\vec{r}\cdot\vec{B} = \frac{4\pi i k}{c} \sum_{\ell m} h_\ell^{(1)}(kr)\, Y_{\ell m}(\Omega) \int d^3r'\, j_\ell(kr')\, Y_{\ell m}^*(\Omega')\, \vec{r}'\cdot\vec{\nabla}'\times\vec{J}(\vec{r}')$$

But,
$$\vec{r}\cdot\vec{B} = \frac{1}{k}\vec{L}^2 \psi^M$$

$$= \frac{1}{k} \sum_{\ell m} \psi^M_{\ell m}\, h_\ell^{(1)}(kr)\, Y_{\ell m}(\Omega)\, \ell(\ell+1)$$

where we have used: $\vec{r}\cdot(\vec{\nabla}\times\vec{L}) = i\,\vec{L}^2$ and $\vec{L}^2 Y_{\ell m}(\Omega) = \ell(\ell+1) Y_{\ell m}$.

Consider

$$\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + k^2 - \frac{\ell(\ell+1)}{r^2}\right) g_\ell(r,r') = -\frac{4\pi}{r^2}\delta(r-r')$$

~~Effectively,~~ From its original definition, $g_\ell(r,r')$ is the coefficient of an expansion of a function that is invariant under $r \leftrightarrow r'$. Hence,

$$\cancel{\left(\frac{d^2}{dr'^2} + \frac{2}{r'}\frac{d}{dr'} + k^2 - \frac{\ell(\ell+1)}{r'^2}\right) g_\ell(r',r) = -\frac{4\pi}{r'^2}\delta(r'-r)}$$

$$0 = \sum_{\ell m} \left[g_\ell(r,r') - g_\ell(r',r) \right] Y^*_{\ell m}(\Omega') Y_{\ell m}(\Omega).$$

~~[illegible]~~

By symmetry,

$$g_\ell(r,r') = g_\ell(r',r)$$

Now, suppose $r > r'$. Then,

$$g_\ell(r,r') = A(r') h^{(1)}_\ell(kr)$$

since we require outgoing spherical waves as $r \to \infty$. On the other hand, if $r < r'$, then

$$g_\ell(r,r') = B(r') j_\ell(kr)$$

in order to ensure non-singular behavior at $r=0$. Using $g_\ell(r,r') = g_\ell(r',r)$, it then follows that

$$g_\ell(r,r') = C j_\ell(kr_<) h^{(1)}_\ell(kr_>). \qquad r \neq r'.$$

Now, integrate the differential equation from $r = r'-\varepsilon$ to $r = r'+\varepsilon$. Then,

$$\left.\frac{dg_\ell}{dr}\right|_{r=r'+\varepsilon} - \left.\frac{dg_\ell}{dr}\right|_{r=r'-\varepsilon} + \frac{2}{r}\left[g_\ell(r,r'+\varepsilon) - g_\ell(r,r'-\varepsilon) \right]$$

$$+ \int_{r'-\varepsilon}^{r'+\varepsilon} \left(k^2 - \frac{\ell(\ell+1)}{r^2} \right) g_\ell(r,r')\, dr = -\frac{4\pi}{r'^2}$$

$g_\ell(r,r')$ is clearly continuous at $r=r'$ since $g_\ell(r,r) = C j_\ell(kr) h^{(1)}_\ell(kr)$. But its derivative is discontinuous.

Thus,

$$\frac{d}{dr}C\,j_\ell(kr')h_\ell(kr)\Big|_{r=r'} - \frac{d}{dr}C\,j_\ell(kr)h_\ell(kr')\Big|_{r=r'} = -\frac{4\pi}{r'^2}$$

This must be an identity for all r'. If so, it would be valid for $kr' \ll 1$. Thus, it is sufficient to consider the small argument expressions:

$$j_\ell(kr) = \frac{(kr)^\ell}{(2\ell+1)!!}$$

$$h_\ell(kr) = j_\ell(kr) + i n_\ell(kr) = \frac{-i\,(2\ell-1)!!}{(kr)^{\ell+1}}$$

Then,

$$\frac{-4\pi}{Cr^2} = \frac{(kr)^\ell}{(2\ell+1)!!}\,k\,\frac{(-i)(2\ell-1)!!}{(kr)^{\ell+2}}(-\ell-1)$$

$$- \frac{k\,\ell\,(kr)^{\ell-1}}{(2\ell+1)!!}\,\frac{(-i)(2\ell-1)!!}{(kr)^{\ell+1}}$$

$$= \frac{+i}{k}(2\ell+1)\,\frac{(2\ell-1)!!}{(2\ell+1)!!}\,\frac{1}{r^2}$$

Thus, $C = 4\pi i k$

Hence,

$$g_\ell(r,r') = 4\pi i k\, j_\ell(kr_<)\,h_\ell^{(1)}(kr_>)$$

We therefore identify:

$$\Psi_{\ell m}^{M} = \frac{4\pi i k^2}{c\,\ell(\ell+1)} \int d^3r'\, j_\ell(kr')\, Y_{\ell m}^*(\Omega')\, \vec{r}\,'\cdot\vec{\nabla}'\times\vec{J}(\vec{r}\,')$$

Since $kd \ll 1$, we can use the small ~~arg~~ argument result for

$$j_\ell(kr') \simeq \frac{(kr')^\ell}{(2\ell+1)!!}$$

$$\Psi_{\ell m}^{M} = \frac{4\pi i k^{\ell+2}}{c\,\ell(\ell+1)\,(2\ell+1)!!} \int d^3r\, r^\ell\, Y_{\ell m}^*(\Omega)\, \vec{r}\cdot\vec{\nabla}\times\vec{J}(\vec{r})$$

Note: $\vec{\nabla}\cdot(\vec{r}\times\vec{J}) = \vec{J}\cdot(\vec{\nabla}\times\vec{r}) - \vec{r}\cdot\vec{\nabla}\times\vec{J}$

$$= -\vec{r}\cdot\vec{\nabla}\times\vec{J}$$

So, we can write

$$\boxed{\Psi_{\ell m}^{M} = \frac{4\pi i k^{\ell+2}}{\ell\,(2\ell+1)!!} M_{\ell m}} \qquad (kd \ll 1)$$

where

$$M_{\ell m} = -\frac{1}{c(\ell+1)} \int d^3r\, r^\ell\, Y_{\ell m}^*(\Omega)\, \vec{\nabla}\cdot(\vec{r}\times\vec{J}) \qquad \text{(Jackson's form)}$$

are the magnetic multiple moments, which we can rewrite as

$$\boxed{M_{\ell m} = \frac{1}{c\,\ell(\ell+1)} \int d^3r\, \vec{r}\times\vec{J}\cdot\vec{\nabla}\left(r^\ell Y_{\ell m}^*(\Omega)\right)}$$

Consider ~~ℓ~~ $m=0$. Then, clearly $M_{\ell m}=0$. No surprise here. Anyway, the sums over ℓ start with $\ell=1$.

<u>$\ell=1$:</u>

$$r\,Y_{1m}(\Omega) = \left(\frac{3}{4\pi}\right)^{1/2} r_m \qquad r_0 = z,\quad r_{\pm 1} = \mp\frac{(x\pm iy)}{\sqrt{2}}$$

Now
$$\vec{\nabla}\vec{r} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)(x\hat{x} + y\hat{y} + z\hat{z})$$
$$= \hat{x}\hat{x} + \hat{y}\hat{y} + \hat{z}\hat{z}$$
$$= \overleftrightarrow{1}$$

So,
$$M_{1m} = \mu_m \left(\frac{3}{4\pi}\right)^{1/2}$$

where
$$\vec{\mu} = \frac{1}{2c}\int d^3r\; \vec{r}\times\vec{J}(\vec{r})$$

To get the electric multipole moments, the same analysis yields:

$$\vec{r}\cdot\vec{E} = -\frac{1}{k}\vec{L}^2\psi^E$$
$$= -\frac{1}{k}\sum_{\ell m}\psi^E_{\ell m}\,h_\ell^{(1)}(kr)\,Y_{\ell m}(\Omega)\,\ell(\ell+1)$$

implying

$$\psi^E_{\ell m} = \frac{-4\pi i k^2}{c\,\ell(\ell+1)}\int d^3r\, j_\ell(kr)\,Y^*_{\ell m}(\Omega)\left[ik\,\vec{r}\cdot\vec{J}(\vec{r}) - c(2 + \vec{r}\cdot\vec{\nabla})\rho(\vec{r})\right]$$

An equivalent form:

$$\psi^E_{\ell m} = +\frac{4\pi i k^2}{c\,\ell(\ell+1)} \int d^3r\, j_\ell(kr)\, Y^*_{\ell m}(\Omega) \left[\frac{i}{k} \vec{\nabla}^2(\vec{r}\cdot\vec{J}) + c(2+\vec{r}\cdot\vec{\nabla})\rho(\vec{r}) \right]$$

Note that

$$\int d^3r\, \frac{i}{k} \vec{\nabla}^2(\vec{r}\cdot\vec{J})\, j_\ell(kr)\, Y^*_{\ell m}(\Omega)$$ [see F. Low p226]

$$= \int d^3r\, \vec{r}\cdot\vec{J}\, \frac{i}{k} \vec{\nabla}^2\left(j_\ell(kr) Y^*_{\ell m}(\Omega)\right)$$

after twice integration by parts, dropping surface terms. But

$$(\vec{\nabla}^2 + k^2)\left(j_\ell(kr) Y^*_{\ell m}(\Omega)\right) = 0$$

so we get

$$= -\int d^3r\, ik\, \vec{r}\cdot\vec{J}\, j_\ell(kr) Y^*_{\ell m}(\Omega)$$

which confirms our claim above.

If we just up the leading behavior in r, $\quad j_\ell(kr) \simeq \frac{(kr)^\ell}{(2\ell+1)!!}$, then

$$\int d^3r\, \vec{r}\cdot\vec{J}\, \frac{i}{k} \vec{\nabla}^2\left(j_\ell(kr) Y^*_{\ell m}(\Omega)\right)$$

$$= \frac{i}{k} \frac{k^\ell}{(2\ell+1)!!} \int d^3r\, \vec{r}\cdot\vec{J}\, \vec{\nabla}^2\left(r^\ell Y^*_{\ell m}(\Omega)\right)$$

$$= 0$$

since $\vec{\nabla}^2(r^\ell Y^*_{\ell m}(\Omega)) = \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{\vec{L}^2}{r^2}\right)(r^\ell Y^*_{\ell m})$

$$= \left[\ell(\ell-1) + 2\ell - \ell(\ell+1)\right](r^{\ell-2} Y^*_{\ell m})$$

$$= 0$$

Thus, we need to go one extra order in the r expansion.

$$j_\ell(kr) = \sum_{n=0}^{\infty} \frac{(-1)^n (kr)^{\ell+2n}}{(2n+2\ell+1)!!\, n!\, 2^n}$$

$$= \frac{(kr)^\ell}{(2\ell+1)!!} - \frac{(kr)^{\ell+2}}{2(2\ell+3)!!} + \ldots$$

Note that $(\vec{\nabla}^2+k^2)\left[\frac{(kr)^\ell}{(2\ell+1)!!} - \frac{(kr)^{\ell+2}}{2(2\ell+3)!!} + \ldots\right] Y_{\ell m}^*(\Omega)$

$$= \frac{k^2(kr)^\ell Y_{\ell m}^*}{(2\ell+1)!!} - \frac{k^{\ell+2}}{2(2\ell+3)!!}\vec{\nabla}^2\left(r^{\ell+2}Y_{\ell m}^*(\Omega)\right) + \ldots$$

But, $\vec{\nabla}^2(r^{\ell+2}Y_{\ell m}^*) = \left[(\ell+2)(\ell+1) + 2(\ell+2) - \ell(\ell+1)\right] r^\ell Y_{\ell m}^*$

$$= 2(2\ell+3)\, r^\ell Y_{\ell m}^*$$

so, the above result does yield zero as expected.

Then,

$$\int d^3r\, \vec{r}\cdot\vec{J}\, \frac{i}{k}\vec{\nabla}^2\left(j_\ell(kr) Y_{\ell m}^*(\Omega)\right) = -\int d^3r\, ik\vec{r}\cdot\vec{J}\, j_\ell(kr)\, Y_{\ell m}^*(\Omega)$$

$$= -\frac{ik^{\ell+1}}{(2\ell+1)!!}\int d^3r\, \vec{r}\cdot\vec{J}\, r^\ell Y_{\ell m}^*(\Omega)$$

Now, using

$$\partial_k(r^2 J_k) = 2\vec{r}\cdot\vec{J} + r^2\vec{\nabla}\cdot\vec{J}$$

$$= 2\vec{r}\cdot\vec{J} + i\omega r^2\rho$$

$$= \frac{-ik^{\ell+1}}{2(2\ell+1)!!}\int d^3r\left[\partial_k(r^2J_k) - i\omega r^2\rho\right] r^\ell Y_{\ell m}^*(\Omega)$$

$\omega = ck$

$$= \frac{-ik^{\ell+1}}{2(2\ell+1)!!}\int d^3r\, r^\ell Y_{\ell m}^*(\Omega)\left[\partial_k(r^2J_k) - ick\, r^2\rho(\vec{r})\right]$$

Thus, the term proportional to

$$\frac{-4\pi i k^2}{c\,l(l+1)}\int d^3r\, j_l(kr)\, Y^*_{lm}(\Omega)\, ik\,\vec{r}\cdot\vec{J}(\vec{r}) \sim O\left(k^{l+4}\right)$$

<u>Question:</u> the leading order behavior of this term is subdominant.
But if $Q_{1m}=0$ and $Q_{2m}\neq 0$, does the $l=1$ piece of the above contribute at the same order as the $l=2$ piece due to Q_{2m}?
How does this match with the electric quadrupole as calculated in rectangular coordinates?

Some of this is addressed by Bellotti and Bornatici, J. Phys. A Math Gen 30 (1997) 4273.

See in particular the bottom of p4279 and the top of p4280.

$$= \frac{-4\pi i k^{\ell+2}}{c\,\ell(\ell+1)(2\ell+1)!!}\int d^3r\, r^\ell Y^*_{\ell m}(\Omega)\left[ik\vec{r}\cdot\vec{J}(\vec{r}) - c(2+\vec{r}\cdot\vec{\nabla})\rho(\vec{r})\right]$$

In the limit of $kd \ll 1$, we can drop the $k\vec{r}\cdot\vec{J}(\vec{r})$ term relative to $\rho(\vec{r})$.

Next, $\vec{r}\cdot\vec{\nabla} = r\frac{\partial}{\partial r}$. Integrate by parts, using

$$\frac{\partial}{\partial r}\left[Y^*_{\ell m}(\Omega)\, r^{\ell+3}\right] = Y^*_{\ell m}(\Omega)\, r^{\ell+2}(\ell+3)$$

$$\psi^E_{\ell m} = \frac{4\pi i k^{\ell+2}}{\ell(\ell+1)(2\ell+1)!!}\int r^\ell Y^*_{\ell m}(\Omega)\left[2\rho(\vec{r}) + r\frac{\partial\rho(\vec{r})}{\partial r}\right] r^2\, dr\, d\Omega$$

$$= \frac{4\pi i k^{\ell+2}}{\ell(\ell+1)(2\ell+1)!!}(-\ell-1)\int r^\ell Y^*_{\ell m}(\Omega)\,\rho(\vec{r})\, d^3r$$

$$\boxed{\psi^E_{\ell m} = \frac{-4\pi i k^{\ell+2}}{\ell(2\ell+1)!!} Q_{\ell m}} \qquad (kd \ll 1)$$

where

$$\boxed{Q_{\ell m} = \int d^3r\, r^\ell Y^*_{\ell m}(\Omega)\,\rho(\vec{r})}$$

are the usual electric dipole moments

Thus,

$$\frac{dP^{E\ell m}}{d\Omega} = \frac{c}{8\pi k^2}\,\ell(\ell+1)\,\frac{16\pi^2 k^{2\ell+4}}{\ell^2[(2\ell+1)!!]^2}\,|Q_{\ell m}|^2\,|\vec{X}_{\ell m}|^2$$

$$= \frac{2\pi c k^{2\ell+2}(\ell+1)}{\ell[(2\ell+1)!!]^2}\,|Q_{\ell m}|^2\,|\vec{X}_{\ell m}|^2$$

For $\dfrac{dP^{M\ell m}}{d\Omega}$, $Q_{\ell m} \to M_{\ell m}$.

Check $\ell=1$, $m=0$ case:

$$|\vec{X}_{10}|^2 = \frac{3}{8\pi}\sin^2\theta \,, \qquad Q_{10} = \left(\frac{3}{4\pi}\right)^{1/2} p$$

$$\frac{dP}{d\Omega} = 2\pi c k^4\,\frac{2}{9}\,p^2\sin^2\theta\,\frac{3}{8\pi}\cdot\frac{3}{4\pi}$$

$$= \frac{ck^4}{8\pi}p^2\sin^2\theta \qquad \checkmark$$

$P = \frac{1}{3}ck^4p^2$

For $\ell=1$, $m=1$,

$$|\vec{X}_{11}|^2 = \frac{3}{16\pi}(1+\cos^2\theta)\,, \qquad Q_{1,1} = -\left(\frac{3}{2\pi}\right)^{1/2} p$$

$$\frac{dP}{d\Omega} = \frac{ck^4}{8\pi}p^2(1+\cos^2\theta)$$

$$P = \frac{2ck^4p^2}{\ }$$

(Factor of 2 due to fact that for $\vec{p} = qa(\hat{x}+i\hat{y})$, $|\vec{p}|^2 = 2q^2a^2 = 2p^2$)

examples: electric dipole radiation

(a) oscillation of a charge q along the z-axis

$$\vec{r}(t) = \vec{a}\cos\omega t$$

The dipole moment is

$$\vec{p}(t) = q\vec{a}\cos\omega t = \mathrm{Re}\, q\vec{a}e^{-i\omega t}$$

or $\quad \vec{p} = q\vec{a}$

In the spherical basis

$$Q_{1m} = \int d^3r\, r\, Y^*_{1m}\, q\, \delta^3(\vec{r} - \vec{a}e^{-i\omega t})$$

$$= \left(\frac{3}{4\pi}\right)^{1/2} q\, r_m^* \Big|_{\vec{r} = \vec{a}e^{-i\omega t}}$$

$$r_0 = z \qquad r_{\pm 1} = \mp\left(\frac{x \pm iy}{\sqrt{2}}\right)$$

Choosing the z-axis to lie along $\vec{a}$

$$Q_{1m} = \left(\frac{3}{4\pi}\right)^{1/2} q\, a\, \delta_{m0}$$

$$\frac{dP}{d\Omega} = \frac{ck^4}{8\pi} q^2 a^2 \sin^2\theta$$

(b) rotating charge in the x-y plane

$$\vec{r}(t) = a(\hat{x} + i\hat{y})\, e^{-i\omega t}$$

$$\vec{p} = qa(\hat{x} + i\hat{y}) \qquad |\vec{p}|^2 = 2q^2a^2$$

$$\cancel{Q_{1m} = \left(\frac{3}{8\pi}\right)^{1/2} q a\, \delta_{m,1}}.$$

$$\frac{dP}{d\Omega} = \frac{ck^4}{8\pi} q^2 a^2 (1 + \cos^2\theta)$$

~~factor of 2 problem~~!

Now,

$$Q_{11} = -\left(\frac{3}{8\pi}\right)^{1/2} q(x - iy) = -\left(\frac{3}{8\pi}\right)^{1/2}(p_x - ip_y)$$

$$Q_{1-1} = \left(\frac{3}{8\pi}\right)^{1/2} q(x + iy) = \left(\frac{3}{8\pi}\right)^{1/2}(p_x + ip_y)$$

$$Q_{10} = \left(\frac{3}{4\pi}\right)^{1/2} qz = \left(\frac{3}{4\pi}\right)^{1/2} p_z$$

where $\vec{p} = q\vec{r}$. Then, in our example

$$p_x = qa$$
$$p_y = iqa$$

Thus,

$$Q_{11} = -\left(\frac{3}{8\pi}\right)^{1/2} 2qa$$

$$= -\left(\frac{3}{2\pi}\right)^{1/2} qa$$

$$Q_{10} = Q_{1-1} = 0$$

i.e. $$Q_{1m} = -\left(\frac{3}{2\pi}\right)^{1/2} qa\,\delta_{m1}$$

Rotating charge in the x-y plane revisited.

Write:

$$\rho(\vec{r},t) = q\,\delta(r - \vec{r}_0(t))$$

$$= \frac{q}{r^2}\,\delta(r-a)\,\delta(\cos\theta)\,\delta(\phi - \omega t)$$

$$Q_{1m}^{(t)} = \int d^3r\, r\, Y_{1m}^*(\Omega)\,\rho(\vec{r},t)$$

$$= q\int r\,dr\,d\cos\theta\,d\phi\; Y_{1m}^*(\theta,\phi)\,\delta(r-a)\,\delta(\cos\theta)\,\delta(\phi-\omega t)$$

$$= qa\,Y_{1m}^*\left(\tfrac{\pi}{2}, \omega t\right)$$

$$Y_{1,\pm1}(\Omega) = \mp\sqrt{\frac{3}{8\pi}}\,\sin\theta\, e^{\pm i\phi}$$

$$Y_{10}(\Omega) = \sqrt{\frac{3}{4\pi}}\,\cos\theta$$

Thus,

$$Y_{1,\pm1}\left(\tfrac{\pi}{2}, \omega t\right) = \mp\sqrt{\frac{3}{8\pi}}\, e^{\pm i\omega t}$$

$$Y_{10}\left(\tfrac{\pi}{2}, \omega t\right) = 0$$

Hence,

$$Q_{1m}(t) = \mp\sqrt{\frac{3}{8\pi}}\, qa\, e^{\mp i\omega t}, \qquad m = \pm1$$

__Question:__ Can we simply insert this result into $\dfrac{dP^{Elm}}{d\Omega} = \dfrac{2\pi c k^{2\ell+2}(\ell+1)}{\ell[(2\ell+1)!!]^2}|Q_{\ell m}|^2|\vec{X}_{\ell m}|^2$

__Answer:__ No. The latter formula was derived under the assumption that

$$Q_{\ell m}^{(t)} \propto e^{-i\omega t}$$

which is definitely not the case here. In fact, the previous assumption that $Q_{\ell m}^{(t)} \propto e^{-i\omega t}$ followed from $\rho(\vec{r},t) = \rho(\vec{r})e^{-i\omega t}$ which is not true for the present computation. In fact, $\rho(\vec{r},t)$ is manifestly real in this calculation.

Instead of rederiving the multipole expansion for arbitrary t-dependence of $Q_{\ell m}$, we can use the following trick. Instead of computing $Q_{1m}(t)$, we compute

$$\vec{p}(t) = \int d^3r\, \vec{r}\, \rho(\vec{r},t)$$

$$= q\int d^3r\,(\hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta)$$

$$\times \frac{1}{r^2}\,\delta(r-a)\,\delta(\cos\theta)\,\delta(\phi-\omega t)$$

$$\vec{p}(t) = qa(\hat{x}\cos\omega t + \hat{y}\sin\omega t)$$

In fact, it is easy to see that this is compatible with our previous result since,

$$Q_{11}(t) = -\left(\frac{3}{8\pi}\right)^{1/2}(p_x - ip_y) = -\left(\frac{3}{8\pi}\right)^{1/2} e^{-i\omega t}\, qa$$

$$Q_{1-1}(t) = \left(\frac{3}{8\pi}\right)^{1/2}(p_x + ip_y) = \left(\frac{3}{8\pi}\right)^{1/2} e^{i\omega t}\, qa$$

$$Q_{10}(t) = \left(\frac{3}{4\pi}\right)^{1/2} p_z = 0$$

However, we note that we can write $\vec{p}(t) = \mathrm{Re}\; qa(\hat{x}+i\hat{y})e^{-i\omega t}$

That is, we can ~~write~~ define a complex vector $\vec{p}e^{-i\omega t}$ where

$$\vec{p} = qa(\hat{x}+i\hat{y})$$

Then, likewise we define the complex tensor $Q_{1m}e^{-i\omega t}$ where

$$Q_{1,\pm1} = \mp\left(\frac{3}{8\pi}\right)^{1/2}(p_x \mp ip_y) = \mp\left(\frac{3}{8\pi}\right)^{1/2} qa(1\pm1)$$

That is, $Q_{11} = -\left(\frac{3}{2\pi}\right)^{1/2} qa$, $\quad Q_{10} = Q_{1-1} = 0$. This tensor satisfies the requirements of our derivations, so we may conclude that:

$$\frac{dP}{d\Omega} = \frac{4\pi c k^4}{9}\left(\frac{3}{2\pi}\right) q^2 a^2 |X_{11}|^2 \qquad |X_{11}|^2 = \frac{3}{16\pi}(1+\cos^2\theta)$$

$$= \frac{ck^4}{8\pi}\, q^2a^2(1+\cos^2\theta).$$

In fact, the correct rigorous procedure is outlined in problem 9.1 in Jackson. Given

$$\rho(\vec{r},t) = \frac{q}{r^2}\delta(r-a)\delta(\cos\theta)\delta(\phi-\omega t)$$

we note that $\rho(\vec{r},t+T) = \rho(\vec{r},t)$ where $T = \frac{2\pi}{\omega}$. ($\omega > 0$ by definition.)

Thus, we can write

$$\rho(\vec{r},t) = \sum_{n=-\infty}^{\infty} \rho_n(\vec{r}) e^{-in\omega t}$$

which respects $\rho(\vec{r}, t + \frac{2\pi m}{\omega}) = \rho(\vec{r},t)$ for any integer m.

We now solve for $\rho_n(\vec{r})$.

$$\frac{1}{T}\int_0^T \rho(\vec{r},t) e^{im\omega t} dt = \sum_{n=-\infty}^{\infty} \rho_n(\vec{r}) \underbrace{\frac{1}{T}\int_0^T e^{i(m-n)\omega t} dt}_{\delta_{mn}}$$

$$= \rho_m(\vec{r}).$$

Note that $\rho_{-m}(\vec{r}) = \rho_m^*(\vec{r})$. Hence,

$$\rho(\vec{r},t) = \rho_0(\vec{r}) + \sum_{n=1}^{\infty} \rho_n(\vec{r}) e^{-in\omega t} + c.c.$$

$$= \frac{1}{T}\int_0^T \rho(\vec{r},t)\,dt + \sum_{n=1}^{\infty} 2\,\mathrm{Re}\left[\rho_n(\vec{r}) e^{-in\omega t}\right]$$

Inserting the expression at the top of the page

$$\rho_n(\vec{r}) = \frac{1}{T}\int_0^T \rho(\vec{r},t) e^{in\omega t} dt$$

$$= \frac{q}{a^2 T}\delta(r-a)\delta(\cos\theta)\int_0^T e^{in\omega t}\delta(\phi-\omega t)\,dt$$

$$= \frac{q}{2\pi a^2}\delta(r-a)\delta(\cos\theta) e^{in\phi}$$ where we used $T = \frac{2\pi}{\omega}$.

Here,

$$\rho(\vec{r},t) = \frac{8}{2\pi a^2}\,\delta(r-a)\,\delta(\cos\theta)\left[1 + \sum_{n=1}^{\infty} \mathrm{Re}\; 2e^{in(\phi-\omega t)}\right]$$

Note that $\rho(\vec{r},t)$ is a superposition of terms, all of which are proportional to:

$$\rho(\vec{r},t) \propto e^{-in\omega t} \qquad n=0,1,2,\dots$$

So, we may treat each mode separately.

Case 1: $n=1$.

$$\rho(\vec{r},t) = \mathrm{Re}\;\frac{8}{\pi a^2}\,\delta(r-a)\,\delta(\cos\theta)\,e^{in\phi}e^{-i\omega t}$$

So, we may now use our previous formalism, which assumes that $\rho(\vec{r},t) = \mathrm{Re}(\rho(\vec{r})e^{-i\omega t})$. Here,

$$\rho(\vec{r}) = \frac{8}{\pi a^2}\,\delta(r-a)\,\delta(\cos\theta)\,e^{in\phi}$$

Then,

$$Q_{1m} = \frac{8}{\pi a^2}\int d^3r\; r\, Y_{1m}^{*}(\Omega)\,\delta(r-a)\,\delta(\cos\theta)\,e^{in\phi}$$

$$= \frac{8a}{\pi}\int d\cos\theta\, d\phi\; Y_{1m}^{*}(\theta,\phi)\,\delta(\cos\theta)\,e^{in\phi}$$

$$= \frac{8a}{\pi}\int_0^{2\pi} d\phi\; Y_{1m}^{*}\left(\frac{\pi}{2},\phi\right)e^{in\phi}$$

So, $Q_{10} = Q_{1,-1} = 0$ and

$$Q_{11} = \frac{8a}{\pi}\left(-\sqrt{\frac{3}{8\pi}}\right)\int_0^{2\pi} d\phi = -8a\sqrt{\frac{3}{2\pi}}$$

which confirms our previous result.

Case 2: $n=2,3,4,\dots$

$$Q_{1m} \propto \frac{1}{2\pi}\int_0^{2\pi} d\phi\, e^{-im\phi}e^{in\phi} = \delta_{mn} = 0 \quad \text{since } n>|m|.$$

So, indeed we only have to keep $n=1$ if we are interested in the dominant dipole contribution.

This result also shows that for the higher multipoles $Q_{\ell m}$, we must consider all possible $n=1,2,\ldots,\ell$. Actually, by looking at the $\cos\theta$ dependence, it is obvious that only the case of $n=\ell$ contributes. Thus, the charge distribution in this problem yields all the electric multipole moments.

The other lesson we learn from this is that when we compute $Q_{\ell m}(t)$ using the first method:

$$Q_{\ell m}(t) = \int d^3r\, r^\ell Y^*_{\ell m}(\Omega)\, \rho(\vec{r},t)$$

one finds the following non-zero results:

$$Q_{\ell \pm\ell}(t) = e^{\mp i\omega\ell t}(\ldots)$$

The origin of the $e^{+i\omega\ell t}$ dependence is the negative n in

$$\rho(\vec{r},t) = \sum_{n=-\infty}^{\infty} \rho_n(\vec{r}) e^{-in\omega t}$$

By rewriting this as

$$\rho(\vec{r},t) = \rho_0(\vec{r}) + \sum_{n=1}^{\infty} 2\,\mathrm{Re}\left[\rho_n(\vec{r}) e^{-in\omega t}\right]$$

this implies that for $Q_{\ell m}(t) = Q_{\ell m}\, e^{-i\omega t \ell}$, we identify

$$Q_{\ell\ell} = 2(\ldots)$$

$$Q_{\ell -\ell} = 0$$

which is what we found.