CHAPTER XIV

THE PASSAGE OF FAST PARTICLES THROUGH MATTER

§113. Ionization losses by fast particles in matter: the non-relativistic case

A FAST charged particle, in passing through matter, ionizes the atoms and thereby loses energy.† In gases, the ionization losses can be regarded as being due to collisions between the fast particle and individual atoms. In a solid or liquid medium, however, several atoms may interact simultaneously with the particle. The effect of this on the energy loss by the particle can be macroscopically regarded as resulting from the dielectric polarization of the medium by the charge. Let us first consider this effect for non-relativistic velocities of the particle. We shall see that the polarization of the medium then has only a slight effect on the losses. The derivation of this result is of interest because the method can be extended to other cases.

Let us first of all ascertain the conditions under which the phenomenon can be macroscopically considered. The spectral resolution of the field produced at a distance r from the path of a particle moving with velocity v consists chiefly of terms whose frequency is of the order v/r (the reciprocal of the "collision time"). The ionization of an atom can be effected by field components of frequency $\omega \gtrsim \omega_0$, where ω_0 is some mean frequency corresponding to the motion of the majority of the electrons in the atom. The particle therefore interacts simultaneously with many atoms if v/ω_0 is large compared with the interatomic distances. In solids and liquids these distances are of the same order of magnitude as the dimension a of the atoms themselves. Thus we obtain the condition $v \gg a\omega_0$, i.e. the velocity of the ionizing particle must be large compared with the velocities of the atomic electrons (or at least of the majority of them).‡

Let us now determine the field produced by a charged particle moving through matter. In the non-relativistic case it is sufficient to consider only the electric field, defined by the scalar potential ϕ. This potential satisfies Poisson's equation

$$\hat{\varepsilon}\triangle\phi = -4\pi e\delta(\mathbf{r}-\mathbf{v}t), \qquad (113.1)$$

in which the permittivity is written as an operator, and the expression $e\delta(\mathbf{r}-\mathbf{v}t)$ on the right-hand side is the density due to a point charge e moving with constant velocity $\mathbf{v}$.§

† We speak, as is customary, of "ionization losses", but these are, of course, understood to include losses due to the excitation of atoms to discrete energy levels.

‡ The corresponding condition for the energy E of the particle is $E \gg MI/m$, where M is the mass of the particle, m that of the electron, and I some mean ionization energy for the majority of the electrons in the atom.

§ We assume that the particle moves in a straight line, and thereby neglect scattering, as is always permissible in problems of this type.

If the charge on the particle is ze, then all the formulae pertaining to energy loss in this and the following sections should be multiplied by z^2.

We expand ϕ as a Fourier space integral:

$$\phi = \int_{-\infty}^{\infty} \phi_{\mathbf{k}} \exp(i\mathbf{k}\cdot\mathbf{r}) \frac{d^3k}{(2\pi)^3} \qquad (113.2)$$

Taking the Laplacian of this equation, we find that the Fourier component of $\triangle\phi$ is $(\triangle\phi)_{\mathbf{k}} = -k^2\phi_{\mathbf{k}}$.

Taking the Fourier component of equation (113.1) gives

$$\hat{\varepsilon}(\triangle\phi)_{\mathbf{k}} = -\int 4\pi e\delta(\mathbf{r}-\mathbf{v}t)\exp(-i\mathbf{k}\cdot\mathbf{r})\,dV$$
$$= -4\pi e \exp(-it\mathbf{v}\cdot\mathbf{k}).$$

Thus $\hat{\varepsilon}\phi_{\mathbf{k}} = (4\pi e/k^2)\exp(-it\mathbf{v}\cdot\mathbf{k})$, and $\phi_{\mathbf{k}}$ therefore depends on time through a factor $\exp(-it\mathbf{v}\cdot\mathbf{k})$. The operator $\hat{\varepsilon}$ acting on a function $\exp(-i\omega t)$ multiplies it by $\varepsilon(\omega)$. Hence

$$\phi_{\mathbf{k}} = \frac{4\pi e}{k^2\varepsilon(\mathbf{k}\cdot\mathbf{v})}\exp(-it\mathbf{v}\cdot\mathbf{k}).$$

The Fourier components of the field and of the potential are related by $\mathbf{E}_{\mathbf{k}}\exp(i\mathbf{k}\cdot\mathbf{r}) = -\mathbf{grad}\,[\phi_{\mathbf{k}}\exp(i\mathbf{k}\cdot\mathbf{r})] = -i\mathbf{k}\phi_{\mathbf{k}}\exp(i\mathbf{k}\cdot\mathbf{r})$. Thus

$$\mathbf{E}_{\mathbf{k}} = -i\mathbf{k}\phi_{\mathbf{k}} = -\frac{4\pi i e\mathbf{k}}{k^2\varepsilon(\mathbf{k}\cdot\mathbf{v})}\exp(-it\mathbf{v}\cdot\mathbf{k}). \qquad (113.3)$$

The total field strength is obtained by inverting the Fourier transform:

$$\mathbf{E} = \int_{-\infty}^{\infty} \mathbf{E}_{\mathbf{k}}\exp(i\mathbf{k}\cdot\mathbf{r})\frac{d^3k}{(2\pi)^3}. \qquad (113.4)$$

The energy loss by the moving particle is just the work done by the force $e\mathbf{E}$ exerted on the particle by the field which it produces. Taking the value of the field at the point occupied by the particle, namely $\mathbf{r} = \mathbf{v}t$, we obtain in the integrand in (113.4) a factor $\exp(it\mathbf{v}\cdot\mathbf{k})$ which cancels with the factor $\exp(-it\mathbf{v}\cdot\mathbf{k})$ in the expression (113.3) for $\mathbf{E}_{\mathbf{k}}$. Hence the force $\mathbf{F}$ is

$$\mathbf{F} = -4\pi i e^2 \int_{-\infty}^{\infty} \frac{\mathbf{k}}{k^2\varepsilon(\mathbf{k}\cdot\mathbf{v})}\frac{d^3k}{(2\pi)^3}$$

It is evident that the direction of the force $\mathbf{F}$ is opposite to that of the velocity $\mathbf{v}$; let the latter be the x-direction. Putting $k_x v = \omega$, $q = \sqrt{(k_y{}^2 + k_z{}^2)}$ and replacing $dk_y dk_z$ by $2\pi q\,dq$, we can write the magnitude of $\mathbf{F}$ as

$$F = \frac{ie^2}{\pi}\int_{-\infty}^{\infty}\int_{0}^{q_0}\frac{q\omega\,dq\,d\omega}{\varepsilon(\omega)(q^2v^2+\omega^2)}. \qquad (113.5)$$

The choice of q_0 is discussed below.

The following remark should be made concerning the integration with respect to ω in formula (113.5). As $\omega \to \infty$ the function $\varepsilon(\omega) \to 1$, and the integral is logarithmically divergent. This happens because we ought to have subtracted from the field **E** the field which would be present if the particle were moving in a vacuum (i.e. if $\varepsilon = 1$); this field evidently does not affect the energy lost by the particle in matter.

If this subtraction were effected, $1/\varepsilon$ in the integrand of (113.5) would become $1/\varepsilon - 1$, and the integral would converge. The same result can be obtained by taking the integration from $-\Omega$ to $+\Omega$ and then letting Ω tend to infinity. Since the function $\varepsilon'(\omega)$ is even, the real part of the integrand is an odd function of the frequency, and gives zero. The integral of the imaginary part of the integrand converges.

In what follows we shall sometimes find it convenient to use the notation

$$1/\varepsilon(\omega) = \eta(\omega) = \eta' + i\eta'', \tag{113.6}$$

with $\eta'(\omega)$ and $\eta''(\omega)$ respectively even and odd functions, and $\eta'' = -\varepsilon''/|\varepsilon|^2 < 0$. Formula (113.5) can be rewritten in the explicitly real form

$$F = \frac{2e^2}{\pi} \int_0^\infty \int_0^{q_0} \frac{q\omega|\eta''(\omega)|}{(q^2 v^2 + \omega^2)} \, dq \, d\omega. \tag{113.7}$$

The energy loss per unit path length is the work done by the force over that distance, which is just F; it is called the *stopping power* of the substance with respect to the particle.

According to the general rules of quantum mechanics, the Fourier component of the field whose wave vector is **k** transmits to the δ-electron released in ionization a momentum $\hbar\mathbf{k}$. For sufficiently large q ($\gg \omega_0/v$) we have $k^2 = q^2 + \omega^2/v^2 \approx q^2$, so that the momentum transferred is approximately $\hbar\mathbf{q}$. A given value of **q** corresponds to collisions with impact parameter $\sim 1/q$. Hence the condition for the macroscopic treatment to be valid is $1/q \gg a$. Accordingly, we take as the upper limit of integration a value q_0 such that $\omega_0/v \ll q_0 \ll 1/a$. The quantity $F(q_0)$ is the energy loss of a fast particle with transfer of momentum not exceeding $\hbar q_0$ to the atomic electron.

Integrating with respect to q in (113.7), we obtain

$$F(q_0) = \frac{2e^2}{\pi v^2} \int_0^\infty \omega|\eta''(\omega)| \log \frac{q_0 v}{\omega} \, d\omega. \tag{113.8}$$

This formula cannot be further transformed in a general manner, but it can be written in a more convenient form as follows. We first calculate the integral

$$\int_0^\infty \omega\eta''(\omega)\, d\omega = -\tfrac{1}{2} i \int_{-\infty}^\infty (\omega/\varepsilon)\, d\omega.$$

To do so, we notice that, if the integration is taken in the complex ω-plane along a contour consisting of the real axis and a very large semicircle σ in the upper half-plane, the integral is zero, since the integrand has no poles in the upper half-plane. For large values of ω, the function $\varepsilon(\omega)$ is given by formula (78.1):

$$\varepsilon(\omega) = 1 - \frac{4\pi e^2 N}{m\omega^2}. \tag{113.9}$$

The integration along the large semicircle σ can be carried out by using this formula, the result is†

$$-\int_0^\infty \omega\eta''(\omega)\, d\omega = -\frac{2\pi i N e^2}{m} \int_\sigma \frac{d\omega}{\omega} = 2\pi^2 N e^2/m. \qquad (113$$

We define a mean frequency of the motion of the atomic electrons by

$$\log \bar{\omega} = \frac{\int_0^\infty \omega\eta''(\omega) \log \omega \, d\omega}{\int_0^\infty \omega\eta''(\omega)\, d\omega} = \frac{m}{2\pi^2 N e^2} \int_0^\infty \omega|\eta''(\omega)| \log \omega \, d\omega. \qquad (113.$$

Then formula (113.8) can be written

$$F(q_0) = (4\pi N e^4/mv^2) \log (q_0 v/\bar{\omega}). \qquad (113.$$

The following remark should be made here. It might seem from the form of (113.7) (113.11) that the main contribution to the ionization losses (113.12) comes frc frequencies at which there is considerable absorption. This is not so; these formulae m contain a considerable contribution from ranges in which ε'' is small. The reason is that such ranges the function $\varepsilon(\omega) \approx \varepsilon'(\omega)$ may pass through zero. It is seen from formu (113.5) that the zeros of $\varepsilon(\omega)$ are poles of the integrand. In reality, of course, $\varepsilon''(\omega)$ is n exactly zero, and so the zeros of $\varepsilon(\omega)$ are not on the real axis but just below it. Hence, wh the expression used for $\varepsilon(\omega)$ is real and passes through zero, the contour must be indent upwards at the pole of the integrand, and so a contribution to the integral occurs. F example, if the function $\varepsilon(\omega)$ is given by (84.5), the contribution to the energy loss (113.1 from the poles $\pm\omega_1$ (where $\varepsilon(\omega_1) = 0$) is easily seen, by direct calculation from (113.7), be $(4\pi N e^4 m v^2 a^2) \log (q_0 v/\omega_1)$.

In order to find the energy loss $F(q_1)$ with transfer of momentum not exceedir some value $\hbar q_1 > \hbar q_0$, we must "join" formula (113.12) to that given by the quantu theory of collisions, corresponding to energy loss by collisions with single atoms. Th can be done by using the fact that the ranges of applicability of the two formulae overla As we know from the theory of collisions, the energy loss with transfer of momentum in range of $\hbar dq$ is

$$dF = (4\pi N e^4/mv^2)\, dq/q, \qquad (113.1$$

and this formula is applicable (in the non-relativistic case) for any value of $q \gg \omega_0/v$ whic is compatible with the laws of conservation of momentum and energy, provided that th energy transferred is small compared with the initial energy of the fast particle.‡ Th

† This is the same as (82.12), as it should be, since, as $|\omega| \to \infty$, $|\varepsilon| \to 1$ and $\eta'' \to -\varepsilon''$.

‡ See *QM*, §149. The "effective retardation" used there differs from F by a factor $N_a = N/Z$, the numbe density of atoms.

Formula (113.13) applies to collisions with free electrons. Its range of applicability as hitherto determine ($q \gg \omega_0/v$), however, extends to values of q for which the atomic electrons cannot be regarded as free. Th condition for this is $q \gg \omega_0/v_0$, where v_0 is the order of magnitude of the velocity of the majority of the atomi electrons; the energy $\hbar^2 q^2/2m$ of the δ-electron is then large compared with atomic energies.

energy loss with all values of q between q_0 and q_1 is accordingly $(4\pi Ne^4/mv^2)\log(q_1/q_0)$. When this quantity is added to formula (113.12), q_0 is replaced by q_1, so that

$$F(q_1) = (4\pi Ne^4/mv^2)\log(q_1 v/\bar{\omega}). \qquad (113.14)$$

If a momentum $\hbar q_1$ large compared with the atomic momenta is given to an atomic electron, its energy is $E_1 = \hbar^2 {q_1}^2/2m$. Thus we can write

$$F(E_1) = (2\pi Ne^4/mv^2)\log(2mv^2 E_1/\hbar^2\bar{\omega}^2). \qquad (113.15)$$

Formulae (113.14) and (113.15) give the energy loss of a fast particle by ionization with a transfer of energy not exceeding a value E_1 that is small compared with the original energy of the particle. It must be emphasized that with this condition the formulae are equally valid for fast electrons and fast heavy particles. Formula (113.15) differs from the formula derived from a microscopic discussion, neglecting interactions between atoms (*QM*, (149.14)) only by the definition of the "ionization energy" I, which is here represented by $\hbar\bar{\omega}$. The mean (with respect to the electrons) ionization energy of an atom is usually almost independent of its interaction with other atoms, being determined mainly by the electrons of the inner shells, which are almost unaffected by that interaction. Moreover, this quantity appears here only in a logarithm, and so the exact definition of it has even less effect on the magnitude of the energy loss.

In a collision between a heavy particle and an electron, even the maximum transferable momentum $\hbar q_{max}$ is small compared with the momentum Mv of the particle. The change in the energy of the heavy particle is therefore $\mathbf{v}\cdot\hbar\mathbf{q}$; equating this to the energy of the electron gives $\hbar^2 q^2/2m = \hbar\mathbf{q}\cdot\mathbf{v} \leqslant \hbar qv$, whence $\hbar q_{max} = 2mv$, and $E_{1,max} = 2mv^2$. Substituting for E_1 in (113.15), we obtain as the total ionization energy loss by the fast particle

$$F = \frac{4\pi Ne^4}{mv^2}\log\frac{2mv^2}{\hbar\bar{\omega}}. \qquad (113.16)$$

This differs from the usual expression (*QM*, (150.10)) only in the definition of the ionization energy $\hbar\bar{\omega}$.

We can see how $\hbar\bar{\omega}$ defined by (113.11) becomes, in a rarefied medium, the mean ionization energy of a single atom given by *QM*, (149.11). To do so, we note that in a rarefied gas, which for simplicity we suppose to consist of uniform atoms, the permittivity is $\varepsilon = 1 + 4\pi N_a\alpha(\omega)$, where N_a is the number of atoms per unit volume, $\alpha(\omega)$ the polarizability of one atom; here $|\varepsilon - 1| \ll 1$. The imaginary part of $\eta = 1/\varepsilon$ is $|\eta''| \cong 4\pi N_a\alpha''(\omega)$. The polarizability of the atom is given by *QED*, (85.13); separating the imaginary part by means of *QED* (75.19), we have when $\omega > 0$

$$|\eta''| = \tfrac{4}{3}\pi N_a \sum_n |\mathbf{d}_{0n}|^2\,\delta(E_n - E_0 - \hbar\omega),$$

where E_0 and E_n are the energies of the ground state and excited states of the atom. Substitution of this expression in (113.11) and carrying out the integration, with $N = N_a Z$, gives the definition in *QM*, (149.11).

§114. Ionization losses by fast particles in matter: the relativistic case

At velocities comparable with that of light, the effect of the polarization of the mediu on its stopping power with respect to a fast particle may become very important even gases.†

To derive the appropriate formulae, we use a method analogous to that used in §113, b it is now necessary to begin from the complete Maxwell's equations. When extraneo charges are present with volume density ρ_{ex}, and extraneous currents wih density $\mathbf{j}_{ex}$, the equations are‡

$$\operatorname{div}\mathbf{H} = 0, \qquad \mathbf{curl}\,\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t}, \qquad (114.$$

$$\operatorname{div}\hat{\varepsilon}\mathbf{E} = 4\pi\rho_{ex}, \qquad \mathbf{curl}\,\mathbf{H} = \frac{1}{c}\frac{\partial\hat{\varepsilon}\mathbf{E}}{\partial t} + \frac{4\pi}{c}\mathbf{j}_{ex}. \qquad (114.$$

In the present case the extraneous charge and current distribution are given by

$$\rho_{ex} = e\delta(\mathbf{r} - \mathbf{v}t), \qquad \mathbf{j}_{ex} = e\mathbf{v}\,\delta(\mathbf{r} - \mathbf{v}t). \qquad (114.$$

We introduce scalar and vector potentials, with the usual definitions:

$$\mathbf{H} = \mathbf{curl}\,\mathbf{A}, \qquad \mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} - \mathbf{grad}\,\phi, \qquad (114.4$$

so that equations (114.1) are satisfied identically. The additional condition

$$\operatorname{div}\mathbf{A} + \frac{1}{c}\frac{\partial\hat{\varepsilon}\phi}{\partial t} = 0 \qquad (114.5$$

is imposed on the potentials $\mathbf{A}$ and ϕ; this is a generalization of the usual *Lorentz conditio* in the theory of radiation. Then, substituting (114.4) in (114.2), we obtain the followin equations for the potentials:

$$\left.\begin{aligned} \triangle\mathbf{A} - \frac{\hat{\varepsilon}}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} &= -\frac{4\pi}{c}e\mathbf{v}\,\delta(\mathbf{r} - \mathbf{v}t), \\ \hat{\varepsilon}\left(\triangle\phi - \frac{\hat{\varepsilon}}{c^2}\frac{\partial^2\phi}{\partial t^2}\right) &= -4\pi e\,\delta(\mathbf{r} - \mathbf{v}t). \end{aligned}\right\} \qquad (114.6$$

We expand $\mathbf{A}$ and ϕ as Fourier space integrals. Taking the Fourier components o equations (114.6), we have

$$k^2\mathbf{A}_\mathbf{k} + \frac{\hat{\varepsilon}}{c^2}\frac{\partial^2\mathbf{A}_\mathbf{k}}{\partial t^2} = \frac{4\pi e\mathbf{v}}{c}\exp(-it\mathbf{v}\cdot\mathbf{k}),$$

$$\hat{\varepsilon}\left(k^2\phi_\mathbf{k} + \frac{\hat{\varepsilon}}{c^2}\frac{\partial^2\phi_\mathbf{k}}{\partial t^2}\right) = 4\pi e\exp(-it\mathbf{v}\cdot\mathbf{k}).$$

† This effect was pointed out by E. Fermi (1940), who performed the calculation for the particular case of a gas whose atoms are regarded as harmonic oscillators. The general derivation given here is due to L. Landau.

‡ We put $\mu(\omega) \equiv 1$, since matter does not exhibit magnetic properties at the frequencies important as regards ionization losses.

Hence we see that $\mathbf{A_k}$ and $\phi_\mathbf{k}$ depend on time through a factor $\exp(-it\mathbf{v}\cdot\mathbf{k})$. We again put $\omega = \mathbf{k}\cdot\mathbf{v} = k_x v$, and obtain

$$\left.\begin{aligned} \mathbf{A_k} &= \frac{4\pi e}{c}\frac{\mathbf{v}}{k^2-\omega^2\varepsilon(\omega)/c^2}e^{-i\omega t},\\ \phi_\mathbf{k} &= \frac{4\pi e}{\varepsilon(\omega)}\frac{1}{k^2-\omega^2\varepsilon(\omega)/c^2}e^{-i\omega t}. \end{aligned}\right\} \tag{114.7}$$

The Fourier component of the electric field is

$$\mathbf{E_k} = i\omega\mathbf{A_k}/c - i\mathbf{k}\phi_\mathbf{k}. \tag{114.8}$$

From these formulae the force $\mathbf{F} = e\mathbf{E}$ acting on the particle is found in the same way as in §113.† Using the same notation, we now have

$$F = \frac{ie^2}{\pi}\int_{-\infty}^{\infty}\int_0^{q_0}\frac{\left(\frac{1}{v^2}-\frac{\varepsilon}{c^2}\right)\omega q\,dq\,d\omega}{\varepsilon\left[q^2+\omega^2\left(\frac{1}{v^2}-\frac{\varepsilon}{c^2}\right)\right]}. \tag{114.9}$$

As $c\to\infty$ this formula tends, of course, to (113.5).

Let us first carry out the integration with respect to frequency. In order to effect an integration in the complex ω-plane, we first ascertain the poles of the integrand in the upper half-plane. The function $\varepsilon(\omega)$ has no singularity and no zero in this half-plane, and so the required poles can only be the zeros of the expression

$$\omega^2\left(\frac{\varepsilon}{c^2}-\frac{1}{v^2}\right)-q^2.$$

We shall show that, for any value of the positive real quantity q^2, this expression vanishes for only one value of ω.

To prove this,‡ we use a theorem in the theory of functions of a complex variable: the integral

$$\frac{1}{2\pi i}\int_C\frac{df(\omega)}{d\omega}\frac{d\omega}{f(\omega)-a}, \tag{114.10}$$

taken along a closed contour C, is equal to the difference between the numbers of zeros and poles of $f(\omega)-a$ in the region bounded by C. Let

$$f(\omega) = \omega^2\left(\frac{\varepsilon(\omega)}{c^2}-\frac{1}{v^2}\right),$$

$a = q^2$ be a positive real number, and C be a contour consisting of the real axis and a very large semicircle (Fig. 61). The function $f(\omega)$ has no pole in the upper half-plane

† The magnetic force $e\mathbf{v}\times\mathbf{H}/c$ is seen by symmetry to be zero, and in any case is perpendicular to the velocity of the particle and so does no work on it.

‡ The following argument is analogous to the proof (*SP* 1, §123) that $\varepsilon(\omega)$ has no zero in the upper half-plane.

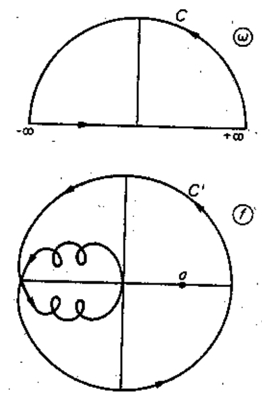

Fig. 61

or on the real axis†; the integral (114.10) therefore gives the number of zeros of the function $f(\omega)-a$ in the upper half-plane. To calculate its value, we write it as

$$\frac{1}{2\pi i}\int_{C'}\frac{df}{f-a}, \tag{114.11}$$

the integration being taken along a contour C' in the plane of the complex variable f which maps the contour C from the ω-plane. For $\omega = 0$, $f = 0$. For positive real ω we have $\operatorname{im} f > 0$, and for negative real ω, $\operatorname{im} f < 0$. At infinity $f \to -\omega^2[(1/v^2)-(1/c^2)]$, and therefore f goes round a large circle when ω goes round the large semicircle. Hence we see that the path of integration C' in the f-plane is of the kind shown schematically in Fig. 61. When a is real and positive, as in Fig. 61, in going round C' the argument of the complex number changes by 2π, and the integral (114.11) is equal to unity. This completes the proof.‡

Furthermore, it is easy to see that this single root of the equation $f(\omega)-q^2 = 0$ lies on the imaginary ω-axis: for purely imaginary ω the function $f(\omega)$, like $\varepsilon(\omega)$, is real and takes all values from 0 to ∞, including q^2.

Let us now return to the integral with respect to ω in (114.9):

$$\int_{-\infty}^{\infty}\frac{\left(\frac{1}{\varepsilon v^2}-\frac{1}{c^2}\right)\omega\,d\omega}{q^2-\omega^2\left(\frac{\varepsilon}{c^2}-\frac{1}{v^2}\right)}.$$

† For metals $\varepsilon(\omega)$ has a pole at $\omega = 0$, but $\omega^2\varepsilon$ always tends to zero with ω.

‡ If a is negative the argument of $f-a$ changes by 4π on going round C', so that the integral (114.11) is equal to 2, i.e. the function $f(\omega)+|a|$ has two zeros in the upper half-plane.

This can be written as the difference between the integral along the contour C and that along the large semicircle. The latter is $\int d\omega/\omega = i\pi$, and the former is $2\pi i$ times the residue of the integrand at its only pole. Let $\omega(q)$ be the function defined by the equation

$$\omega^2\left(\frac{\varepsilon}{c^2}-\frac{1}{v^2}\right)=q^2. \tag{114.12}$$

Then, since the residue of an expression $f(z)/\phi(z)$ at a pole $z = z_0$ is $f(z_0)/\phi'(z_0)$, the integral along C is

$$2\pi i \frac{\omega\left(\dfrac{1}{\varepsilon v^2}-\dfrac{1}{c^2}\right)}{-\dfrac{d}{d\omega}\left[\omega^2\left(\dfrac{\varepsilon}{c^2}-\dfrac{1}{v^2}\right)\right]} = 2\pi i \frac{\omega\left(\dfrac{1}{\varepsilon v^2}-\dfrac{1}{c^2}\right)}{-dq^2/d\omega}.$$

Collecting these expressions and substituting in (114.9), we have

$$F = e^2\int_0^{q_0}\left[\frac{\omega\left(\dfrac{1}{\varepsilon v^2}-\dfrac{1}{c^2}\right)}{q\,dq/d\omega}+1\right]q\,dq$$

or, replacing the integration with respect to q in the first term by one with respect to ω,

$$\begin{aligned}F &= e^2\int_{\omega(0)}^{\omega(q_0)}\left[\frac{1}{v^2\varepsilon(\omega)}-\frac{1}{c^2}\right]\omega\,d\omega+\tfrac{1}{2}e^2q_0^{\,2}\\ &= \frac{e^2}{v^2}\int_{\omega(0)}^{\omega(q_0)}\left[\frac{1}{\varepsilon(\omega)}-1\right]\omega\,d\omega+\tfrac{1}{2}e^2q_0^{\,2}\\ &\quad+\tfrac{1}{2}e^2\left(\frac{1}{v^2}-\frac{1}{c^2}\right)[\omega^2(q_0)-\omega^2(0)].\end{aligned} \tag{114.13}$$

Large values of q correspond to large absolute values ω of the root of equation (114.12). Using therefore the expression (113.9) for $\varepsilon(\omega)$, we find

$$\omega^2(q_0) = -v^2\gamma^2\left(q_0^{\,2}+\frac{4\pi Ne^2}{mc^2}\right),$$

where we have put $\gamma = 1/\sqrt{[1-v^2/c^2]}$. Substitution in (114.13) gives

$$F = \frac{e^2}{v^2}\int_{\omega(0)}^{iv q_0\gamma}\left[\frac{1}{\varepsilon(\omega)}-1\right]\omega\,d\omega-\frac{2\pi Ne^4}{mc^2}-\frac{e^2}{2v^2\gamma^2}\omega^2(0); \tag{114.14}$$

in the integral, only the leading term $ivq_0\gamma$ need be retained in $\omega(q_0)$.

The integration in (114.14) is over purely imaginary values of ω. We use the real variable $\omega'' \equiv \omega/i$, with the lower limit $\xi \equiv \omega(0)/i$, and again put $1/\varepsilon = \eta$ (113.6). The required integral is

$$-\int_{\xi}^{vq_0\gamma}[\eta(i\omega'')-1]\omega''\,d\omega''.$$

The values of the function $\eta(\omega)$ on the imaginary axis can be expressed in terms of imaginary part on the real axis:

$$\eta(i\omega'')-1 = \frac{2}{\pi}\int_0^{\infty}\frac{x\eta''(x)}{x^2+\omega''^2}dx$$

(cf. (82.15)). Hence the integral is (if we neglect x in comparison with $vq_0\gamma$)

$$\frac{2}{\pi}\int_0^{\infty}\int_{\xi}^{vq_0\gamma}\frac{x|\eta''(x)|\omega''\,d\omega''\,dx}{x^2+\omega''^2} = \frac{1}{\pi}\int_0^{\infty}x|\eta''(x)|\log\frac{v^2q_0^{\,2}\gamma^2}{x^2+\xi^2}dx.$$

We substitute this result in (114.14), and for simplicity put

$$\log\Omega \equiv \overline{\tfrac{1}{2}\log(\omega^2+\xi^2)}, \tag{114.15}$$

where the bar denotes an averaging with weight $\omega|\eta''(\omega)|$, as in (113.11). Then

$$F(q_0) = \frac{4\pi Ne^4}{mv^2}\log\frac{vq_0\gamma}{\Omega}-\frac{2\pi Ne^4}{mc^2}+\frac{e^2}{2v^2\gamma^2}\xi^2. \tag{114.16}$$

Two cases must be considered in the further examination of this formula. Let us first suppose that the medium is a dielectric, and that the velocity of the particle satisfies the condition

$$v^2 < c^2/\varepsilon_0, \tag{114.17}$$

where $\varepsilon_0 = \varepsilon(0)$ is the electrostatic value of the permittivity. On the imaginary axis the function $\varepsilon(\omega)$ decreases monotonically from $\varepsilon_0 > 1$ for $\omega = 0$ to 1 for $\omega = i\infty$. The expression on the left-hand side of equation (114.12) therefore increases monotonically from 0 to ∞, and for $q = 0$ (114.12) gives $\omega = 0$. Thus we must put $\xi = 0$ in (114.16); then Ω becomes the mean atomic frequency $\bar{\omega}$ (113.11), and

$$F(q_0) = \frac{4\pi Ne^4}{mv^2}\left[\log\frac{vq_0\gamma}{\bar{\omega}}-\frac{v^2}{2c^2}\right]. \tag{114.18}$$

For $v \ll c$ this formula becomes (113.12), as it should.

The value of q_0 is such that $q_0 \ll 1/a$, where a is the order of magnitude of the interatomic distances (in solids and liquids equal to the dimension of the atoms). In order to extend the formula to higher values of the transferred momentum and energy, it must be "joined" to the formulae of the ordinary theory of collisions, as in §113, but the joining must now be carried out in two stages, First, using formula (113.13), we enter the range of q corresponding to energy transfers large compared with atomic energies but not yet relativistic. Formula (114.18) is unchanged in form, but may now involve the δ-electron energy $\hbar^2q_1^{\,2}/2m$. Calling this E_1, we have

$$F(E_1) = \frac{2\pi Ne^4}{mv^2}\left[\log\frac{2mv^2E_1\gamma^2}{\hbar^2\bar{\omega}^2}-\frac{v^2}{c^2}\right]. \tag{114.19}$$

We can now go on to the relativistic values of E_1 by using a formula of relativistic collision theory, according to which the stopping power with energy transfer between E' and $E' + dE'$ is

$$(2\pi Ne^4/mv^2)\,dE'/E' \qquad (114.20)$$

if E' is small compared with the maximum transfer $E_{1,\max}$ compatible with the laws of conservation of momentum and energy for a collision between the fast particle concerned and a free electron.† Since the integration of (114.20) gives a term in $\log E'$, it is clear that formula (114.19) is unchanged in form, and it is therefore valid for all $E_1 \ll E_{1,\max}$.

In the retardation of a fast heavy particle (with mass $M \gg m$ and energy E which, though relativistic, is such that $E \ll M^2c^2/m$), the maximum energy transfer to an electron is $E_{1,\max} \cong 2mv^2\gamma^2$ and is still small in comparison with E (see *QED*, (82.23)). For such particles the differential expression for the energy lost to free electrons is

$$\frac{2\pi Ne^4}{mv^2}\left(\frac{1}{E'} - \frac{1}{2mc^2\gamma^2}\right)dE'$$

for all E'; see *QED*, (82.24). The energy loss additional to (114.19), with energy transfer from E_1 to $E_{1,\max}$ (with $E_1 \ll E_{1,\max}$) is then

$$\frac{2\pi Ne^4}{mv^2}\left(\log\frac{E_{1,\max}}{E_1} - \frac{E_{1,\max}}{2mc^2\gamma^2}\right) = \frac{2\pi Ne^4}{mv^2}\left(\log\frac{2mv^2\gamma^2}{E_1} - \frac{v^2}{c^2}\right). \qquad (114.21)$$

Adding this to (114.19), we find the total stopping power with respect to the heavy particle:

$$F = \frac{4\pi Ne^4}{mv^2}\left(\log\frac{2mv^2\gamma^2}{\hbar\bar{\omega}} - \frac{v^2}{c^2}\right). \qquad (114.22)$$

Formula (114.22) differs from that of the usual theory only in that the "ionization energy" is $\hbar\bar{\omega}$; cf. *QED*, (82.26).

Let us now turn to the second case, namely that where

$$v^2 > c^2/\varepsilon_0, \qquad (114.23)$$

which, in particular, always holds for metals, where $\varepsilon(0) = \infty$. The expression $\omega^2(\varepsilon/c^2 - 1/v^2)$ on the left-hand side of equation (114.12) then has two zeros on the imaginary ω-axis, one at $\omega = 0$ and the other at $\omega = i\xi$, where ξ is defined by

$$\varepsilon(i\xi) = c^2/v^2. \qquad (114.24)$$

In the range from 0 to $i\xi$ the expression $\omega^2(\varepsilon/c^2 - 1/v^2)$ is negative, and for $|\omega| > \xi$ it takes all positive values from 0 to ∞. As $q \to 0$, therefore, the root of equation (114.12) in this case tends to ξ, which is the value to be substituted in (114.15) and (114.16).

Two limiting cases may be considered. If ξ is small compared with the atomic frequencies ω_0, then the last term in (114.16) may be neglected, and $\Omega \cong \bar{\omega}$. Thus we return to formula (114.18). The opposite limiting case, where $\xi \gg \omega_0$, is of particular interest. Since, for large ξ, the function $\varepsilon(i\xi)$ tends to 1, it is evident from (114.24) that this case corresponds to ultra-relativistic velocities of the particle. Using formula (113.9) for $\varepsilon(\omega)$, we can write from equation (114.24)

$$\xi^2 = 4\pi Ne^2v^2\gamma^2/mc^2 \cong 4\pi Ne^2\gamma^2/m.$$

† See *QED*, (81.15) and (82.24). The stopping power F is obtained on multiplying these expressions for the cross-section by the energy loss $m\Delta$ and by N.

As the velocity of the particle increases, the condition $\xi \gg \omega_0$ is ultimately fulfilled in any medium, i.e. whatever the electron density N (even in a gas). The velocity required is, however, the greater, the smaller N, i.e. the more rarefied the medium.

From (114.15) we then have simply $\Omega \cong \xi$. Putting also $v \cong c$, we find that the last two terms in (114.16) cancel, leaving

$$F(q_0) = (2\pi Ne^4/mc^2)\log(mc^2q_0^2/4\pi Ne^2).$$

Extending this formula, in the same manner as above, to large values of the momentum and energy transfer, we find the following expression for the energy loss of an ultra-relativistic particle with an energy transfer not exceeding E_1 ($\ll E_{1,\max}$):

$$F(E_1) = (2\pi Ne^4/mc^2)\log(m^2c^2E_1/2\pi Ne^2\hbar^2). \qquad (114.25)$$

This result is considerably different from that obtained in the ordinary theory, which neglects the polarization of the medium. According to that theory (see *QED*, §82), in the ultra-relativistic range the stopping power $F(E_1)$ continues to increase (though only logarithmically) with the energy of the particle:†

$$F(E_1) = \frac{2\pi Ne^4}{mc^2}\log\left(\frac{2mc^2\gamma^2E_1}{I^2} - 2\right).$$

The polarization of the medium results in a screening of the charge, and the increase in the losses is thereby finally stopped; it tends to the constant value (independent of γ) given by formula (114.25).

For heavy particles a formula can also be derived for the total stopping power with any energy transfer up to $E_{1,\max}$ (if the latter is small compared with the energy of the particle itself). Again using the expression (114.21), in which we can now put $v = c$, we find

$$F = \frac{2\pi Ne^4}{mc^2}\left[\log\frac{m^3c^4\gamma^2}{\pi Ne^2\hbar^2} - 1\right]. \qquad (114.26)$$

We see that the total stopping power continues to increase with the velocity of the particle, owing to close collisions with a large energy transfer, for which the polarization of the medium has no screening effect. This increase, however, is rather slower than that given by the theory when the polarization is neglected. According to that theory,

$$F = \frac{4\pi Ne^4}{mc^2}\left[\log\frac{2mc^2\gamma^2}{I} - 1\right];$$

see *QED*, (82.28). The coefficient of the $\log\gamma$ term here is twice that in (114.26).

It may also be noted that the presence of the electron density N in the argument of the logarithm in formulae (114.25) and (114.26) results in the following property of energy losses of ultra-relativistic particles: when such a particle passes through layers of different substances containing the same number of electrons per unit surface area, the losses are smaller in media with larger N.

† This formula is obtained by adding *QED* (82.20) and (82.25), with E_1 for $m\Delta_{\max}$ in the latter. For a small energy transfer E_1, the formulae apply to both fast electrons and fast heavy particles.

§115. Cherenkov radiation

A charged particle moving in a transparent medium emits, in certain circumstances, an unusual type of radiation, first observed by P. A. Cherenkov and S. I. Vavilov, and theoretically interpreted by I. E. Tamm and I. M. Frank (1937). It must be emphasized that this radiation is entirely unrelated to the bremsstrahlung which is almost always emitted by a rapidly moving electron. The latter radiation is emitted by the moving electron itself when it collides with atoms. The Cherenkov effect, however, involves radiation emitted by the medium under the action of the field of the particle moving in it. The distinction between the two types of radiation appears with particular clarity when the particle has a very large mass: the bremsstrahlung disappears, but the Cherenkov radiation is unaffected.

The wave number and frequency of an electromagnetic wave propagated in a transparent medium are related by $k = n\omega/c$, where $n = \sqrt{\varepsilon}$ is the refractive index, which is real. We again suppose the medium isotropic and non-magnetic. We have seen that the frequency of the Fourier component of the field of a particle moving uniformly in the x-direction in a medium is related to the x-component of the wave vector by $\omega = k_x v$. If this component is a freely propagated wave, these two relations must be consistent. Since $k > k_x$, it follows that we must have

$$v > c/n(\omega). \tag{115.1}$$

Thus radiation of frequency ω occurs if the velocity of the particle exceeds the phase velocity of waves of that frequency in the medium concerned.†

Let θ be the angle between the direction of motion of the particle and the direction of emission. We have $k_x = k \cos\theta = (n\omega/c)\cos\theta$ and, since $k_x = \omega/v$, we find that

$$\cos\theta = c/nv. \tag{115.2}$$

Thus a definite value of the angle θ corresponds to radiation of a given frequency. That is, the radiation of each frequency is emitted forwards, and is distributed over the surface of a cone with vertical angle 2θ, where θ is given by (115.2). The distributions of the radiation in angle and in frequency are thus related in a definite manner.

The emission of electromagnetic waves, if it occurs, involves a loss of energy by the moving particle. This loss forms part, though a small part, of the total losses calculated in §114. (The bremsstrahlung is not included therein.) In this sense the term "ionization losses" is not quite accurate. We shall now find the corresponding part of the total losses, and thus determine the intensity of the Cherenkov radiation.

According to (114.9), the energy loss in the frequency interval $d\omega$ is

$$dF = -d\omega \frac{ie^2}{\pi} \sum \omega \left(\frac{1}{c^2} - \frac{1}{\varepsilon v^2}\right) \int \frac{q\,dq}{q^2 - \omega^2\left(\dfrac{\varepsilon}{c^2} - \dfrac{1}{v^2}\right)},$$

where the summation is over terms with $\omega = \pm|\omega|$. We introduce as a new variable

$$\xi = q^2 - \omega^2\left(\frac{\varepsilon}{c^2} - \frac{1}{v^2}\right).$$

† The problem of radiation from an electron moving uniformly in a vacuum at a velocity $v > c$ was discussed by A. Sommerfeld (1904) before the theory of relativity became known.

Then

$$dF = -d\omega \frac{ie^2}{2\pi} \sum \omega \left(\frac{1}{c^2} - \frac{1}{\varepsilon v^2}\right) \int \frac{d\xi}{\xi}.$$

In integrating along the real ξ-axis we must pass round the singular point $\xi = 0$ (for whi $q^2 + k_x^2 = k^2$) in some manner, which is determined by the fact that, although we suppo $\varepsilon(\omega)$ real (the medium being transparent), it actually has a small imaginary part, which positive for $\omega > 0$ and negative for $\omega < 0$. Accordingly, ξ has a small negative or positi imaginary part, and the path of integration ought to pass below or above the real ax respectively. This means that, when the path of integration is displaced to the real axis, w must pass below or above the singular point respectively. This gives a contribution to dI and the real parts cancel in the sum. Indenting the path of integration with infinitesim semicircles, we find

$$\sum \omega \int d\xi/\xi = \omega \left\{ \int_{\cup} d\xi/\xi - \int_{\cap} d\xi/\xi \right\} = 2i\pi\omega.$$

Thus the final formula is

$$dF = \frac{e^2}{c^2}\left(1 - \frac{c^2}{v^2 n^2}\right)\omega\, d\omega, \tag{115.3}$$

which gives the intensity of the radiation in a frequency interval $d\omega$. According to (115.2) this radiation is emitted in an angle interval.

$$d\theta = \frac{c}{vn^2 \sin\theta} \frac{dn}{d\omega} d\omega. \tag{115.4}$$

The total intensity of the radiation is obtained by integrating (115.3) over all frequencies for which the medium is transparent.

It is easy to determine the polarization of the Cherenkov radiation. As we see from (114.7) the vector potential of the radiation field is parallel to the velocity $\mathbf{v}$. The magnetic field $\mathbf{H}_k = i\mathbf{k} \times \mathbf{A}_k$ is therefore perpendicular to the plane containing $\mathbf{v}$ and the ray direction $\mathbf{k}$. The electric field (in the "wave region") is perpendicular to the magnetic field, and therefore lies in that plane.

PROBLEM

Find the cone of Cherenkov radiation wave vectors for a particle moving uniformly in a uniaxial non-magnetic crystal: (a) along the optical axis, (b) at right angles to the optical axis (V. L. Ginzburg, 1940).

SOLUTION. (a) When a charge moves in a uniaxial crystal, the Cherenkov radiation is in general on two cones corresponding to the ordinary and extraordinary waves. In motion along the optical axis, however, the ordinary wave is not emitted, even though a condition such as (115.1) may be satisfied: this wave always has linear polarization with the vector $\mathbf{E}$ perpendicular to the principal cross-section (that is, the plane through the optical axis—which we take as the z-axis—and the direction of any given $\mathbf{k}$), and the emission of such a wave in the case concerned is evidently impossible, since the work $e\mathbf{E}\cdot\mathbf{v} = 0$ and the particle does not lose energy. The extraordinary radiation cone is found by substituting in (98.5) the value of n from (115.2), which is valid even if the medium is not isotropic; in the present case, the angle θ between $\mathbf{k}$ and $\mathbf{v}$ is the same as the angle between $\mathbf{k}$ and the optical axis. The result is

$$\tan^2\theta = (\varepsilon_\parallel/\varepsilon_\perp)(v^2\varepsilon_\perp/c^2 - 1),$$

and we must have $v > c/\sqrt{\varepsilon_\perp}$. This is a circular cone on which the intensity distribution is uniform over the generators (as is in any case obvious from symmetry). The vertical angle 2ϑ of the ray vector cone is related to θ by $\tan\vartheta = (\varepsilon_\perp/\varepsilon_\parallel)\tan\theta$.