Physics 216. Quantum Mechanics. Professor Dine

Spring, 2004. Homework Set 4. Due Tues., May 18.

Radiation Theory (Also a Handout!)

Make sure you read Shankar, p. 499-520.

Let's remember the form of the lagrangian for the electromagnetic field. In a covariant form (see Jackson) the action is:

$$S = \int dt d^3x \mathcal{L} = \int dt L = \int d^4x - \frac{1}{4} F_{\mu\nu}^2 \tag{1}$$
$$= \int d^4x \frac{1}{2} (\vec{E}^2 - \vec{B}^2) = \int d^4x \frac{1}{2} ((\frac{d}{dt}\vec{A})^2 - (\partial_i A_j)^2).$$

Exercise (1.): Verify the last two equalities, and check that for vanishing charges and currents, in the gauge $\vec{\nabla} \cdot \vec{A} = 0$ that this gives the wave equation for $\vec{A}$ (you may set $A_o = \phi = 0$). Determine the canonical momentum, and show that the Hamiltonian is

$$H = \int d^3x \frac{1}{2} ((\frac{d}{dt}\vec{A})^2 + (\partial_i A_j)^2) \tag{2}$$

We want to show that this is a collection of harmonic oscillators. Consider the system in a cubic box of volume V. We can Fourier-analyze the vector potential:

$$A = \sum_{\vec{k}} \frac{1}{\sqrt{V}} e^{i\vec{k}\cdot\vec{x}} \vec{A}(\vec{k}, t) \tag{3}$$

Substituting back in the lagrangian:

$$L = \sum_{\vec{k}} \frac{1}{2} (\frac{d}{dt}\vec{A}(\vec{k}) \cdot \frac{d}{dt}\vec{A}(-\vec{k}) - \vec{k}^2 \vec{A}(k) \cdot \vec{A}(-\vec{k})). \tag{4}$$

Exercise (2). Verify this equation. Show that $\vec{A}(-\vec{k}) = \vec{A}(\vec{k})^*$. Use this, and write $\vec{A} = \vec{A}_1 + i\vec{A}_2$, to reexpress the lagrangian as:

$$L = \sum_{\vec{k}} \frac{1}{2} ((\frac{d}{dt}\ A_1(\vec{k}))^2 - \vec{k}^2 \vec{A}_1(k)^2) + \frac{1}{2} ((\frac{d}{dt}\vec{A}_2(\vec{k}))^2 - \vec{k}^2 \vec{A}_2(k)^2). \tag{5}$$

Interpret this as a lagrangian for harmonic oscillators. In this form, are there restrictions on the sum over $\vec{k}$?

Exercise (3). What are the restrictions on the $A_i(\vec{k})$'s? (remember the gauge condition). To implement this restriction, write

$$A_i(\vec{k}) = \sum_{\lambda} \epsilon_i(k, \lambda) Q_i(k, \lambda) \tag{6}$$

with a suitable restriction on $\vec{\epsilon}$. For this part of the discussion, you may want to restrict ϵ to be real.

Exercise (4): Now make the usual correspondence between harmonic oscillator coordinates and momenta and raising and lowering (creation and annihilation) operators. Starting with the form of the Hamiltonian in terms of such operators, show that the Hamiltonian for the electromagnetic field can be written in the form:

$$H = \sum_{\vec{k},\lambda} \omega_k (a^\dagger(\vec{k},\lambda) a(\vec{k},\lambda) + \frac{1}{2}). \tag{7}$$

Here there is no restriction on $\vec{k}$ (explain).

A more useful way to do this is to immediately expand $\vec{A}$ in terms of creation and annihilation operators. We can think of $\vec{A}$ as a Heisenberg operator, and write the expansion in the form

$$\vec{A}(\vec{x},t) = \sum_k \frac{1}{\sqrt{V} 2\omega_k} [\epsilon(\vec{k},\lambda) a(\vec{k},\lambda) e^{-i\omega t + i\vec{k}\cdot\vec{x}} + \epsilon^*(\vec{k},\lambda) a^\dagger(\vec{k},\lambda) e^{i\omega t - i\vec{k}\cdot\vec{x}}] \tag{8}$$

We can plug this into our expression for the Hamiltonian, yielding, again:

$$H = \sum_{\vec{k},\lambda} \omega_k (a^\dagger(\vec{k},\lambda) a(\vec{k},\lambda) + \frac{1}{2}). \tag{9}$$

We can interpret this by saying that the eigenvalues of the number operators, $a^\dagger(\vec{k},\lambda)a(\vec{k},\lambda)$ are the occupation numbers of the state (the number of photons of momentum $\vec{k}$ in the state). The energy of the ground state is unobservable (except in the presence of gravity); it is known as the zero point energy, and represents the fact that in quantum mechanics one can't simply have a state with $\vec{A} = \vec{E} = 0$.

Exercise 5: Do the same thing for the momentum of the electromagnetic field:

$$\vec{P} = \int d^3x \vec{E} \times \vec{B} \tag{10}$$

(Hint: it is important to use: $\epsilon(\vec{k},\lambda) \cdot \epsilon(\vec{k},\lambda') = \delta_{\lambda,\lambda'}$). Also, at some stage, you may need to use $\sum \vec{k} = 0$.)

Exercise 6: Show that, for the photoelectric effect, the cross section for a single photon to ionize an atom is the same as that we found in the semiclassical approximation. *You do not need to redo the calculation we did there; just show that the formulas are the same.*

Exercise 7: (This problem concerns molecular physics.) A model potential often used for the internuclear potential of diatomic molecules is the Morse potential:

$$U(R) = U_o(e^{-2(R-R_o)/a} - 2e^{-(R-R_o)/a}) \tag{11}$$

Discuss plausible values for the parameters R_o and a. Describe the rotational and vibrational spectra in terms of these parameters.

Exercise 8: Justify the use of the adiabatic approximation in molecular physics by estimating the distance travelled by the nuclei of a molecule during a period of the electronic motion.