Physics 216. Quantum Mechanics. Professor Dine

Spring, 2004. A SCATTERING THEORY PRIMER

Also Homework Set 5. Due Thursday, May 27.

Scattering theory is central to quantum mechanics. It can be a confusing subject. These notes attempt to summarize the basic ideas in a concise way.

1 Time-Dependent Perturbation Theory Approach

The quickest way to get to answers is to set up scattering as a problem in time-dependent perturbation theory. In practice, this is the most useful way to get results when there are complicated possible final states (think of a high energy proton hitting another proton, and the amplitude to produce a proton, 45 pi mesons, two electrons, three photons...)

The basic idea is to suppose that the interaction is short range, so if the projectile and the target are well separated, the interaction turns off. So for two particles interacting with a potential, one takes the interaction Hamiltonian to be:

$$H'(t) = V(r)e^{\eta t} \tag{1}$$

where one will take $\eta \to 0$ at the end of any computation. Proceeding with first order perturbation theory yields for the transition rate per unit time:

$$R = \frac{2\pi}{\hbar} | < f|V|i > |^2 \delta(E_f - E_i). \tag{2}$$

Exercise: Derive this. Note that the initial and final states here are general.
Exercise: Consider two particle, elastic scattering. Define center of mass and relative coordinates. Discuss separation of variables in the Schrodinger equation, and argue that you can ignore the center of mass coordinates.

To obtain a cross section, one needs to integrate over some set of final states (this gets rid of the δ-function) and divide by the flux. For incoming plane waves of momentum $\vec{k}$, the flux is $\vec{j} = \vec{v}$ (with box normalization, there is a $1/V$; there is also a V in the density of states; these factors are cancelled by a $1/V^2$ in the matrix element-squared). For elastic scattering, the final state can be taken to have momentum $\vec{k}'$, and the integration over final states involves $\int \frac{d^3k}{(2\pi)^3}$. This gives the cross section formula:

$$\frac{d\sigma}{d\Omega} = \frac{m^2}{4\pi^2} < \vec{k}'|V(r)|\vec{k} > |^2. \tag{3}$$

The matrix element is just the Fourier transform of the potential with respect to the "momentum transfer", $\vec{q} = \vec{k} - \vec{k}'$. For a Yukawa potential,

$$V(r) = \frac{ge^{-\mu r}}{r} \tag{4}$$

the fourier transform is:

$$f(\theta) = -\frac{2mg}{m^2 + q^2} \tag{5}$$

If θ is the scattering angle,

$$q^2 = 4k^2 \sin^2(\theta/2). \tag{6}$$

Exercise: Consider the problem of the photoelectric effect in this language, using the quantized electromagnetic field. The initial state is the state of the incoming photon and the hydrogen atom in its ground state. The outgoing state is the ion and the electron. You can take the photon flux to be c. You can copy any integrals needed for the matrix element from our earlier discussion.

2 Time-Independent Approach to Scattering Problems

Here we will focus mainly on elastic scattering. The idea here is that the continuum states (e.g. the $E > 0$ states in the square well or the hydrogen atom) are associated with scattering. For a short range potential, the solutions have the form, for large r,

$$\psi_{\vec{k}}(\vec{x}) = e^{i\vec{k}\cdot\vec{x}} + f(\theta,\phi)\frac{e^{ikr}}{r} \tag{7}$$

You can check that at large r this is a solution. It corresponds to an incoming plane wave and an outgoing spherical wave. The proper treatment of this problem uses wave packets. A linear superposition of these states will behave, in the far past, as a free, incoming wave, and in the far future as a free outgoing wave plus an outgoing spherical wave.

We can obtain the cross section by the following heuristic device (justified by the full wave packet treatment, to be described in class). Treat the first and second terms as non-interfering. In practice, this will be ok if the wave is sufficiently localized transverse to the beam, and one studies scattering away from the beam direction. Calculate the incoming flux, the outgoing flux at θ, ϕ to give the ratio:

$$\frac{d\sigma}{d\Omega} = |f(\theta,\phi)|^2. \tag{8}$$

Exercise: Derive this result. You will need, for the outgoing flux, the gradient in spherical coordinates.

So for this problem, we know everything if we know $f(\theta,\phi)$. Up to a phase, we can guess f from our time-dependent discussion. We can find it, in principle, by solving the Schrodinger equation exactly. But again, it is most useful to develop a perturbation series (again this generalizes to complicated problems for which one cannot, say, work out a numerical solution easily). To find f, one can write a formal solution of the Schrodinger equation in terms of a Green's function for the free-particle problem.

$$(\vec{\nabla}^2 + \vec{k}^2)G^o(\vec{r}) = \delta(\vec{r}). \tag{9}$$

This is actually the Green's function for the Helmholtz equation, which you have encountered in $E\&M$. The solution is:

$$G^o = -\frac{e^{ikr}}{4\pi r}. \tag{10}$$

Using this, we can write a formal solution to the Schrodinger equation:

$$\psi_k(r) = e^{i\vec{k}\cdot\vec{r}} + 2m\int d^3r' G^o(\vec{r},\vec{r}')V(\vec{r}')\psi_k(\vec{r}'). \tag{11}$$

We can solve this in a series of successive approximations. First, if the potential is weak, approximate ψ_k by $e^{i\vec{k}\cdot\vec{x}}$. This is the Born approximation:

$$\psi_k(r) = e^{i\vec{k}\cdot\vec{r}} + 2m \int d^3r' G^o(\vec{r},\vec{r}')V(\vec{r}')e^{i\vec{k}\cdot\vec{r}'}. \tag{12}$$

Higher orders yield the "Born series":

$$\psi_k = e^{i\vec{k}\cdot\vec{r}} + 2mG^oV\psi_o + (2m)^2G^oVG^oV + \ldots \tag{13}$$

where integrations are understood, and $\psi_o = e^{i\vec{k}\cdot\vec{x}}$.

At leading order, we can read off the scattering amplitude, $f(\theta,\phi)$. We want to know the behavior of ψ_k for large r:

$$\psi_k(\vec{r}) \approx e^{ik\cdot\vec{r}} - \frac{e^{ikr}}{r}\frac{2m}{4\pi}\int d^3r' e^{-i\vec{k}'\cdot\vec{r}'}V(\vec{r}')e^{i\vec{k}\cdot\vec{r}'} \tag{14}$$

From this we read of the scattering amplitude:

$$f(\theta,\phi) = \frac{m}{2\pi} < \vec{k}'|V\vec{k} > \tag{15}$$

in agreement with the time-dependent analysis.

Further Problems:

1. Sakurai 7.2.
2. Sakurai 7.3.
3. Sakurai 7.6.
4. Sakurai 7.10.