
INTRODUCTION TO THE QUANTUM THEORY

Third Edition

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in 1929 to his death in 1939. Lee
om 1947 to 1954. Leonard I. Schiff

CHAPTER 14

THE THEORY OF ALPHA DECAY

14.1 ESCAPE OF AN ALPHA PARTICLE

The most characteristic feature of all radioactive decay processes is the exponential law they follow: N , the number of atoms which have not decayed, decreases with time according to

$$N = N_0 e^{-\lambda t} \quad (14.1)$$

In order to see what this law says about any one atom, differentiate it to find

$$\frac{dN}{dt} = -\lambda N \quad (14.2)$$

so that the rate of decay each second is proportional to the number present. It follows that λ represents the probability that any given nucleus will decay in any given unit interval of time. This constant probability, independent of the age of the nucleus, was very hard to understand with the traditional concepts of physics, and it is only with the insights of quantum mechanics that it seems natural.

This chapter will study the emission of alpha particles from a nucleus. The situation here is different from that in beta or gamma decay, for example; those particles do not exist in the nucleus but are formed at the moment of emission. The material constituting the alpha particle, however, is a part of the nucleus, and we are going to assume that because of the extraordinary stability of this configuration (once formed, it takes 20.5 MeV to remove a neutron from an alpha particle in isolation from other nuclear matter), the neutrons and protons inside

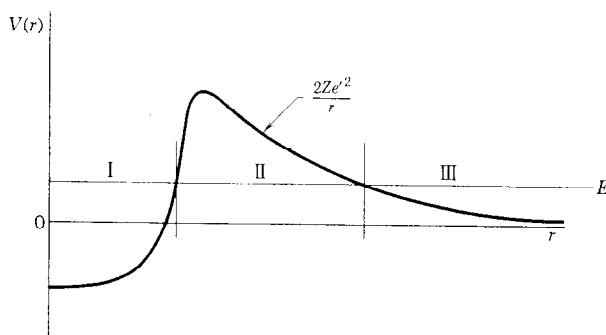


FIGURE 14-1

Qualitative behavior of the potential acting on an alpha particle inside and outside the nucleus.

a nucleus spend an appreciable fraction of their time all made up as alpha particles and ready to be emitted. How they escape from the forces binding the nucleus together was explained by Gamow¹ and others as an example of the quantum-mechanical penetration of a potential barrier. That such a barrier exists can be seen if one looks at it from either side: An alpha particle incident from outside will be repelled by the coulomb field of the nucleus, and on the other hand it is clear that a nucleus would disperse altogether under the influence of the coulomb repulsion of its charges if it were not enclosed by the barrier of some stronger force. The situation is therefore as shown in Fig. 14.1, where only the outer extremity of the barrier can be drawn with certainty, since it is due to a coulomb force (Z is the atomic number of the *daughter* nucleus). Because it is difficult to be more detailed, we will assume only that a single alpha particle is always present in the nucleus, and will calculate its probability of escape.

We are going to evaluate the wave function in the WKB approximation, but this at once raises a difficulty, since the method as we have developed it starts from the time-independent Schrödinger equation for a stationary state, whereas here we are assuming that a bound alpha particle will ultimately leak away and the wave function is therefore not a periodic function of the time. We can, however, treat the problem approximately by making use of the fact that if the barrier is nearly impenetrable, the wave function inside will be nearly that of the corresponding stationary state. The situation is essentially three-dimensional, but by considering only spherically symmetrical wave functions, corresponding to the emission of alpha particles with angular momentum equal to zero, we can do the computation in one dimension. Assume that ψ is a function of $r := (x^2 + y^2 + z^2)^{1/2}$ only, and introduce for convenience a new variable $u(r)$, defined by $\psi(r) = u(r)/r$. Then

$$\nabla^2 \psi = \frac{1}{r} \frac{d^2 u}{dr^2}$$

¹ For Gamow's approach, which differs from the one used here, see Fermi (1950), chap. 3.

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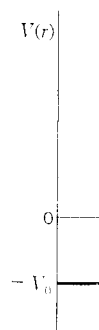
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Simplified

so that Schrödinger's equation becomes

$$\frac{d^2u}{dr^2} + \frac{2m}{\hbar^2} [E - V(r)]u = 0$$

If ψ is finite at the origin, then

$$u(0) = 0 \tag{14.3}$$

In order to avoid the worst computation difficulties, we shall assume a somewhat simplified model due to Bethe, in which the alpha particle is assumed to move in a constant potential, which is the smoothed-out effect of all its nuclear interactions, until it reaches the edge of the nucleus, where an abrupt rise takes place. In the three regions of Fig. 14.2, the WKB approximation gives

$$u_I = A \sin Kr \quad 0 < r < R \tag{14.4}$$

$$u_{II} = C\kappa^{-1/2} \exp\left(-\int_R^r \kappa dr\right) + D\kappa^{-1/2} \exp\left(\int_R^r \kappa dr\right) \quad R < r < b \tag{14.5}$$

$$u_{III} = A'k^{-1/2} \exp\left(i\int_b^r k dr\right) \quad b < r \tag{14.6}$$

Here

$$K = \left[\frac{2m}{\hbar^2} (E + V_0)\right]^{1/2}$$

$$\kappa = \left[\frac{2m}{\hbar^2} \left(\frac{2Ze^2}{r} - E\right)\right]^{1/2}$$

$$k = \left[\frac{2m}{\hbar^2} \left(E - \frac{2Ze^2}{r}\right)\right]^{1/2}$$

We have used the boundary condition (14.3) for u_I and specified an outgoing wave in region III. The WKB approximation used in regions II and III is here

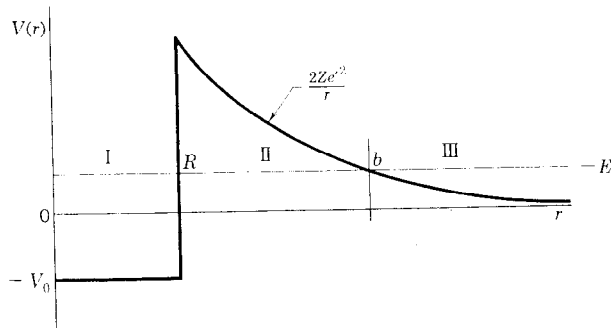


FIGURE 14.2
Simplified version of Fig. 14.1.

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valid for the slowly varying coulomb potential. The second turning point b is given by

$$b = \frac{2Ze^2}{E} \quad (14.7)$$

To join the wave functions at the nuclear radius, we have from the continuity of u

$$A \sin KR = (C + D)[\kappa(R)]^{-1/2} \quad (14.8)$$

and from that of u' ,

$$AK \cos KR = (-C + D)[\kappa(R)]^{1/2} \quad (14.9)$$

where in evaluating the derivative on the right we have included the rapid dependence of u on r through the exponential and neglected its relatively slow dependence through the amplitude. To match wave functions at $r = b$, write

$$u_{II} = Ce^{-\sigma} \kappa^{-1/2} \exp\left(-\int_b^r \kappa dr\right) + De^{\sigma} \kappa^{-1/2} \exp\left(\int_b^r \kappa dr\right) \quad (14.10)$$

where

$$\sigma := \int_R^b \kappa(r) dr \quad (14.11)$$

and comparison with (4.52b) and (14.6) then shows that

$$C = \mathfrak{g}^* e^{\sigma} A' \quad D = \frac{1}{2} \mathfrak{g} e^{-\sigma} A' \quad \mathfrak{g} := e^{i\pi/4} \quad (14.12)$$

We shall see later that σ is at least of the order of 10, so that D is much smaller than C , as one would expect from (14.5). The alpha particle escapes only after a very long time, and its wave function is therefore essentially that of a particle in a stationary state, which would have D equal to zero (Fig. 14.3).

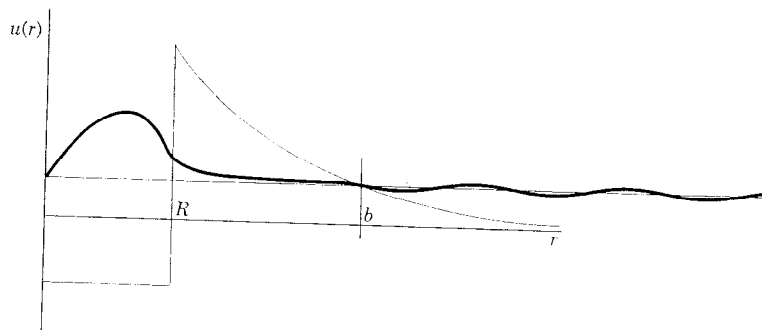


FIGURE 14.3

Wave function for the simplified version. The amplitude of the outer wave is greatly exaggerated.

The probability per second of decay is denoted by λ and is equal to

$$\lambda = \frac{4\pi\hbar|A'|^2}{m} \tag{14.13}$$

Problem 14.1. Derive the last formula.

By (14.12), we have

$$\lambda = 4\pi \frac{\hbar}{m} e^{-2\sigma} |C|^2 \tag{14.14}$$

and from (14.9), neglecting D ,

$$\lambda = 4\pi \frac{\hbar K^2}{m\kappa(R)} |A|^2 e^{-2\sigma} \cos^2 KR \tag{14.15}$$

This formula can be simplified if we assume that the alpha particle within the nucleus is in the lowest quantum state. The nuclear barrier then reduces the wave function almost to zero at the edge of the nucleus, and we have

$$KR \approx \pi \quad 4\pi|A|^2 \approx \frac{2}{R} \tag{14.16}$$

Problem 14.2. Justify (14.16).

Thus finally, in this approximation,

$$\lambda = \frac{2\pi^2\hbar}{mR^3\kappa(R)} e^{-2\sigma}$$

In order to gain some physical insight into this formula we can rewrite it (using $KR \approx \pi$) as

$$\lambda = 2 \frac{\hbar K}{mR} \frac{K}{\kappa(R)} e^{-2\sigma}$$

The quantity $\hbar K/m$ is the velocity v_{in} of the alpha particle within the nucleus, and $K/\kappa(R)$ is in the general neighborhood of unity. Thus, roughly,

$$\lambda \approx \frac{v_{in}}{R} e^{-2\sigma} \tag{14.17}$$

which can be given a simple interpretation: The alpha particle oscillates around in the nucleus, hitting the nuclear barrier about v_{in}/R times per second. At each impact, there is a probability $e^{-2\sigma}$ of penetrating the barrier, so that the probability of penetration per second is approximately that given by λ . Because the exponential varies so rapidly, the $e^{-2\sigma}$ contains most of the result and approximations in the factor in front of it are unimportant.

Problem 14.3. Show that $e^{-2\sigma}$ is in fact the probability of penetrating the barrier.

Problem 14.6. Extend the calculation by including the next terms of the expansions in (14.20). How does this affect the estimate for R ? In deriving (14.17) it was assumed that $\kappa(R) \approx K$. Recalculate (14.21) using the correct value for $\kappa(R)$.

Problem 14.7. Assuming that nuclei of the same mass number have equal radii, find the formula which compares $\log \tau_{1/2}$ for a pair of such nuclei as a function of their different Z 's and E 's. ^{226}Ra emits a 4.78-MeV alpha with a half-life of 1622 y. What is the half-life of the 6.33-MeV alpha from ^{226}Th ? (The experimental value is 30.9 min.)

Problem 14.8. An alpha-active nucleus usually emits alphas of several different energies. Show that, other things being equal, the ratio of the intensities at two different energies E_1 and E_2 is given by $\exp[-3.84Z(E_1^{-1/2} - E_2^{-1/2})]$, and compare this formula with experimental values found in the literature.

Although the approximations which we have made preclude any claim to exactness, it is clear that they give insight that is better than qualitative. Historically this theory was the first successful application of quantum mechanics to a nuclear problem, and it had special importance as a successful calculation of a process that according to classical physics could not possibly happen. Many modern devices, such as those involving the Josephson effects (Chap. 6), depend on the tunneling of electrons through an oxide film.

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