The Lippmann-Schwinger Eqn., The S-Matrix, T-matrix, and Unitarity

In scattering theory, we are interested in states which, asymptotically, are free particle states. Write

$$H = H_0 + V$$

$$H_0 |\phi\rangle = E|\phi\rangle = \frac{p^2}{2m}|\phi\rangle$$

$$(H_0 + V)|\psi\rangle = E|\psi\rangle$$

Formal soln:

$$|\psi\rangle = \frac{1}{E - H_0} V|\psi\rangle + |\phi\rangle$$

Check: multiply by $E - H_0$:

$$(E - H_0)|\psi\rangle = V|\psi\rangle$$

$\frac{1}{E-H_0}$ is singular; $\Rightarrow$ $\frac{1}{E - H_0 + i\epsilon}$

In coordinate representation,

$$\langle \vec{x} | \frac{1}{E - H_0 + i\epsilon} | \vec{x}' \rangle = \int \frac{d^3p}{(2\pi)^3} \frac{e^{i\vec{p}\cdot(\vec{x}' - \vec{x})}}{[E - \vec{P}^2/2m + i\epsilon]}$$

This is the integral we encountered earlier;

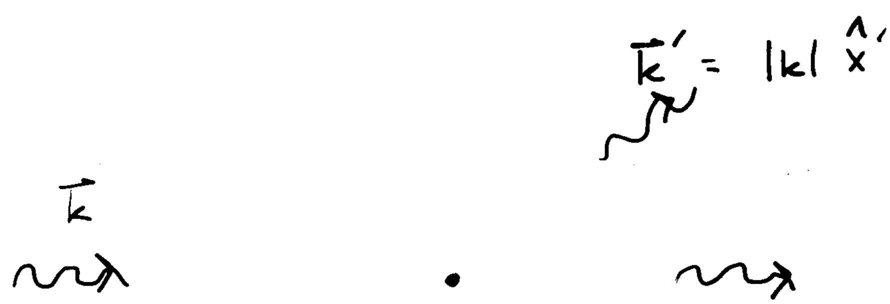
$$G = -\frac{1}{4\pi} \frac{2m}{\hbar^2} \frac{e^{ik|\vec{x}-\hat{x}'|}}{|\vec{x}-\hat{x}'|} \simeq -\frac{1}{4\pi} \frac{2m}{\hbar^2} \frac{1}{r} e^{ikr} e^{-ik\hat{x}\cdot}$$

We see that this choice of bd'y condition ($i\epsilon$ prescription) the asymptotic wave function has desired form,

$$\psi \underset{|x|\to\infty}{=} e^{i\vec{p}\cdot x} + \frac{f(\Omega)}{r} e^{ikr}$$

with $f(\Omega) = -\frac{1}{4\pi} \frac{2m}{\hbar^2} \int d^3x' \, e^{ik\hat{x}\cdot\vec{x}'} \, V(\vec{x}') \, \psi(\vec{x}')$

We have seen, in wave packet description, that from incoming plane wave, we get transmitted wave + outgoing scattered wave



Call

$$\langle \vec{k}' | S | \vec{k} \rangle = \langle \vec{k}' | U(\infty, -\infty) | \vec{k} \rangle$$

It is natural to break up S in a way which reflects transmitted, reflected wave. Call

$$V | \psi^{(+)} \rangle = T | \phi \rangle \qquad \psi\rangle = T|\vec{p}\rangle$$

Then $f(\vec{k}', \vec{k}) \equiv f(x) = -\frac{1}{4\pi} \frac{2m}{\hbar^2}$ [illegible] $\langle \vec{k}' | T | \vec{k} \rangle$

$$S(k, \vec{k}') = (2\pi)^3 \delta^{(3)}(\vec{p}' - \vec{p}) - \frac{1}{E - \frac{\vec{p}'^2}{2m} + i\epsilon} \langle p' | T | \vec{p} \rangle$$

$$\frac{1}{E - \frac{\vec{p}'^2}{2m} + i\epsilon} = 2\pi i\, \delta(E_{p'} - E_p)$$

[Check: $\int \frac{dx}{x_0 - x + i\epsilon} f(x) = 2\pi i f(x_0)$ [res. theorem]

$$S(\vec{k}, \vec{k}') = (2\pi)^3 \delta^{(3)}(\vec{p}' - \vec{p}) - 2\pi i\, \delta(E_p' - E_p)\, t(\vec{p} \to \vec{p}')$$

$$t(\vec{p} \to \vec{p}') = \langle \vec{p}' | T | \vec{p} \rangle$$

In terms of T, we can rewrite LS eqn:

$$T|\phi\rangle = V|\phi\rangle + V\frac{1}{E-H_0+i\epsilon}T|\phi\rangle \qquad \equiv T|\psi\rangle$$

In this form, the equation is readily solved by iteration:

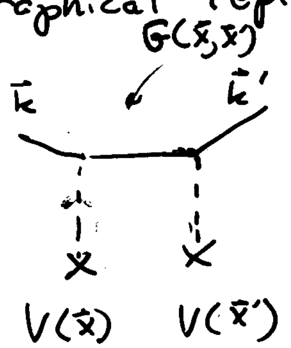
$$T = V + V\frac{1}{E-H_0+i\epsilon}V + V\frac{1}{E-H_0+i\epsilon}V\frac{1}{E-H_0+i\epsilon}V + \dots$$

$$f^{(1)}(\vec{k},\vec{k}') = -\frac{1}{4\pi}\frac{2m}{\hbar^2}(2\pi)^3\langle\vec{k}'|V|\vec{k}\rangle$$

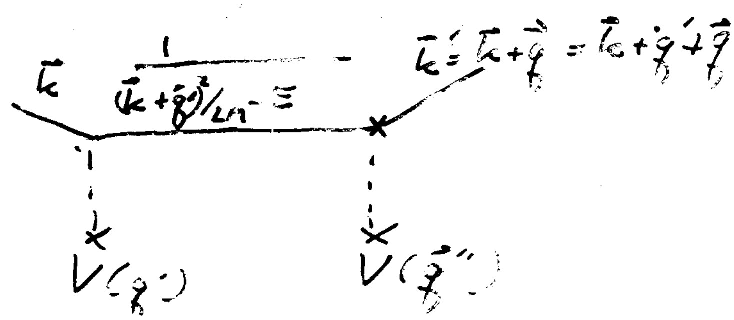
$$f^{(2)}(\vec{k},\vec{k}') = -\frac{1}{4\pi}\frac{2m}{\hbar^2}(2\pi)^3\langle\vec{k}'|V\frac{1}{E-H_0+i\epsilon}V|\vec{k}\rangle$$

$$= -\frac{1}{4\pi}\frac{2m}{\hbar^2}(2\pi)^3\int d^3x\, d^3x'\, e^{i\vec{k}\cdot\vec{x}}V(\vec{x})\,G(\vec{x},\vec{x}')$$
$$\times V(\vec{x}')\,e^{-i\vec{k}\cdot\vec{x}'}$$

Graphical rep:



$\Longrightarrow$



Unitarity of S-matrix $\Longleftrightarrow$ Optical Theorem

$$S^{+}S = 1$$

$$\int \frac{d^3p''}{(2\pi)^3}\, S^{+}(\vec{p},\vec{p}'')\, S(\vec{p}'',\vec{p}') = (2\pi)^3\, \delta^{(3)}(\vec{p}'-\vec{p})$$

$$- 2\pi i\, \delta(E_{p'}-E_p)\left[t(\vec{p}\to\vec{p}') - t^{*}(\vec{p}\to p')\right]$$

$$- \frac{(4\pi)^2}{(2\pi)^3}\, \delta(E_{p'}-E_p) \int d^3p''\, \delta(E_p - E_{p''})\, |t(\vec{p}',\vec{p}'')|^2$$

Recall, $|t|^2 = [(2\pi)^2 m]^{-2}\, |f|^2$

$$dp = \frac{m}{p}\, dE$$

$d^3p'' =$ [illegible] dE, gives

$\propto \int d\Omega\, \frac{p}{4m}\, |f|^2$ for last term

setting $\vec{p} = \vec{p}'$, δ-function drops out,

$$\mathrm{Im}\, f(\theta=0) = \frac{p/\hbar\; \sigma_{tot}}{4\pi} \qquad (f(\theta=0) = f(\vec{k},\vec{k}))$$

Resonances & Wave Packets

Recall scattered wave packet had form

$$\chi\left(\hat{k}(r - vt) + \vec{s} - \vec{b}\right) \qquad s = \nabla\alpha$$

$$f = |f| e^{i\alpha}$$

From our formula for σ_ℓ, assuming at resonance a single partial wave dominates cross section, if $\Gamma \ll E_r$, cross section has sharp peak.

$$f(\theta) \simeq -\frac{2\ell+1}{k} P_\ell(\cos\theta) \frac{\Gamma}{2(E-E_r)+i\Gamma}$$

$$\frac{d}{dE} \arg f(\theta) = \frac{d}{dE} \tan^{-1}\left[\frac{\Gamma}{2(E-E_r)}\right]$$

$$= \frac{2\Gamma}{4(E-E_r)^2+\Gamma^2}$$

At resonant energy, time delay of order Γ^{-1}.
Note energy spread of metastable state $\sim \Gamma$, consistent with time/energy uncertainty relation.

Observing metastable state:

Suppose $\Delta E \gg \Gamma$ in wave packet (opposite to our earlier discussion)

Set $b = 0$ (you can fix!)

$$\Psi_{sc} = \int d^3k' \, A(\vec{k}' - \vec{k}) \, f_k(\theta, \phi) \, \frac{e^{ik'r - E't/\hbar}}{r}$$

$$= \Sigma (2l+1) P_l(\cos\theta) \frac{\Gamma}{2} \frac{e^{ikr - Et/\hbar}}{r} \, I$$

$$\text{where} \quad I = \int \frac{d^3k' \, A}{k' [E' - E_r + \frac{1}{2} i \Gamma]} \exp[i(k' - k_r) r - i(E' - E_r$$

A assumed roughly constant in resonant region, so average over angles only

$$I \cong m A_r \, F(t - r/v_r) \qquad A_r = \int A(k_r \hat{k}' - \vec{k}) \, d\Omega$$

$$F(\tau) = \int_0^\infty dE \; \frac{e^{-i(E' - E_r)\tau}}{(E' - E_r + \frac{1}{2} i\Gamma)} \, dE'$$

For:

$|\tau| >> \hbar/\Delta E$: replace lower limit by $-\infty$.

Do as contour integral:

$$F(\tau) = \begin{cases} 0 & \tau << -\hbar/\Delta E \\ -2\pi i \, e^{-\Gamma\tau/2} & \tau >> \hbar/\Delta E \end{cases}$$

$$\psi_{sc} \cong -(2\ell+1) P_\ell(\cos\theta) \, \frac{m A_r}{2\hbar^2} \, \Gamma \, F(t - r/v_2) \, \frac{e^{i[k_r r - l}}{r}$$

At fixed r (away from [illegible]) see

nothing, then for $\frac{-\hbar}{\Delta t} < t < \hbar/\Delta E$

see strong signal which decays exponentially.