Description of Scattering in Terms of Wave Packets

Free wave packets first:

$|\psi\rangle$: state vector at $t=0$

$$\frac{d}{dt}\langle\vec{p}\rangle = 0$$

Under what circumstances does wave propagat without distortion?

$$\psi(\vec{x},t) = \psi(\vec{x}-\vec{v}t, 0) \times (\text{phase})$$

$$\vec{v} = \frac{\langle\vec{p}\rangle}{m}$$

Phase: expect $e^{-im\vec{v}^2/2\,t}$

Consider: $|\bar{\psi}\rangle = e^{-im\vec{v}^2/2\,t}\, e^{-i\vec{p}\cdot\vec{v}\,t/\hbar}|\psi\rangle$

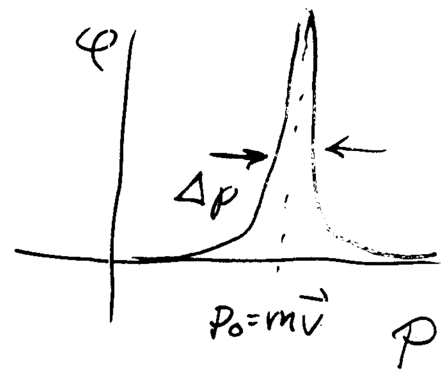
Study

$$|\langle\psi|\bar{\psi}\rangle| = |\langle\psi|\exp\left(\tfrac{i}{\hbar}\tfrac{\vec{p}^2}{2m}t\right)e^{-i\vec{p}\cdot\vec{v}\,t/\hbar}\,e^{-im\vec{v}^2\frac{t}{2\hbar}}$$

$$= |\langle\psi|\exp\left(\tfrac{i}{2m\hbar}(\vec{p}-m\vec{v})^2 t\right)|\psi\rangle$$

$$= \int\frac{d^3p}{(2\pi)^3}\,|\varphi(\vec{p})|^2\exp\{i(\vec{p}-m\vec{v})^2 t/2m\hbar\}d$$

$\varphi(\vec{p})$: momentum space wave function

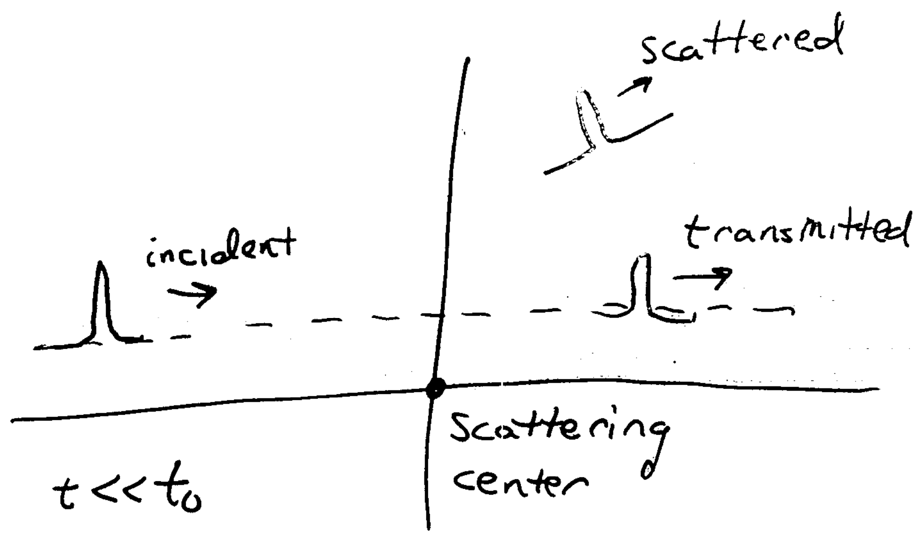


Exponential approximately unity provided $\frac{\Delta p^2 t}{2m\hbar} \ll 1$

or $\quad \frac{\Delta p\, t}{m} \ll \frac{2\hbar}{\Delta p} \cong 2\Delta x$

spreading $\ll$ width

Scattering problem



$\langle \vec{r} \rangle = \vec{b} + \vec{V}(t - t_0) \qquad t \ll t_0$

$\vec{b}$: impact parameter

t_0: time at which particle would reach scattering plane if free

Incident beam: collection of wave packets differing slightly in $\vec{b}$, shape, $\vec{v}$, etc. Nothing depends on shape (we will see). Take all to have same shape + $\vec{V}$; different $\vec{b}$.

Standard wave packet

$$\chi(\vec{r}) = \int \frac{d^3k}{(2\pi)^3}\, A(\vec{k})\, e^{i\vec{k}\cdot\vec{r}}$$

$\langle \vec{r} \rangle_\chi = 0$. Small transverse size (compared to apparatus). $A(\vec{k})$ peaked about $\vec{k}=0$.

Solution of full, time-dep. Schrodinger eqn:

$$\Psi_b(\vec{r},t) = \int \frac{d^3k'}{(2\pi)^3}\, A(\vec{k}'-\vec{k})\, e^{-i\vec{k}'\cdot\vec{b}}\, \psi_{k'}(\vec{r})\, e^{-iE't/\hbar}$$

$$E' = \frac{\hbar^2 k'^2}{2m}; \quad \psi_{k'}(\vec{r}) = e^{i\vec{k}'\cdot\vec{r}} + f_{k'}(\Omega)\, \frac{e^{ik'r}}{r}$$

Consider behavior of ψ_b as $t \to \begin{cases} -\infty \\ +\infty \end{cases}$

$t \to -\infty$: phase oscillates wildly except at stationary pts; occurs for large r, so can use asymptotic form of ψ:

$$\Psi_b(\vec r, t) = \int \frac{d^3k'}{(2\pi)^3}\, e^{-i\vec k'\cdot\vec b}\, e^{-iE't/\hbar}\, e^{i\vec k'\cdot r}\, A(\vec k' - \vec k)$$

$$+ \int \frac{d^3k'}{(2\pi)^3}\, e^{-i\vec k'\cdot\vec b}\, e^{-E't/\hbar}\, \frac{f(\theta,\phi)}{r}\, e^{ik'r}\, A(\vec k' - \vec k)$$

Phase of first term stationary for

$$-\vec b - \frac{\hbar\vec k'}{m} t + \vec r = 0, \text{ i.e.}$$

$$\vec r = \vec b + \frac{\hbar\vec k}{m} t \qquad (\vec k' \cong \vec k)$$

2^{nd} term: $\quad -\vec b - \frac{\hbar k}{m} t + \hat k' r = 0$

But $r > 0$, $t \to -\infty$, so <u>not</u> <u>satisfied</u>.

To see shape of wavepacket, shift $\vec k' \to \vec k' + \vec k$

$$E' \doteq \frac{\hbar^2 k^2}{2m} + \hbar v(\vec k - \vec k') + \mathcal{O}(\vec k - \vec k')^2$$

$$\Rightarrow \frac{\hbar^2 k^2}{2m} - \hbar v \vec k'$$

$$\Psi_b = \int \frac{d^3k'}{(2\pi)^3} A(\bar{k}'-\bar{k}) e^{-i\bar{k}'\cdot\bar{b}} e^{i\bar{k}'\cdot\bar{r}} e^{-im\frac{v^2}{2}t/\hbar} e^{i\bar{v}\cdot(\bar{k}-$$

$$= \int \frac{d^3k'}{(2\pi)^3} A(\bar{k}') e^{i(\bar{k}'\cdot(\bar{r}-\bar{b}-\bar{v}t))} e^{-iEt/\hbar} e^{i\bar{k}\cdot(\bar{r}-}$$

$$= e^{-iE_0 t/\hbar} e^{i\bar{k}\cdot(\bar{r}-\bar{b})} \chi(\bar{r}-\bar{v}t-\bar{b})$$

Now consider $t\to\infty$ Ψ_b a sum of two terms.

First is transmitted wave, $e^{-iE_0 t/\hbar} e^{i\bar{k}\cdot(\bar{r}-\bar{b})} \chi(\bar{r}-\bar{v}t-\bar{b}$

2^{nd} term now contributes:

$$\hat{k}' r = \bar{b} + \frac{\hbar \bar{k}'}{m} t \qquad (r \cong |\bar{v}|t)$$

To evaluate, write

$$f_{k'} = |f_{k'}| e^{i\alpha_{k'}} \cong |f_k| e^{i\alpha_k + i\vec{\nabla}_k \alpha_{k'}(\bar{k}-\bar{k})}$$

$$= f_k e^{i\nabla\alpha_k\cdot(\bar{k}-\bar{k}')}$$

2^{nd} term:

$$\frac{f_k(\theta,\phi)}{r} e^{-imv^2/2t} e^{i\bar{k}\cdot\bar{v}t} \int \frac{d^3k'}{(2\pi)^3} A(\bar{k}-\bar{k}')$$

$$\times e^{i\bar{k}'\cdot\left(\frac{\bar{k}r}{|\bar{k}|} - \bar{b} - \bar{v}t\right)} e^{i\vec{\nabla}\alpha_k\cdot(\bar{k}-\bar{k}'}$$

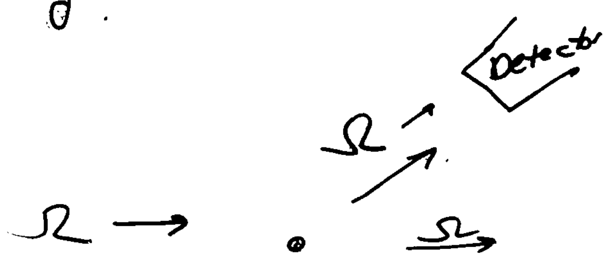
Shift

$$\vec{k}' \to \vec{k}' + \vec{k}$$

$$= \frac{f_k(\theta,\phi)}{r} e^{ikr} e^{-i\vec{k}\cdot b} \int \frac{d^3k}{(2\pi)^3} A(k') e^{ik'\cdot(\hat{k}r - \vec{b} - vt - i\vec{\nabla}\alpha_k)}$$

$$= \frac{f_k(\theta,\phi)}{r} e^{ikr} e^{-i\vec{k}\cdot\vec{b}} \chi(r\hat{k} - \vec{b} - \vec{v}t + \underbrace{\vec{\nabla}\alpha_k}_{\text{time delay}})$$

Calculating the cross section:



Provided wave peaked in space, detector not too close to beam, only scattered wave detected. Assume uniform distribution in $\vec{b}$. Ask probability packet has entered ~~detector~~ after some large time.

$$P_{\Delta\Omega} = \Delta\Omega \int_0^\infty dr\, r^2 |\psi_0(r,t)|^2 = |f_k(\theta,\phi)|^2 \Delta\Omega$$

$$\times \int_0^\infty dr\, |\chi(\hat{k}(r - vt) + \overset{\vec{\nabla}\alpha}{\vec{s}} - \vec{b}|$$

Need to integrate over $\vec{b}$.

$\hat{k}$: along z axis.

Call $z = r - vt$

$dr = dz$

$dz\, d^2b \equiv d^3x$

$-vt < z < \infty$

Extend to $-\infty$.

$$\text{Rate} = j\, \Delta\Omega\, |f_k(\theta,\phi)|^2 \int d^3x\, |\psi(\vec{x})|^2$$

$$\frac{d\sigma}{d\Omega} = |f_k(\theta,\phi)|^2$$