# The Dirac Field

Physics 217 2013, Quantum Field Theory

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October 2013

# Lorentz Transformation Properties of the Dirac Field

First, rotations. In ordinary quantum mechanics,

$$\psi^\dagger \sigma^i \psi \tag{1}$$

is a vector under rotations. How does this work?

Under infinitesimal rotations,

$$\psi \to (1 + i\omega^i S^i)\psi \tag{2}$$

So

$$\begin{aligned} \psi^\dagger \sigma^i \psi &\to \psi^\dagger \sigma^i \psi + i\omega^j \psi^\dagger [\sigma^i, \sigma^j] \psi \\ &= (\delta^{ik} - \epsilon_{ijk}\omega^j)\psi^\dagger \sigma^k \psi \end{aligned} \tag{3}$$

This is the transformation law for a vector. We seek an analogous construction with $\gamma^\mu$.

First we rewrite the transformation law under rotations in a way that is closer to the form in which we have written infinitesimal Lorentz transformations. Replace $\omega^i$ by

$$\omega_{ij} = \epsilon_{ijk}\omega^k \tag{4}$$

(compare this with rewriting the magnetic field, $\vec{B}$, as an antisymmetric tensor; also useful in considering Lorentz transformation properties, $F_{ij}$).

Similarly, $J_{ij} = \epsilon_{ijk}J^k$ for the angular momentum operators (generators of rotations).

In terms of such tensors, the ordinary orbital angular momentum operator is simply

$$L_{ij} = -i(x_i\partial_j - x_j\partial_i) \tag{5}$$

and similarly for other angular momentum operators. The angular momentum commutation relations are:

$$[J_{ij}, J_{kl}] = i(\delta_{ik}J_{jl} - \delta_{il}J_{jk} - \delta_{jk}J_{il} + \delta_{jl}J_{ik}) \tag{6}$$

One can check these commutation relations for the $L_{ij}$'s, for example. One can think of these as the defining relations of the rotation group. E.g.

$$e^{i\alpha_{ij}J_{ij}} \approx (1 + i\alpha_{ij}L_{ij} + i\alpha_{ij}S_{ij}). \tag{7}$$

# Group Theory of the Lorentz Group

Similarly, we can start with the transformation law for a scalar under Lorentz transformations:

$$\phi'(x) = \phi(\Lambda^{-1}x) \tag{8}$$

where $\Lambda_{\mu\nu} \approx 1 + \omega_{\mu\nu}$. In other words:

$$\delta\phi(x) = -\omega^{\mu\nu} x_\mu \partial_\nu \phi(x) \tag{9}$$

So we can define:

$$L^{\mu\nu} = i(x^\mu \partial^\nu - x^\nu \partial^\mu) \tag{10}$$

as the analog of the generators of orbital rotations; indeed, $L^{ij}$ are just the angular momentum operators. We can evaluate their commutators, and abstract the basic commutation relations for the generators of Lorentz transformations:

$$[J^{\mu\nu}, J^{\rho\sigma}] - i(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho}). \tag{11}$$

# Representations of the Lorentz Group

We have the analog of orbital angular momentum. For ordinary rotations, the spin one generator is $(\mathcal{G})^i_{jk} = -i\epsilon_{ijk}$, or in our antisymmetric index notation:

$$(\mathcal{G}_{ij})_{kl} = i(\delta^i_k \delta^j_l - \delta^i_k \delta^j_\ell) \tag{12}$$

The analog for the Lorentz group, corresponding to the transformation law for vectors, $V^\mu$, is

$$\mathcal{G}^{\mu\nu}_{\alpha\beta} = i(\delta^\mu_\alpha \delta^\nu_\beta - \delta^\mu_\beta \delta^\nu_\alpha) \tag{13}$$

You can check that this:

1. gives the correct infinitesimal transformation for a vector, $x^\mu$
2. obeys the Lorentz group commutation relations.

# Lorentz generators in the spinor representation

Starting with:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \tag{14}$$

we construct the matrices, analogous to the spin-1/2 matrices:

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu] \tag{15}$$

(for $\mu, \nu = i, j$ it is easy to check that these *are* the spin matrices).

These are readily seen to obey the Lie algebra of the Lorentz group.

Now, however, it is not $\psi^\dagger\gamma^\mu\psi$ which transforms as a vector, but $\bar{\psi}\gamma^\mu\psi$, where

$$\bar{\psi} = \psi^\dagger\gamma^0. \tag{16}$$

To check this, we need certain properties of the $\gamma^\mu$ matrices, easily seen to be true in our representations:

1. $(\gamma^0)^\dagger = \gamma^0 \quad (\gamma^i)^\dagger = -\gamma^i$
2. From which it follows that: $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$.

Let's start, in fact, by checking that $\bar{\psi}\psi$ is a *scalar*.

$$\bar{\psi}\psi \to \psi^\dagger(1 - i\omega_{\mu\nu}\frac{-i}{4}\gamma^0[\gamma^\nu,\gamma^\mu]\gamma^0)\gamma^0(1 + i\omega_{\rho\sigma}\frac{i}{4}[\gamma^\rho,\gamma^\sigma])\psi \quad (17)$$
$$= \bar{\psi}\psi.$$

Now let's do the vector:

$$\bar{\psi}\gamma^\mu\psi \to \psi^\dagger(1 - i\omega_{\mu\nu}\frac{-i}{4}\gamma^0[\gamma^\nu,\gamma^\mu])\gamma^0\gamma^0\gamma^\mu(1 + i\omega_{\rho\sigma}\frac{i}{4}[\gamma^\rho,\gamma^\sigma])\psi \quad (18)$$

Using the commutation relations:

$$[\gamma^\mu, S^{\rho\sigma}] = \mathcal{G}^{\rho\sigma}_{\mu_\nu}\gamma^\nu \quad (19)$$

the right hand side of eqn. 18 becomes

$$\bar{\psi}\gamma^\mu\psi - \frac{i}{2}\omega_{\rho\sigma}(\mathcal{G}^{\rho\sigma})^\mu_\nu\bar{\psi}\gamma^\nu\psi. \quad (20)$$

Introducing also the matrix

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{21}$$

which anti commutes with all of the other $\gamma$'s, we can construct the following bilinears in the fermion field which transform as irreducible tensors:

1. Scalar: $\bar{\psi}\psi$
2. Pseudoscalar: $\bar{\psi}\gamma^5\psi$
3. Vector: $\bar{\psi}\gamma^\mu\psi$
4. Pseudo vector: $\bar{\psi}\gamma^\mu\gamma^5\psi$
5. Second rank tensor: $\bar{\psi}\sigma^{\mu\nu}\psi$.

(You will get to familiarize yourselves with these objects for homework; the "pseudo" character will be discussed shortly.)

With these results, it is easy to construct a relatiavitsically invariant lagrangian:

$$\mathcal{L} = i\bar{\psi}\partial_\mu\gamma^\mu\psi - m\bar{\psi}\psi \equiv i\bar{\psi}\not{\partial}\psi - m\bar{\psi}\psi. \tag{22}$$

Euler-Lagrange equations (varying with respect to $\psi, \bar{\psi}$ independently:

$$i\not{\partial}\psi - m\psi = 0 \tag{23}$$

the Dirac equation.
We want to interpret now as quantum fields.

The canonical momentum is curious:

$$\Pi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = i\psi^\dagger \tag{24}$$

With this we can construct the Hamiltonian; it is reminiscent of Dirac's original expression:

$$H = \int d^3x \mathcal{H} = \int d^3x \psi^\dagger(x)(-i\gamma^0\vec{\gamma}\cdot\vec{\nabla} + \beta m)\psi(x). \tag{25}$$

We will take a step which we will see is necessary for a sensible interpretation of the theory: we require

$$\{\psi(\vec{x},t), \Pi(\vec{x}',t)\} = i\delta(\vec{x}-\vec{x}'). \tag{26}$$

In order to develop a momentum space expansion of the fermion fields, as we did for scalars, we first, we need more control of the solutions of the free field equations in momentum space (for the scalar, these were trivial).

# Momentum Space Spinors

In your text, various relations for spinors, including orthogonality relations and spin sums, are worked out by looking at explicit solutions. We can short circuit these calculations in a variety of ways. Here is one:

Two things slightly different than your text:

1. Dirac matrices: It will be helpful to have an explicit representation of the Dirac matrices, or more specifically of Dirac's matrices, somewhat different than the one in your text:

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \tag{27}$$

# Quick calculation of spin sums, Normalizations, etc.

We first consider the positive energy apinors, $p^0 = \sqrt{\vec{p}^2 + m^2}$. If $\chi$ is a constant spinor,

$$u(p) = N(\not{p} + m)\chi$$

solves the Dirac equation. Now take $\chi$ to be a solution of the Dirac equation with $\vec{p} = 0$. We can work, for this discussion, in any basis, so let's choose our original basis, where the $\vec{p} = 0$ spinors are particularly simple, and take the two linearly-independent spinors to be

$$\chi_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} ; \chi_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Let's first get the normalizations straight. We will require:

$$\bar{u}(p)u(p) = 2m$$

(note this is Lorentz invariant). With our solution the left hand side is

$$N^2\chi^\dagger(\not{p}^\dagger + m)\gamma^o(\not{p} + m)\chi$$

$$= N^2\chi^\dagger\gamma^o\gamma^o(\not{p}^\dagger + m)\gamma^o(\not{p} + m)\chi$$

$$= N^2\chi^\dagger\gamma^o(\not{p} + m)(\not{p} + m)\chi$$

$$= N^2\chi^\dagger 2m(\not{p} + m)\chi$$

From the explicit form of the Dirac matrices and the $\chi$'s, $\chi^\dagger \not{p}\chi = E$. So

$$N^2 = \frac{1}{(E + m)}.$$

With this we can do the spin sums. First note that for the $\chi$'s, looking at their explicit form:

$$\sum_s \chi\chi^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \frac{1}{2}(1+\gamma^o).$$

So now

$$\sum_s u(p,s)\bar{u}(p,s) = \sum_s u(p,s)u^\dagger(p,s)\gamma^o$$

$$= \frac{1}{2}N^2(\not{p}+m)(1+\gamma^o)(\not{p}^\dagger+m)\gamma^o$$

$$= \frac{1}{2}N^2(\not{p}+m)(1+\gamma^o)\gamma^o\gamma^o(\not{p}^\dagger+m)\gamma^o$$

$$= \frac{1}{2}N^2(\not{p}+m)(1+\gamma^o)(\not{p}+m)$$

$$= \frac{1}{2}N^2 2(m+p^o)(\not{p}+m)$$

$$= (\not{p}+m).$$

(in the next to last step, just multiply out the terms).

Finally, we can compute the inner products:

$$
\begin{aligned}
u^\dagger(p,s)u(p,s') &= N^2\chi^\dagger(\not{p}^\dagger + m)(\not{p} + m)\chi \\
&= N^2\chi^\dagger\gamma^o(\not{p} + m)\gamma^o(\not{p} + m)\chi \\
&= N^2\chi^\dagger\gamma^o(p^2 - m^2 + 2p^o(\not{p} + m)\gamma^o\chi \\
&= 2E\delta_{s,s'}
\end{aligned}
$$

**Exercise:**

Work out the corresponding relations for the negative energy spinors, $v(p,s)$, including the spin sums, normalization, and orthogonality relations.

$$\sum_s u_\alpha(p,s)\bar{u}_\beta(p,s) = (\not{p}+m)_{\alpha\beta} \quad \sum_s v_\alpha(p,s)\bar{v}_\beta(p,s) = (\not{p}-m)_{\alpha\beta}$$

$$u_\alpha^\dagger(p,s)u_\alpha(p,s') = 2E_p\delta_{ss'} = v_\alpha^\dagger(p,s)v_\alpha(p,s')$$

$$\bar{u}_\alpha(p,s)u_\alpha(p,s') = 2m\delta_{ss'} = \bar{v}_\alpha(p,s)v_\alpha(p,s')$$

Now we want to write momentum space expansions of the spinor fields, analogous to those for scalar fields. Suppose we have operators $a$, $a^\dagger$ which obey the anticommutation relations:

$$\{a, a^\dagger\} = 1; \{a, a\} = 0; \{a^\dagger, a^\dagger\} = 0$$

Then construct the “number operator"

$$N = a^\dagger a$$

and the state $|0\rangle$ by the condition

$$a|0\rangle = 0$$

($a$ is a destruction operator). Then

$$Na^\dagger|0\rangle = a^\dagger a a^\dagger|0\rangle.$$

Using the anticommutation relations to move $a$ to the right,

$$= -a^\dagger a^\dagger a|0\rangle + a^\dagger|0\rangle.$$

In other words

$$Na^\dagger|0\rangle = |0\rangle$$

so $a^\dagger$ creates a one particle state. But since $(a^\dagger)^2 = 0$ (from the anticommutation relations) there is no two particle state.

So anticommutation relations of this kind build in the exclusion principle; only the zero and one-particle states are allowed. Consider, first, finite volume, as we did for the scalar field. The field operator $\Psi(\vec{x}, t)$ should satisfy the Dirac equation. So we write

$$\psi_\alpha = \sum_s \sum_p \frac{1}{\sqrt{2E_p}} \left( a(p,s) u_\alpha(p,s) e^{-ip\cdot x} + b^\dagger(p,s) v_\alpha(p,s) e^{ip\cdot x} \right) . \tag{28}$$

$$\psi^\dagger_\alpha = \sum_s \sum_p \frac{1}{\sqrt{2E_p}} \left( a^\dagger(p,s) u^\dagger_\alpha(p,s) e^{-ip\cdot x} + b(p,s) v^\dagger_\alpha(p,s) e^{ip\cdot x} \right) . \tag{29}$$

Taking

$$\{a(\vec{p}, s), a^\dagger(\vec{p}', s')\} = \delta_{ss'}\delta_{\vec{p},\vec{p}'}$$

satisfies the (anti) commutation relations above.
For the Hamiltonian we obtain (as we will see in a moment),

$$H = \sum_{\vec{p},s} E(p)(a^\dagger(p, s)a(p, s) + b^\dagger(p, s)b(p, s)) + \infty$$

only because we assumed anti commutation relations. So we see that we can have states with zero or one electron and zero or one positron for each momentum and spin.

# Momentum Space Expansion of the Dirac Field, Infinite volume

Infinite volume:

$$\psi_\alpha = \sum_s \int \frac{d^3p}{(2\pi)^3\sqrt{2}E_p}(a(p,s)u_\alpha(p,s)e^{-ip\cdot x} \tag{30}$$

$$+b^\dagger(p,s)v_\alpha(p,s)e^{ip\cdot x}).$$

$$\psi_\alpha^\dagger = \sum_s \int \frac{d^3p}{(2\pi)^3\sqrt{2}E_p}(a^\dagger(p,s)u_\alpha^\dagger(p,s)e^{-ip\cdot x} \tag{31}$$

$$+b(p,s)v_\alpha^\dagger(p,s)e^{ip\cdot x}).$$

Now consider the Hamiltonian:

$$H = \int d^3x \bar{\psi} \left( -i\vec{\gamma} \cdot \vec{\nabla} + m \right) \psi. \tag{32}$$

Plug in the expansions of the fields, and use:

$$(\vec{\gamma} \cdot \vec{p} + m)u = p^0 \gamma^0 u \qquad (-\vec{\gamma} \cdot \vec{p} + m)v = -p^0 \gamma^0 v \tag{33}$$

and

$$u^\dagger(p, s)u(p, s') = v^\dagger(p, s)v(p, s') = 2p^0 \delta_{s,s'} \tag{34}$$

to find:

$$H = \int \frac{d^3p}{(2\pi)^3} \sum_s E_p \left[ a^\dagger(p, s)a(p, s) - b(p, s)b^\dagger(p, s) \right]. \tag{35}$$

Now if we quantize as for the scalar field with commutators, we obtain a negative contribution from the negative energy states, and the energy is unbounded below. If we quantize with *anti commutators*, we have:

$$H = \int \frac{d^3p}{(2\pi)^3} \sum_s E_p \left[ a^\dagger(p,s)a(p,s) + b^\dagger(p,s)b(p,s) - 1 \right] . \tag{36}$$

So we now have a sensible expression in terms of number operators, with an infinite contribution which we can think of as representing the energy of the Dirac sea.

# The Dirac Propagator

Just as Dirac fields obey *anti commutation relations*, the time-ordered product for fermions is designed with an extra minus sign, for example:

$$T(\psi(x_1)\psi(x_2)) = \theta(x_1^0 - x_2^0)\psi(x_1)\psi(x_2) - \theta(x_2^0 - x_1^0)\psi(x_2)\psi(x_1). \tag{37}$$

The basic fermion propagator is:

$$S_F(x_1 - x_2) = T\langle 0|\psi(x)\bar{\psi}(y)|0\rangle. \tag{38}$$

Let's take a particular time ordering, $x_1^0 > x_2^0$,

$$S_F = \sum_{s,s'} \int d^4x \frac{d^3p d^3p'}{(2\pi)^6 \sqrt{E(p)E(p')}} e^{-ip\cdot(x_1-x_2)} u(p,s)\bar{u}(p',s'). \tag{39}$$

The $x$ integral gives $\delta(p-p')$; then we have, from the sum over spins, $\not{p}+m$. Indeed, up to the factor of $\not{p}+m$, this is what we had for the same time ordering for the scalar propagator.

For the other time ordering, we obtain, again, the same result as for the scalar, except with a factor $-\not{p}+m$, from the spin sum. Changing $p \to -p$ gives

$$S_F(p) = \frac{\not{p}+m}{p^2-m^2+i\epsilon}. \tag{40}$$

# The Discrete Symmetries P and C

**Parity**: The Dirac lagrangian is unchanged if we make the replacement:

$$\psi(\vec{x}, t) \to \gamma^o \psi(-\vec{x}, t) \tag{41}$$

Let's see what effect this has on the creation and annihilation operators, $a$, $b$, etc.

$$\psi_p(\vec{x}, t) = \gamma^o \psi(-\vec{x}, t) \tag{42}$$

$$= \int \frac{d^3p}{(2\pi)^3 \sqrt{E_p}} (a(\vec{p}, s)\gamma^o u(\vec{p}, s) e^{-ip^o x^o - i\vec{p}\cdot\vec{x}} + b^\dagger(\vec{p}, s)\gamma^o v(\vec{p}, s) e^{ip^o x^o + i\vec{p}\cdot\vec{x}}).$$

We can easily determine what $\gamma^o$ does to $u$ and $v$ using our explicit expressions (ignoring the normalization factor, which is unimportant for this discussion):

$$\gamma^o(\not{p} + m)\chi = (p^o\gamma^o + \vec{p}\cdot\vec{\gamma} + m)\gamma^o\chi \tag{43}$$

$$= u(-\vec{p}, s)$$

$$\gamma^o v(\vec{p}, s) = -v(-\vec{p}, s) \tag{44}$$

So making the change of variables $\vec{p} \to -\vec{p}$ in our expression for $\psi_p$, gives

$$\psi_p(\vec{x}, t) = \gamma^o \psi(-\vec{x}, t) \tag{45}$$

$$= \int \frac{d^3p}{(2\pi)^3 \sqrt{E_p}} (a(-\vec{p}, s)u(\vec{p}, s)e^{-ip\cdot x} + b^\dagger(-\vec{p}, s)v(\vec{p}, s)e^{ip\cdot x})$$

**Charge Conjugation:** Now we can do the same thing for C. Here:

$$\psi_c(x) = \gamma^2 \psi^*(x) \tag{46}$$

$$= \int \frac{d^3p}{(2\pi)^3 \sqrt{E_p}} (a^\dagger(\vec{p}, s) \gamma^2 u^*(\vec{p}, s) e^{ip\cdot x} + b(\vec{p}, s) \gamma^2 v^*(\vec{p}, s) e^{-ip\cdot x}).$$

Now we consider the action of $\gamma^2$ on $u^*, v^*$:

$$\gamma^2 u^* = \gamma^2 (\not{p}^* + m)\chi^* = \gamma^2 (p^o \gamma^o - p^1 \gamma^1 + p^2 \gamma^2 - p^3 \gamma^3 + m)\chi \tag{47}$$

$$= (-\not{p} + m)\gamma^2 \chi.$$

Here we have not been ashamed to use the explicit properties of the $\gamma$ matrices; $\gamma^1$ and $\gamma^3$ are real, while $\gamma^2$ is imaginary; the first two anticommute with $\gamma^2$ while the third commutes. Now we use the explicit form of $\gamma^2$ to see that it takes the positive energy $\chi$ to the negative energy $\chi$, with the opposite spin. So, indeed, we have that

$$a_c(p,s) = b(p,-s) \quad b_c(p,s) = a(p,-s) \tag{48}$$

i.e. it reverses particles and antiparticles and flips the spin.

**Exercise:** Determine the action of $P$ and $C$ on particle and antiparitcle states of definite momentum.