Physics 217. Quantum Field Theory. Professor Dine

Free Scalar Fields and Homework Set 1

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1 Field Theory in a Box and Passing to the Infinite Volume Limit

Consider a real scalar field, with lagrangian

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu \phi)^2 - m^2\phi^2. \tag{1}$$

The equations of motion are

$$\partial^2\phi + m^2\phi = 0. \tag{2}$$

These are solved by

$$\phi(\vec{x}, t) = \sum_{\vec{k}} \frac{1}{\sqrt{V}} \frac{1}{\sqrt{2\omega(k)}}[a(\vec{k})e^{ik\cdot x} + a^\dagger(\vec{k})e^{-ik\cdot x}]. \tag{3}$$

Imposing canonical commutation relations on $\phi, \dot{\phi}$, gives

$$[a(\vec{k}), a^\dagger(\vec{k})] = \delta_{\vec{k},\vec{k}'}. \tag{4}$$

$N(\vec{k}) = a^\dagger(\vec{k})a(\vec{k})$ is a conventionally normalized number operator.

The Hamiltonian
Starting with the Lagrangian and the canonical momenta, the Hamiltonian is

$$H = \int d^3x \mathcal{H} = \int d^3x \frac{1}{2}\left[\dot{\phi}^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2.\right] \tag{5}$$

Let's express this in terms of momentum space components, still in a box.

$$H = \int \frac{d^3x}{V} \sum_{\vec{k}} \sum_{\vec{\ell}} \frac{1}{\sqrt{\omega_k\omega_\ell}}[i^2\omega_k\omega_\ell(a(\vec{k})e^{i\vec{k}\cdot\vec{x}} - a^\dagger(\vec{k})e^{-i\vec{k}\cdot\vec{x}})(a(\vec{\ell})e^{i\vec{\ell}\cdot\vec{x}} - a^\dagger(\vec{\ell})e^{-i\vec{\ell}\cdot\vec{x}}) \tag{6}$$

$$+ i^2k^i\ell^i(a(\vec{k})e^{i\vec{k}\cdot\vec{x}} - a^\dagger(\vec{k})e^{-i\vec{k}\cdot\vec{x}})(a(\vec{\ell})e^{i\vec{\ell}\cdot\vec{x}} - a^\dagger(\vec{\ell})e^{-i\vec{\ell}\cdot\vec{x}})$$

$$+ m^2(a(\vec{k})e^{i\vec{k}\cdot\vec{x}} + a^\dagger(\vec{k})e^{-i\vec{k}\cdot\vec{x}})(a(\vec{\ell})e^{i\vec{\ell}\cdot\vec{x}} + a^\dagger(\vec{\ell})e^{-i\vec{\ell}\cdot\vec{x}})].$$

Now we simplify. The $\vec{x}$ integral gives kronecker δ's, $\delta_{\vec{k},\vec{\ell}}$, and $\delta_{\vec{k},-\vec{\ell}}$, times V. We can rearrange the a and $a^\dagger$ terms to look like number operators, using the commutation relations. Terms involving two a's or two $a^\dagger$'s cancel because of the extra minus sign in the $k_i\ell_i$ factor. We are left with:

$$H = \sum_{\vec{k}} \sqrt{\vec{k}^2 + m^2}\left(N(\vec{k}) + \frac{1}{2}\right). \tag{7}$$

I.e. we have a collection of harmonic oscillators.

Exercise: Verify eqn. 7.

Exercise: Repeat the quantization in infinite volume;

$$\sum_{\vec{k}} \to \int \frac{d^3k}{(2\pi)^3 V} \tag{8}$$

Argue that the continuum limit of

$$[a(\vec{k}), a^\dagger(\vec{k}')] = \delta_{\vec{k}\vec{k}'} \tag{9}$$

is

$$(2\pi^3)\delta(\vec{k} - \vec{k}'). \tag{10}$$

With this, repeat the calculation of the Hamiltonian. You can throw away c-number terms (i.e. terms not involving a and $a^\dagger$), but you might want to think about the connection to the zero point term in your calculation above.