Physics 217. Quantum Field Theory. Professor Dine

Fall, 2013. Homework Set 2

SOLUTIONS

1. Do the exercises on the handout on the Dirac equation.

   Solutions: Most of these are very straightforward. It is perhaps worth stressing one in particular: **Exercise:** Determine the action of P and C on particle and antiparitcle states of definite momentum. I want to stress this one because it illustrates the virtues of the form of solution we have employed. We have seen, rather trivially, that under P:

$$a(\vec{p}, s) \rightarrow a(-\vec{p}, s); \quad v(\vec{p}, s) \rightarrow v(-\vec{p}, s). \tag{1}$$

So we see that the state

$$|e, \ vecp, s\rangle = a^{\dagger}(\vec{p}, s)|0\rangle \rightarrow= |e, -\vec{p}, s\rangle$$

$$|\bar{e}, \vec{p}, s\rangle = b^{\dagger}(\vec{p}, s)|0\rangle \rightarrow= -|\bar{e}, -\vec{p}, s\rangle$$

Similarly, for C, we have seen that (repeating the material from the notes: **Charge Conjugation:** Now we can do the same thing for C. Here:

$$\psi_c(x) = \gamma^2 \psi^*(x) \tag{2}$$

$$= \int \frac{d^3p}{(2\pi)^3\sqrt{E_p}} (a^{\dagger}(\vec{p}, s)\gamma^2 u^*(\vec{p}, s)e^{ip\cdot x} + b(\vec{p}, s)\gamma^2 v^*(\vec{p}, s)e^{-ip\cdot x}).$$

Now we consider the action of γ^2 on u^*, v^*:

$$\gamma^2 u^* = \gamma^2(\not{p}^* + m)\chi^* = \gamma^2(p^o\gamma^o - p^1\gamma^1 + p^2\gamma^2 - p^3\gamma^3 + m)\chi \tag{3}$$

$$= (-\not{p} + m)\gamma^2\chi.$$

Here we have not been ashamed to use the explicit properties of the γ matrices; γ^1 and γ^3 are real, while γ^2 is imaginary; the first two anticommute with γ^2 while the third commutes. Now we use the explicit form of γ^2 to see that it takes the positive energy χ to the negative energy χ, with the opposite spin. So, indeed, we have that

$$a_c(p, s) = b(p, -s) \quad b_c(p, s) = a(p, -s) \tag{4}$$

i.e. it reverses particles and antiparticles and flips the spin. In the notation above,

$$|e, \vec{p}, s\rangle \rightarrow |\bar{e}, \vec{p}, -s\rangle \tag{5}$$

2. Construct the energy-momentum tensor for the free Dirac field. Verify that it reproduces the Hamiltonian we used in class. Construct the momentum operator in terms of creation and annihilation operators for electrons and positrons.

Solution: I'll construct the energy momentum tensor as in class. Consider the change in the action under

$$x_\mu \to x_\mu + \epsilon_\mu(x). \tag{6}$$

The variation of

$$\mathcal{L} = i\bar{\psi}\ \partial\!\!\!/\psi - m\bar{\psi}\psi \tag{7}$$

includes the piece which would arise if ϵ was a constant,

$$\mathcal{L} \to \mathcal{L} + \epsilon^\mu \partial_\mu \mathcal{L} \tag{8}$$

and an additional term, from

$$\partial_\mu \psi(x+\epsilon) \to \partial_\mu(\psi(x) + \epsilon^\nu \partial_\nu \psi) \tag{9}$$

$$= \partial_\mu \psi + \epsilon^\nu \partial_\mu \partial_\nu \psi + \partial_\mu \epsilon^\nu \partial_\nu \psi.$$

So, after an integration by parts (in the action)

$$\delta\mathcal{L} = -\partial_\mu \epsilon^\mu \mathcal{L} + \partial_\mu \epsilon^\nu i\bar{\psi}\gamma^\mu \partial_\nu \psi. \tag{10}$$

From this we read off the energy-momentum tensor:

$$\mathcal{T}_{\mu\nu} = i\bar{\psi}\gamma_\mu \partial_\nu \psi - g_{\mu\nu}\mathcal{L}. \tag{11}$$

In particular:

$$T_{00} = \bar{\psi}(-i\vec{\nabla}\cdot\vec{\gamma} + m)\psi \tag{12}$$

as we originally formulated the Dirac Hamiltonian. In terms of creation and annihilation operators, the Hamiltonian is evaluated in the on-line lecture notes.

Similarly

$$T^{i0} = \bar{\psi}(-i\partial_i)\psi. \tag{13}$$

So substituting the mode expansions back in:

$$P^i = \int d^3x \int \frac{d^3p d^3p'}{\sqrt{2E(p)2E(p')(\pi)^6}} \sum_{s,s'} \left(a^\dagger(p',s')e^{-ip'\cdot x}u_\alpha(p',s') + b(p',s')e^{ip'\cdot x}v^\dagger_\alpha(p',s')\right)$$

$$\left(p_i a(p,s)e^{-ip\cdot x}u_\alpha(p,s) + -p_i b^\dagger(p,s)e^{ip\cdot x}v_\alpha(p,s)\right) \tag{14}$$

This simplifies since the integral over x gives $(2\pi)^3\delta(\vec{p}-\vec{p}'$ or $(2\pi)^3\delta(\vec{p}+\vec{p}'$ and the inner products of the u's and v's give $2E\delta_{ss'}$ time factors. Normal ordering gives

$$P^i = \int \frac{d^3p}{(2\pi)^3}\sum_s p^i\left(a^\dagger(p,s)a(b,s) + b^\dagger(p,s)b(p,s)\right). \tag{15}$$

The normal ordering contribution is $\delta(0)p^i$, which vanishes when integrated over $\vec{p}$ (since an odd function).

3. More on the Dirac equation: Verify the commutation relations of $J^{\mu\nu} = i(x^\mu \partial^\nu - x^\nu \partial^\mu)$ as we wrote them in class. Similarly check the commutation relations for the matrices

$$(\mathcal{S}^{\mu\nu})^\alpha_\beta = i(g^{\mu\alpha} g^\nu_\beta - g^\mu_\beta g^{\nu\alpha}).$$

Finally, verify the commutation relations for

$$S^{\mu\nu} = \frac{-i}{4}[\gamma^\mu, \gamma^\nu]$$

, the Lorentz generator constructed in class in terms of the Dirac matrices. Verify that $\bar{\psi}\gamma^\mu\gamma^\nu\psi$ transforms as a second-rank tensor.

4. PS 3.2

Solution: I am not certain that this is the most efficient way to do this, but it works. Start with

$$\bar{u}(p')\gamma^\mu \not{p}' u(p) = -m\bar{u}(p')\gamma^\mu u(p) + 2p'^\mu \bar{u}(p')u(p) \tag{16}$$

where we have anticommuted γ^μ and $\not{p}'$ and used the Dirac equation. Similarly:

$$\bar{u}(p') \not{p}\gamma^\mu / u(p) = -m\bar{u}(p')\gamma^\mu u(p) + 2p^\mu \bar{u}(p')u(p). \tag{17}$$

Adding these two equations gives:

$$2(p'^\mu + p^\mu)\bar{u}(p')u(p) - 2m\bar{u}(p')\gamma^\mu u(p) \tag{18}$$

$$= \bar{u}(p')\gamma^\mu(\not{p} + \not{q})u(p) + \bar{u}(p')(\not{p}' - \not{q})\gamma^\mu u(p).$$

Using the Dirac equation this is:

$$2m\bar{u}(p')\gamma^\mu u() + \bar{u}(p')[\gamma^\mu, q^\nu\gamma_\nu]u(p).$$

Grouping terms,

$$4m\bar{u}(p')\gamma^m uu(p) = 2\bar{u}(p')u(p)(p^\mu + p'^\mu) - \bar{u}(p')q_\nu[\gamma\mu, \gamma^\nu] \tag{19}$$

which is the identity to be proved.