Time Ordered Perturbation Theory

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Quantization of the Free Electromagnetic Field

We have so far quantized the free scalar field and the free Dirac field. We also would like to understand the free electromagnetic field, before proceeding to interacting theories. Your text deals with this very briefly, simply guessing a form for the propagator. We can be a little more systematic, in a fashion which is instructive.

A Tension

In a gauge theory like electromagnetism, there is a tension between two basic principles: Lorentz invariance and unitarity (unitarity is the statement that in quantum mechanics, time evolution is described by a unitary operator, as a result of which probability is conserved). If we choose Lorentz gauge:

$$\partial_\mu A^\mu = 0 \tag{1}$$

and follow our usual quantization procedure, we will be lead to write:

$$A^\mu(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}(a^\mu(p)e^{ip\cdot x} + a^{\dagger\ \mu}e^{-ip\cdot x}) \tag{2}$$

$$[a^\mu(p), a^{\nu\ \dagger}(p)] = g^{\mu\nu}(2\pi)^3\delta(\vec{p} - \vec{p}'). \tag{3}$$

The problem is with the a^o commutator. The states $a^{o\dagger}|0>$ have *negative norm* (check!). Doesn't sound good for quantum mechanics. It turns out that the states with negative norm are never produced in scattering processes, but proving this is a bit involved. An alternative approach gives up manifest Lorentz invariance. One chooses the Coulomb (or “transverse" or “radiation") gauge:

$$\vec{\nabla} \cdot \vec{A} = 0 \tag{4}$$

Note that in writing this condition, we are making a choice of Lorentz frame. The expansion of the gauge field is now:

$$A^i(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}(a(p,\lambda)\epsilon^i(p,\lambda)e^{ip\cdot x} + a^\dagger(p,\lambda)\epsilon^{i\,*}(p,\lambda)e^{-ip\cdot x}). \tag{5}$$

From the gauge condition,

$$\vec{p} \cdot \vec{\epsilon}(p, \lambda) = 0. \tag{6}$$

The commutation relations of the a's are just what you might expect:

$$[a(p, \lambda), a^\dagger(p', \lambda')] = \delta_{\lambda,\lambda'}(2\pi)^3 \delta(\vec{p} - \vec{p}'). \tag{7}$$

From these expressions we can work out the propagator. In this computation, analogous to what we saw in the Dirac case, one encounters:

$$\sum_{\lambda} \epsilon^{i}(p,\lambda)\epsilon^{*j}(p,\lambda) = P_{ij}(p) = (\delta_{ij} - \frac{p^i p^j}{\vec{p}^2}). \tag{8}$$

Then (**Exercise: check**):

$$T < A^i(x)A^j(y) >= \int \frac{d^4p}{(2\pi)^4} e^{ip\cdot x} \frac{iP_{ij}(\vec{p})}{p^2 + i\epsilon} \tag{9}$$

It is also natural to define a propagator for the scalar potential, remembering that propagators are just Green's functions. In momentum space, this is just

$$< A_o A_o >= i \frac{1}{\vec{p}^2}. \qquad (10)$$

Not surprisingly, these propagators don't look very Lorentz invariant. But we can fix this by noting that the full propagator can be written (in momentum space, using the "west coast metric"):

$$D^{\mu\nu} = -\frac{g^{\mu\nu}}{p^2 + i\epsilon} - \frac{p^\mu p^\nu}{\vec{p}^2(p^2 + i\epsilon)} + \frac{\eta^\mu p^\nu + \eta^\nu p^\mu}{(p^2 + i\epsilon)} \qquad (11)$$

where $\eta = (\frac{p^o}{\vec{p}^2}, 0, 0, 0)$ is a fixed four vector.

Exercise: Check this. Don't worry about the $i\epsilon$'s.

Now in electrodynamics, A^μ couples to j^μ, a conserved current. So p^μ always multiplies $j^\mu(p)$, and thus these terms vanish by current conservation. We will actually see how this works in scattering amplitudes later. As a result, we can use the covariant propagator. Note that this is the propagator one might have written in Lorentz gauge by analogy with the propagator for a scalar field.

The fact that in the end one can write manifestly Lorentz invariant Feynman rules means that the non-Lorentz invariant gauge choice doesn't matter in the end. It is possible to prove that, in Coulomb gauge, there are a nice set of operators which generate Lorentz invariance. But this is rather involved and, for the moment, not particularly instructive.

Interacting Field Theories

Consider, first, our scalar field. Lorentz invariance allows many other terms:

$$\mathcal{L} = \frac{1}{2}((\partial_\mu \phi)^2 - m^2\phi^2 - \Gamma^3\phi^3 - \frac{\lambda}{4}\phi^4 - \frac{1}{M}\phi^5 + \ldots . \qquad (12)$$

We will stop at ϕ^4; generally we won't include operators of negative mass dimension ("non-renormalizable").

Let's look at ϕ^3, in the language of old fashioned perturbation theory. We can have processes which change the number of particles.

Quantum Electrodynamics

So far we have free fields:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + \bar{\psi}(\not\partial - m)\psi. \tag{13}$$

To decide how to couple the gauge field, A_μ, to fermions, we will focus on the principle of gauge invariance. We take

$$A_\mu \to A_\mu + \partial_\mu \omega(x) \quad \psi \to e^{i\omega(x)}\psi(x). \tag{14}$$

Then the *covariant derivative* transforms like ψ:

$$D_\mu\psi = (\partial_\mu - iA_\mu)\psi; \quad D_\mu\psi \to e^{i\omega(x)}D_\mu\psi \tag{15}$$

We also introduce a constant e, the electric charge:

$$\mathcal{L} = -\frac{1}{4e^2}F_{\mu\nu}^2 + \bar{\psi}(\not{D} - m)\psi. \tag{16}$$

This lagrangian is Lorentz and gauge invariant.

One often rescales the fields,

$$A_\mu \to eA_\mu \tag{17}$$

in which case

$$A_\mu \to A_\mu + \frac{1}{e}\partial_\mu \omega(x) \quad \psi \to e^{i\omega(x)}\psi(x). \tag{18}$$

$$D_\mu \psi = (\partial_\mu - ieA_\mu)\psi \tag{19}$$

In this form, the role of e as a coupling is clearer, but gauge invariance involves e in an odd way (it will be more odd when we confront renormalization). The alternative form gives gauge invariance a more “geometric" character.

Now, adopting the methods of old fashioned (time-ordered) perturbation theory, we have, for the interaction Hamiltonian:

$$\mathcal{H}_I = e\bar{\psi}A^\mu\gamma_\mu\psi \tag{20}$$

and we have the possibility of creating and destroying electrons, positrons and photons in physical processes.

Our goal is to develop a more covariant treatment of these processes.

Pictures in Quantum Mechanics

We usually work with quantum mechanics in the Schrodinger picture, in which states depend on time, but operators like x and p are independent of time. In the Schrodinger picture:

$$|\Psi_S(t) >= U(t, t_o)|\Psi(t_o) > . \tag{21}$$

The time-development operator is just:

$$U(t, t_o) = e^{-iH(t-t_o)} \tag{22}$$

and it satisfies:

$$i\frac{\partial}{\partial t}U = HU. \tag{23}$$

U has the important properties that:

$$U^\dagger(t,t_o) = U(t_o,t); U(t,t_1)U(t_1,t_o) = U(t,t_o) \tag{24}$$

Another picture which is convenient for certain purposes is the Heisenberg picture. In this picture, the states are independent of time, but operators are now time-dependent:

$$|\Psi_H(t)>= U^\dagger(t,t_o)|\Psi_S(t)>= |\Psi_S(t_o)> \tag{25}$$

while

$$\mathcal{O}_H(t) = U^\dagger(t,t_o)\mathcal{O}_S U(t,t_o). \tag{26}$$

Note that with this definition:

$$< \Psi_H|\mathcal{O}_H|\Psi_H >=< \Psi_S|\mathcal{O}_S|\Psi_S > \tag{27}$$

and

$$i\frac{\partial}{\partial t}\mathcal{O}_H = -U^\dagger H\mathcal{O}_S U + U^\dagger \mathcal{O}_S H U = -[H, \mathcal{O}_H] \tag{28}$$

This is actually an operator version of Ehrnfest's theorem. You can check, for example, that for a particle in a potential, the *operators* obey the classical equations of motion.

As an example, consider the raising and lowering operators for a harmonic oscillator.

$$H = \omega(\frac{1}{2} + a^\dagger a) \quad [a, a^\dagger] = 1. \tag{29}$$

$$[H, a] = -\omega a \tag{30}$$

So

$$\frac{d}{dt} a_H = -i\omega a_H. \tag{31}$$

So

$$a_H(t) = e^{-i\omega t} a_H(0) \tag{32}$$

which is the form that we have become familiar with in our field expansions.

Probably the most useful picture is the “interaction picture." Here we suppose that the Hamiltonian is of the form:

$$H = H_o + H_I \tag{33}$$

where H_o is a Hamiltonian which we know how to diagonalize (in the field theory case, this will be the free Hamiltonian).

The basic assumption of perturbation theory will be that H_I is in some sense small.

Then we remove the part of the time dependence we understand and write

$$|\Psi_I >= e^{iH_o(t-t_o)}|\Psi_S(t) > \tag{34}$$

$$= e^{iH_o(t-t_o)}e^{-iH(t-t_o)}|\Psi_S(t_0) >$$

So we define the time-development operator in the interaction picture:

$$U_I(t, t_o) = e^{iH_o(t-t_o)}e^{-iH(t-t_o)} \tag{35}$$

and define the time-dependent operators:

$$\mathcal{O}_I(t) = e^{iH_o(t-t_o)}\mathcal{O}_S e^{-iH_o(t-t_o)} \tag{36}$$

It is a simple exercise to show that U_I obeys:

$$i\frac{\partial}{\partial t}U_I = H_I(t)U_I;\ \ U_I(t,t) = 1. \tag{37}$$

where

$$\mathcal{H}_I(t) = e^{iH_o(t-t_o)}H_I e^{-iH_o(t-t_o)} \tag{38}$$

U_I has composition properties similar to those of U. These are most easily proven by writing U in a slightly different form:

$$U_I(t,t') = e^{iH_o(t-t_o)}e^{-iH(t-t')}e^{-iH_o(t'-t_o)} \tag{39}$$

This obeys the correct equation and boundary condition (check!). It also manifestly satisfies:

$$U_I^{\dagger}(t,t_o) = U_I(t_o,t); U_I(t,t_1)U_I(t_1,t_2) = U(t,t_2) \tag{40}$$

Solving this equation iteratively (see your favorite quantum mechanics book, e.g. eqns. 18.3.26 in Shankar or p. 85 of PS) yields the standard expansion of time-dependent perturbation theory. For our purposes, however, the most useful form is:

$$U_I(t, t_0) = T\ e^{\left(-i\int_{t_0}^{t} dt' H_I(t')\right)}. \tag{41}$$

The derivation of eqn. 41 proceeds iteratively, starting with the basic differential equation; the goal is a power series expansion of U_I in powers of H_I. First take $U_I \approx 1$ and substitute back in eqn. 37. This yields:

$$U_I(t) \approx 1 + \frac{1}{i} \int_{t_0}^{t} dt_1 H_I(t_1) \tag{42}$$

Repeating (e.g. derive by induction)

$$U_I(t) = 1 + \ldots \tag{43}$$

$$+ \left(\frac{1}{i}\right)^n \int_{t_0}^{t} dt_1 H_I(t_1) \int_{t_0}^{t_1} dt_2 H_I(t_2) \int_{t_0}^{t_2} dt_1 H_I(t_3) \ldots \int_{t_0}^{t_{n-1}} dt_n H_I(t_n) + \ldots$$

In the usual treatment in quantum mechanics, one starts with some initial and final states, and introduces complete sets of energy eigenstates; the time integrals are just integrals of exponentials, yielding energy denominators (in quantum mechanics, one introduces $i\epsilon$ so that the interaction turns off in the past; this damps the integral).

But Dyson's crucial observation was to note that the operators H_I are time ordered in the expression above. So one can take the n'th term above and take the upper limit to be t in each term, if one introduces a time ordering symbol, and divides by $\frac{1}{n!}$ to take care of over counting.

So we are interested in field theory in the object:

$$\int dt H_I = -\int d^4x \mathcal{L}_I \tag{44}$$

which is Lorentz invariant; we have already discussed the Lorentz invariance of the time ordering symbol.

Born Approximation in Quantum Mechanics

The proper derivation of the formulae for the S-matrix considers wave packets (see your textbook, pp. 102-108). But if one is willing to forget niceties, one can derive the usual expressions very quickly.

First, in non-relativistic scattering, we are interested in

$$< \vec{p}_f | U_I(T, -T) | \vec{p}_I >,$$

where it is understood that in the end we want to take the limit $T \to \infty$. To first order, staying away from the forward direction (so that the unit operator piece in U does not contribute, this is

$$< \vec{p}_f | U_I(T, -T) | \vec{p}_I > = < \vec{p}_f | \int_{-T}^{T} dt \; e^{-i(E_f - E_i)t} V | \vec{p}_I >$$

$$= 2\pi \delta(E_f - E_i) < \vec{p}_f | V | \vec{p}_I > .$$

Now comes the sleight of hand. We need to square this. Interpret

$$\delta(E_f - E_i)^2 = \delta(0)\delta(E_f - E_i).$$

$$\delta(0) = \lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} dt$$

$$= \frac{1}{2\pi} T.$$

To get the cross section, divide by T, to get the transition rate per unit time, and integrate over final states (e.g. all momenta in some solid angle, $\Delta\Omega$), and divide by the flux, $\frac{p}{m}$, to give

$$d\sigma = \int_{\Delta\Omega} \frac{d^3 p_f}{(2\pi)^3} | < \vec{p}_f | V | \vec{p}_i > |^2 \frac{m}{p} \delta(E_f - E_i).$$

This is the famous Born approximation for the scattering by a particle in a potential.

Relativistic Generalization

Now we repeat this for field theory. We specialize to processes with two incoming particles, and any number of outgoing particles. So we are interested in

$$< p_1 \ldots p_N | U(T, -T) | k_A k_B > = < p_1 \ldots p_N | S | k_A k_B >$$

(technically, the S matrix is defined to act between in and out states; see your text). Then we define:

$$S = 1 + iT$$

Because of space-time translation invariance, the T matrix always contains a δ function, so we write:

$$< p_1 \dots p_N | iT | k_A k_B >= (2\pi)^4 \delta(k_A + k_B - \sum p_i) i\mathcal{M}(k_A, k_B \to \{p_i\})$$

Note here that all momenta are "on shell", $\vec{p}_i^2 + m^2 = E_i^2$**.**

Don't be nervous about the δ-function; when we actually do computation it will pop out.
Now we square. We interpret $\delta(0) = VT$, in analogy to what we did in the non-relativistic case. Divide now by T, to get the transition rate per unit time, and V, corresponding to a constant target density (non-localized states). Divide also by the flux, which as in the non-relativistic limit goes as $|v_A - v_B|$, but is also multiplied by $2E_A 2E_B$ due to our normalization of the states. You can check that this flux is invariant under boosts along the beam axis. So we have:

$$d\sigma = \frac{1}{2E_A\, 2E_B|v_A - v_B|} \int \Pi \left(\frac{d^3 p_i}{(2\pi^3)} \right) \frac{1}{2E_{p_i}}$$

$$|\mathcal{M}(k_A, k_B \to \{p_i\})|^2 (2\pi)^4 \delta(k_A + k_B - \sum p_i).$$

Exercise: Repeat this exercise for the decay of a particle, and derive Peskin and Schroder's expression 4.86.

It will turn out that the essential information in quantum field theories – spectra, S-matrices and the like – is contained in *correlation functions* or *Green's functions*. We will motivate this gradually, but it will be instructive and simplest to begin by studying these objects. For our scalar field theories, these objects are:

$$G(x_1, \ldots, x_n) = \frac{\langle\Omega|T(\phi(x_1)\phi(x_2)\cdot\phi(x_n))|\Omega\rangle}{\langle\Omega|\Omega\rangle}. \qquad (45)$$

Here the ϕ's are fields in the Heisenberg picture, T is the time ordering symbol; $|\Omega\rangle$ is the ground state of the *full interacting field theory*. Because the ϕ's are scalars and the time-ordering operation is Lorentz invariant (when it matters), this object should have nice Lorentz properties.

To develop perturbation theory, our first goal is to write this in terms of operators in the interaction picture. For this we need to use the operator U to rewrite the fields, ϕ, in terms of interaction picture operators, and $|\Omega\rangle$ in terms of interaction picture states and operators. Let's start with the latter problem. From completeness of *the full Hamiltonian*, with assumed energy eigenstates $|n\rangle$, and corresponding energy eigenvalues E_n, we have

$$e^{-iHT}|0\rangle = e^{-iE_0 T}|\Omega\rangle\langle\Omega|0\rangle + \sum_{n\neq 0} e^{-iE_n T}|n\rangle\langle n|0\rangle. \qquad (46)$$

Letting $T \to \infty(1 - i\epsilon)$ eliminates all states from the sum except the ground state, so we can solve for $|\Omega\rangle$:

$$|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \left(e^{-iE_0 T}\langle\Omega|0\rangle\right)^{-1} e^{-iHT}|0\rangle \tag{47}$$

$$= \lim_{T\to\infty(1-i\epsilon)} \left(e^{-iE_0(T+t_0)}\langle\Omega|0\rangle\right)^{-1} e^{-iH(T+t_0)}|0\rangle$$

$$= \lim_{T\to\infty(1-i\epsilon)} \left(e^{-iE_0(t_0-(-T))}\langle\Omega|0\rangle\right)^{-1} e^{-iH(t_0-(-T))} e^{-iH_0(-T-t_0)}|0\rangle$$

(in the last step we have taken $H_0|0\rangle = 0$).

So we have shown that

$$|\Omega\rangle = \lim AU(t_0, -T)|0\rangle. \tag{48}$$

Here

$$A = \left(e^{-iE_0(t_0-(-T))}\langle\Omega|0\rangle\right)^{-1}. \tag{49}$$

Also

$$\langle\Omega| = A\langle 0|U(T, t_0). \tag{50}$$

The factor A looks nasty, but as we will see it will cancel out between the expressions in the numerator and denominator in our correlation functions.

The *two point function*, for example, is given by:

$$G(x,y) = \frac{T\langle\Omega|\phi(x)\phi(y)|\Omega\rangle}{\langle\Omega|\Omega\rangle} \tag{51}$$

which, for $x_0 > y_0$ is

$$= \frac{\langle 0|U(T,t_0)U^\dagger(x^0,t_0)\phi_I(x)U(x^0,t_0)U^\dagger(y^0,t_0)\phi_I(y)U(y^0,t_0)U(t_0,-T)|0\rangle}{\langle 0|U(T,t_0)U(t_0,-T)|0\rangle}.$$

Using the properties of the time-ordering symbol and the composition properties of the U's this is

$$G(x,y) = \frac{T\langle 0|U(T,x^0)\phi_I(x)U(x^0,y^0)\phi_I(y)U(y^0,-T)|0\rangle}{\langle 0|TU(T,t_0)U(t_0-T)|0\rangle}. \tag{52}$$

Now all factors are time-ordered, and it is understood that we will take the limit $T \to \infty$ in the end, so using our explicit expression for U (noting, now, that there was nothing special about this particular ordering):

$$G(x,y) = \frac{\langle 0|T\phi_I(x)\phi_I(y)U(T,x_0)U(x_0,y_0)U(y_0,-T)|0\rangle}{\langle 0|U(T,-T)|0\rangle} \quad (53)$$

$$= \frac{\langle 0|T\ \phi_I(x)\phi_I(y)exp(-i\int_{-T}^{T} dtH_I(t))|0\rangle}{\langle 0|T\ exp(-i\int_{-T}^{T} dt\ H_I(t))|0\rangle}.$$

This is the essential relation.

Wick's Theorem

This expression is readily evaluated using a result known as "Wick's theorem".

Need, first, notion of *normal ordering*. In general, the interaction picture operators consist of a's and $a^\dagger$'s. For a set of a's, $a^\dagger$'s, we define the normal ordered product as the product with all destruction operators to the right, all creation operators to the left.

We denote this product with the symbol:

$$N(\phi(x_1)\phi(x_2)\ldots\phi(x_n)) \quad \text{or} \quad :\phi(x_1)\phi(x_2)\ldots\phi(x_n): \qquad (54)$$

For example,

$$N(a(p_1)a^\dagger(p_2)a^\dagger(p_3)a(p_4)) = a^\dagger(p_2)a^\dagger(p_3)a(p_1)a(p_4). \qquad (55)$$

Note that the normal ordered product of any set of operators has vanishing expectation value. This will be very important to us.

We write $\phi = \phi^+ + \phi^-$, where ϕ^+ involves positive frequencies (annihilation operators), and ϕ^- negative frequencies (creation operators).

$$\phi^+(x) = \int \frac{d^3p}{(2\pi)^3} a(p) e^{-ip\cdot x}; \quad \phi^+(x) = \int \frac{d^3p}{(2\pi)^3} a(p) e^{-ip\cdot x}. \tag{56}$$

So, e.g.,

$$N(\phi(x_1)\phi(x_2)) = \phi^+(x_1)\phi^+(x_2) + \phi^-(x_1)\phi^+(x_2) \tag{57}$$

$$+\phi^-(x_2)\phi^+(x_1) + \phi^-(x_1)\phi^-(x_2)$$

and similarly for longer strings of operators.

A general product, and in particular a time ordered product of the type of interest to us in the interaction picture, can be related to normal products by commuting creation and destruction operators through each other. These commutators are just c-numbers. Wick's theorem relates time ordered products to products of propagators and normal products of operators.

The basic statement:

$$T(\phi(x_1)\phi(x_2)\ldots\phi(x_n)) = N(\phi(x_1)\phi(x_2)\ldots\phi(x_n)) \tag{58}$$

$$+D(x_1,x_2)N(\phi(x_3)\phi(x_4)\ldots\phi(x_n)) + \text{perms} + \ldots$$

$$+D(x_1,x_2)D(x_3,x_4)\ldots D(x_{n-1},x_n) + \text{perms}.$$

Since we are interested in vacuum expectation values of time ordered products, typically only the last term is of interest to us (but more later).

This is clearly a theorem which must be proven by induction. consider the case of the "two-point function". This is simple because the ordering is of no consequence except for terms involving ϕ^+ and ϕ^-. Choose a particular time ordering ($x_1^0 > x_2^0$). To put things in normal order, it is only necessary to commute $\phi^+(x_1)\phi^-(x_2)$. Since the commutator is a *c*-number, we obtain the same result as for the propagator (for this time ordering); similarly for the reversed time ordering.

Explicitly, for the two point function, if $x_1^0 > x_2^0$:

$$T\,\phi(x_1)\phi(x_2) = \phi^+(x_1)\phi^+(x_2)+\phi^+(x_1)\phi^-(x_2)+\phi^-(x_1)\phi^+(x_2)+\phi^-(x_1)\phi^-(x_2) \tag{59}$$

$$= \phi^+(x_1)\phi^+(x_2)+\phi^-(x_1)\phi^-(x_2)+\phi^-(x_2)\phi^+(x_1)+\phi^-(x_1)\phi^+(x_2)+[\phi(x_1),\phi(x_2)]$$

and the last term, which is a c-number, can be evaluated by taking the vacuum expectation value, and equals

$$T\langle 0|\phi(x_1)\phi(x_2)|0\rangle \tag{60}$$

for this time ordering. The case $x_2^0 > x_1^0$ follows in the same way. I will leave for you the inductive proof (to read about in your text and experiment with on your own). You should at least do for the four point function to get some feeling for what is going on.

Perturbation Theory in ϕ^4 theory

We are ready to apply this to the theory with $\lambda\phi^4$ coupling. We compute the two point function, but including corrections from the interaction.