Physics 217. Quantum Field Theory. Professor Dine

Free Field Propagators: Canonical Quantization

1 Free Scalar Field

Start with

$$D_F(x-y) = T\langle 0|\phi(x)\phi(y)|0\rangle \tag{1}$$

$$= \theta(x_0 - y_0)\langle 0|\phi(x)\phi(y)|0\rangle + \theta(y_0 - x_0)\langle 0|\phi(y)\phi(x)|0\rangle$$

It is easy to evaluate the matrix element

$$\langle 0|\phi(x)\phi(y)|0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega(p)}e^{-ip\cdot(x-y)} \tag{2}$$

where it is important to remember that in this expression, $p_0 = \sqrt{p^2+m^2}$. We can work out the full propagator by noting that

$$\theta(x^0) = \frac{1}{2\pi i}\int \frac{dq^0}{q^0 - i\epsilon}e^{iq^0 x_0} \tag{3}$$

(check using normal rules for contour integrals; ϵ is infinitesimal).

Combining,

$$D_F = \frac{1}{2\pi i}\int \frac{dq^0}{q^0 - i\epsilon}e^{iq^0(x_0-y_0)}\int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega(p)}e^{-ip\cdot(x-y)} + \frac{1}{2\pi i}\int \frac{dq^0}{q^0 - i\epsilon}e^{-iq^0(x_0-y_0)}\int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega(p)}e^{-ip\cdot(y-x)} \tag{4}$$

Remember, again, that $p^0 = \sqrt{p^2+m^2}$. Now we want to make this expression manifestly Lorentz invariant; we want to replace $\int d^3p dq^0$ by $\int d^4p$. We can do this by the following sequence of changes of variables (remember that the q^0 integrals run from $-\infty$ to $+\infty$).

So let $q_0 \rightarrow -q_0 + p_0$ in the first term, and $q_0 \rightarrow q_0 + p_0$ in the second. Also let $\vec{p} \rightarrow -\vec{p}$ in the second term. As a result, the exponential factors are the same in both terms. Putting the rest of the terms over a common denominator one is left with:

$$D_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2 - m^2 + i\epsilon} \tag{5}$$

where we have renamed $q_0 \rightarrow p_0$.

Exercise: Check the algebra above.

2 Green's Function for the Klein-Gordan equation

For $x \neq y$, D_F satisfies the Klein-Gordan equation in x and y, since $\phi(x)$ and $\phi(y)$ do. For $x = y$, you can see, just plugging in the result above, that

$$\left(\partial^2 + m^2\right)D_F(x,y) = i\int \frac{d^4p}{(2\pi)^4}e^{ip\cdot(x-y)} = -i\delta(x-y). \tag{6}$$

We could have found D_F this way; what is interesting (as in ordinary electrodynamics) is the boundary condition. In classical electrodynamics, one usually uses retarded boundary conditions. This leads to an expression which is not manifestly Lorentz invariant. The time ordering symbol yields different boundary conditions, known as "Feynman" boundary conditions. As an exercise, you should work out what this means.