Special Relativity

Physics 217 2013, Quantum Field Theory

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October 2013

Consider, first, boosts along the x axis. These should preserve:

$$c^2 t^2 - x^2. \tag{1}$$

Take units with $c = 1$, and call $x^o = ct$; this is similar to the conservation of vector lengths by ordinary rotations, except for the funny minus sign. So we write the transformation in the form:

$$x = \cosh(\omega)x' + \sinh(\omega)x^{o\prime} \tag{2}$$

$$x^o = \sinh(\omega)x' + \cosh(\omega)x^{o\prime}.$$

We can determine ω by considering the motion of the origin of the primed system, viewed in the unprimed coordinates.

Exercise (a.): Show that $tanh(\omega) = v$ (units with $c = 1$), and derive the standard formulas for the Lorentz transformation.

This analogy with rotations suggests generalizing the notion of a vector (something which transforms in a definite way under rotations) to a four-vector (an object which transforms in a definite way under Lorentz transformations). Calling

$$x^\mu = (t, \vec{x}) = (x^o, \vec{x}) \tag{3}$$

we have the rule:

$$x^o = \gamma(vx' + x^{o\prime}). \tag{4}$$
$$x = \gamma(x' + vx^{o\prime}).$$

It is not hard to generalize these rules to boosts by a velocity $\vec{v}$ in an arbitrary direction:

$$x^0 = \gamma(\vec{v} \cdot x' + x^{0'}). \tag{5}$$

$$\vec{x} = \hat{v}\gamma(\hat{v} \cdot x' + vx^{0'}) + (\vec{x}' - \hat{v}\vec{x}' \cdot \hat{v}).$$

Here $\hat{v} = \vec{v}/v$.
Exercise (b): Check this.

The vectors x^μ, and any vectors which transform in the same way, are called *contravariant* vectors. It is convenient to define covariant vectors,

$$x_\mu = (x^o, -\vec{x}). \tag{6}$$

Note that $x^\mu x_\mu = t^2 - \vec{x}^2$. This is the quantity preserved by Lorentz transformations. We can write the transformations in a matrix form:

$$x^\mu = \Lambda^\mu_\nu x^{\nu\prime}. \tag{7}$$

Exercise (c): Work out the components of Λ for this particular transformation.

$x_\mu y^\mu$ is invariant. A particularly important four vector is the differential, dx^μ. This transforms like any other four vector; it is, after all, the difference of two four vectors.

From the differential, we can construct an invariant

$$dx^\mu dx_\mu = d\tau^2 \tag{8}$$

$$= +dt^2 - d\vec{x}^2$$

In a particle's rest frame, $d\tau$ is just the time which elapses. So $d\tau$ is called the "proper time." While it refers to a special frame, it is a Lorentz-invariant notion.

A useful way to write the proper time in a general frame is:

$$d\tau^2 = dt^2(1 - \left(\frac{d\vec{x}}{dt}\right)^2) \tag{9}$$

or $d\tau = \gamma^{-1}dt$. We see again the time dilation effect.

The notation ds^2 is often used for $d\tau^2$.

Lagrangian and Hamiltonian in Special Relativity

We can try to write a lagrangian for a free particle. In order that the equations of motion for the particle take the same form in any frame, we can try to find a lagrangian which is *Lorentz Invariant*. We have seen that $d\tau$ is Lorentz invariant. So we try, as action, the integral over $d\tau$, multiplied by a constant, which we call $-mc^2$.

$$S = -m \int d\tau = -m \int dt \sqrt{1 - v^2} \qquad (10)$$

$$= \int dt \, L$$

$$L = -m\sqrt{1 - (\frac{d\vec{x}}{dt})^2} \qquad (11)$$

Note that for small velocities,

$$L = -m + \frac{1}{2}m\dot{x}_i^2 \qquad (12)$$

Using the Hamiltonian construction, the momenta are:

$$p^i = \frac{\partial L}{\partial \dot{x}^i} = \frac{m\dot{x}^i}{\sqrt{1 - v^2}} \tag{13}$$

The Hamiltonian is:

$$H = p^i \dot{x}^i - L \tag{14}$$

$$= \frac{mv^2 + m(1 - v^2)}{\sqrt{1 - v^2}}$$

$$= \frac{m}{\sqrt{1 - v^2}}$$

So we have derived $E = \gamma mc^2$ just following the rules of ordinary mechanics.

We are really supposed to express the Hamiltonian in terms of the momenta. This is easy to do here, and gives the important relation:

$$H^2 = E^2 = m^2 + p^2. \tag{15}$$

Four-Velocity and Four-Momentum

We can construct another four-vector by dividing dx^μ by $d\tau$.
The result is the four-velocity:

$$u^\mu = \frac{dx^\mu}{d\tau} \tag{16}$$

with components:

$$u^o = \gamma; u^i = \gamma v^i. \tag{17}$$

The components of u^μ are not all independent;

$$u^\mu u_\mu = (u^o)^2 - \vec{u}^2 = \gamma^2(1 - v^2) = 1. \tag{18}$$

This is a consequence of the definition of u^μ.

The four velocity is closely related to another four-vector, the *energy-momentum four-vector*,

$$p^{\mu} = mu^{\mu}. \tag{19}$$

The components are:

$$p^{o} = m\gamma; p^{i} = mv^{i}\gamma. \tag{20}$$

At low velocities, p^{i} is the ordinary three-momentum. p^{o} is E, the relativistic energy.

The constraint on u^μ is closely related to the energy-momentum relation:

$$(p^o)^2 - \vec{p}^2 = m^2. \tag{21}$$

We see that this is a *relativistically invariant* relation.
One very valuable fact to note: γ is the energy divided by the mass of the particle. This is almost always the easiest way to find the γ factor for the motion of a particle. If you absolutely need to know the velocity for some reason, you can then solve for v in terms of γ. (E.g. SLAC; high energy gamma rays).

One can compute γ quickly by remembering that $\gamma = \frac{E}{m}$. Interesting examples are provided by the LHC ($E = 7$ TeV, so $\gamma \approx 7000$; SLAC: $E = 50$ GeV, so $\gamma = 10^5$; also cosmic rays (think about muon lifetime):

Kinetic energy per particle (nucleus) [GeV]
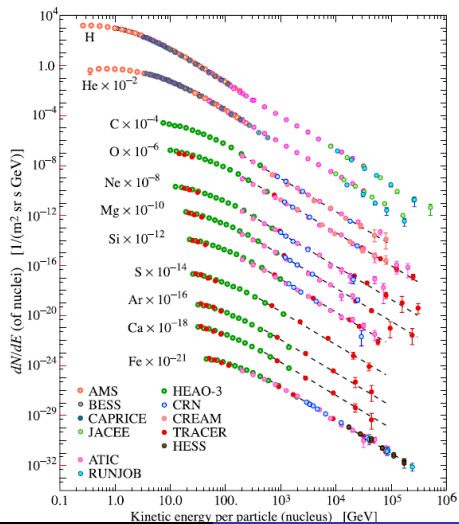
More Four-Vectors

Another very useful four-vector is the gradient. We write this as:

$$\partial_\mu = \frac{\partial}{\partial x^\mu}. \tag{22}$$

This transforms under Lorentz transformations as a *covariant vector* (indices downstairs). One way to see this is to note that

$$\partial_\mu x^\mu = 4 \tag{23}$$

i.e. it is a number, which is Lorentz invariant. But otherwise, you can use the chain rule to check.

With this observation,

$$\Box = \partial_\mu \partial^\mu = \partial_0^2 - \vec{\nabla}^2 \tag{24}$$

is *Lorentz invariant*!

Note, also, that the energy-momentum in quantum mechanics is a four vector:

$$p^\mu = i\hbar \partial^\mu. \tag{25}$$

(Check that this form, with the *i* instead of $-i$, is correct).

Still another useful four vector is the *current*, j^μ. It's components are (again with $c = 1$; you might want to check how to put in the factors of c explicitly):

$$j^o = \rho; \vec{j} = \vec{J}. \tag{26}$$

I won't prove it now, but let's think about it. If we have a charge at rest, then under a boost, we have a charge density and a current. The charge density is increased by a γ factor as a result of the Lorentz contraction. The current is the charge density times the velocity. As one further piece of evidence, this makes the equation of current conservation,

$$\partial_\mu j^\mu = 0 \tag{27}$$

a Lorentz-invariant equation, i.e. true in any frame. **[In another exercise, you will prove that the current of a point particle is a four vector]**

The Vector Potential as a Four-Vector

If we define a four-vector A^μ by

$$A^o = V; \vec{A} = \vec{A} \tag{28}$$

then the Lorentz gauge condition is:

$$\partial_\mu A^\mu = \frac{\partial V}{\partial t} + \vec{\nabla} \cdot \vec{A} = 0. \tag{29}$$

In other words, it is a Lorentz-invariant condition. Moreover, in this gauge, the equations for the potentials can be written in a *manifestly Lorentz-covariant* form:

$$\Box A^\mu = j^\mu. \tag{30}$$

Mechanics again

So far, we wrote the lagrangian for a free particle. We can try and guess a lagrangian for a particle in an electromagnetic field from. We might expect that the lagrangian should involve V. It should also be Lorentz covariant. So we try:

$$S = -m \int d\tau - \int dx^\mu A_\mu. \tag{31}$$

At low velocities, the last term is $- \int dt\phi$. In general, we can write this as:

$$S = \int dt(-m\gamma - q\phi + q\vec{A} \cdot \vec{v}). \tag{32}$$

Our goal is to show that this lagrangian gives the Lorentz force law. We need to work carefully with the Euler-Lagrange equations. First, we work out:

$$\frac{\partial L}{\partial \dot{x}_i} = \gamma m v_i + q A^i. \tag{33}$$

To write the equations of motion, we need to take the *total derivative* with respect to time. A_i has, in general, both explicit time dependence and time dependence coming from its dependence on $x_j(t)$:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = \frac{d\tau}{dt} m \frac{d}{d\tau} u^i + q \dot{A}^i + q v_j \partial_j A^i. \tag{34}$$

In the Euler-Lagrange equations, we also need:

$$\frac{\partial L}{\partial x_i} = -q\partial_i\phi + q(\partial_i A_j)v_j. \tag{35}$$

Putting this all together:

$$m\dot{u}^i = -q(\dot{A}^i + \partial_i\phi) - qv_j(\partial_j A_i - \partial_i A_j) \tag{36}$$

$$= q\vec{E} + q\vec{v} \times \vec{B}.$$

This is the relativistic generalization of the Lorentz force law:

$$m\dot{u}^i = \gamma q[\vec{E} + \vec{v} \times \vec{B}]. \tag{37}$$

More on the Matrices Λ

Now that we are a bit more used to four vector notation, let's return to Λ. Write:

$$x^\mu = \Lambda^\mu_{\ \nu} x^{\nu\prime}; \ y_\mu = y'_\rho \tilde{\Lambda}^\rho_{\ \mu}. \tag{38}$$

So in order that $y_\mu x^\mu$ be invariant, we need:

$$\tilde{\Lambda}^\rho_{\ \mu} \Lambda^\mu_{\ \nu} = \delta^\rho_{\ \nu}. \tag{39}$$

It also follows from their definitions that

$$\tilde{\Lambda}^\mu_{\ \nu} = g^{\mu\rho} g_{\nu\sigma} \Lambda^\sigma_{\ \rho} \tag{40}$$

So

$$g^{\mu\rho} g_{\nu\sigma} \Lambda^\sigma_{\ \rho} \Lambda^\nu_{\ \rho} = \delta^\mu_{\ \rho}. \tag{41}$$

This is the case for our simple Lorentz transformation along the z axis:

$$\Lambda = \begin{pmatrix} \gamma & \gamma v & 0 & 0 \\ \gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{42}$$

$$\tilde{\Lambda} = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{43}$$

(note here that to make the matrices easy to write, I have taken the four vector x^μ to be:

$$\begin{pmatrix} x^o \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}. \tag{44}$$

It is true for the more general transformation we have written above.

In terms of Λ, it is easy to understand the covariance of equations. For example,

$$\partial^2 A^\mu = j^\mu \tag{45}$$

is covariant, since both sides transform in the same fashion under Lorentz transformations.

Infinitesimal Lorentz Transformations

Writing

$$\Lambda^{\mu}_{\ \nu} = \delta^{\mu}_{\ \nu} + \omega^{\mu}_{\ \nu} \tag{46}$$

equation 39 becomes

$$\omega^{\mu}_{\ \nu} \delta_{\mu}^{\ \rho} + \delta^{\mu}_{\ \rho} \omega_{\mu}^{\ \rho} = 0. \tag{47}$$

So

$$\omega^{\rho}_{\ \nu} + \omega_{\nu}^{\ \rho} = 0 \tag{48}$$

or

$$\omega_{\rho\nu} + \omega_{\nu\rho} = 0 \tag{49}$$

i.e. $\omega_{\rho\nu}$ is antisymmetric.

Exercise (d): Determine the components of ω for an infinitesimal Lorentz boost along the *z* axis.
Exercise (e): Show that $d^4 x$ is invariant under arbitrary Lorentz transformations.

Covariant Equations, Covariant Tensors

$$A^\mu = B^\mu \tag{50}$$

Since both sides of this equation transform in the same way under Lorentz transformations, this equation is true in any frame. Similarly,

$$T^{\mu\nu\rho} = \mathcal{J}^{\mu\nu\rho}. \tag{51}$$

Following Einstein, we rewrite Maxwell's equations in a covariant form. The scalar and vector potentials can be grouped as a four vector,

$$A^\mu = (\phi, \vec{A}) \tag{52}$$

Then introduce the field strength (sometimes called Faraday) tensor:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{53}$$

$$F_{oi} = E_i; \ F_{ij} = -\epsilon_{ijk}B_k. \tag{54}$$

These transform as:

$$F^{\mu\nu} = \Lambda^\mu_{\ \rho}\Lambda^\nu_{\ \sigma}F^{\rho\sigma\prime}. \tag{55}$$

There is not much to do but multiply these out. You find the transformation laws for the $\vec{E}$ and $\vec{B}$ fields in your text. What about the Lorentz force law? This should involve $\dot{u}^\mu$, which is a four-vector which would generalize the acceleration. It should be linear in the fields $\vec{E}$ and $\vec{B}$, i.e. linear in $F^{\mu\nu}$. A guess, for a particle of charge e, is:

$$m\dot{u}^\mu = e\,u_\nu F^{\mu\nu}. \tag{56}$$

Exercise f: Check that the components of this equation yield the Lorentz force law.

Other Interactions: Strong, Weak Interactions Look Similar

Strong interactions: $F_{\mu\nu}^a$, $a = 1, \ldots 8$. Equations for the fields:

$$\partial_\nu F^{\mu\nu a} = J^{\mu a}. \tag{57}$$

Currents: *color* of the quarks.

Weak interactions: same thing! ($a = 1 \ldots 3$). Currents: weak charge. But includes an extra piece, due to the *Higgs field*, which leads to a mass:

$$m^2 A_\mu^{i2}.$$

More Exercises:

Exercise (g): Show that $F_{oi} = E_i$, $F_{ij} = \epsilon_{ijk}B_k$.

Exercise (h): Verify that Maxwell's equations take the form:

$$\partial_\mu F^{\mu\nu} = j^\nu.$$

Explain why this result shows that Maxwell's equations are Lorentz covariant.

Exercise (i): The charge and current density, together form a four vector, $j^\mu = (\rho, \vec{j})$. Write the equation for current conservation in a covariant form.

Exercise (j): For a particle of mass m, what is $p^2 = p_\mu p^\mu$?