Lorentz Invariance and Lorentz Group: A Brief Overview

These are things it is important you should know. You can find more background, for example, in Jackson.

Four vectors:

$$x^\mu = (x^o, \vec{x}) = (t, \vec{x}) \qquad x_\mu = (x^o, -\vec{x})$$

$$x_\mu = g_{\mu\nu} x^\nu$$

$$x^\mu x_\mu = x^{o\ 2} - \vec{x}^2 = x^\mu g_{\mu\nu} x^\nu$$

where the metric tensor, $g_{\mu\nu}$ is

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

x^μ, with the index upstairs, is called a contravariant vector; $x_\mu = g_\mu \nu x^\nu$ is called covariant. Note that $g_{\mu\nu}$ is its own inverse. Indices are lowered with $g_{\mu\nu}$ and raised with $g^{\mu\nu}$ which is the inverse of $g_{\mu\nu}$ (and, as a matrix, is the same as $g_{\mu\nu}$. If one raises one index on $g_{\mu\nu}$, one gets

$$g_\mu^\nu = g^{\nu\rho} g_{\mu\rho} = \delta_\mu^\nu.$$

Note we are using the Einstein summation convention, which says that repeated indices are summed.

Lorentz transformations

$$x^{\mu\ \prime} = \Lambda^\mu{}_\nu x^\nu \qquad x'_\mu = \Lambda_\mu{}^\nu x_\nu.$$

(Indices on Λ are raised and lowered with the metric tensor). In general, a four vector is any quantity which transforms like x^μ under Lorentz transformations. Examples include the four velocity, the four momentum, the vector potential and the current density of electrodynamics.

Scalars are invariant. $x \cdot y$ is one example. Others are $p^2 = p \cdot p$, $A_\mu j^\mu$, where A is the vector potential four vector (more below) and j is the electromagnetic current.

Under Lorentz transformations, the dot product of two four vectors, $x \cdot y = x^\mu y_\mu$ is preserved. In terms of Λ, this means:

$$x^{\mu\,\prime} y'_\mu = \Lambda^\mu{}_\nu x^\nu \Lambda_\mu{}^\rho y_\rho$$

so

$$\Lambda^\mu{}_\nu \Lambda_\mu{}^\rho = \Lambda_\mu{}^\rho \Lambda^\mu{}_\nu = g_\nu{}^\rho = \delta_\nu{}^\rho$$

Infinitesmal form of the Lorentz Transformation

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$$

so

$$\omega^\mu{}_\nu \delta_\mu{}^\rho + \delta^\mu{}_\nu \omega_\mu{}^\rho = 0,$$

i.e.

$$\omega^\rho{}_\nu + \omega_\nu{}^\rho = 0.$$

So, lowering indices,

$$\omega_{\rho\nu} + \omega_{\nu\rho} = 0$$

i.e. $\omega_{\rho\nu}$ is antisymmetric.

Exercises:

a. For a Lorentz transformation along the z axis, determine the components of Λ. Note that you can write this nicely by taking:

$$t' = \cosh(\omega)t + \sinh(\omega)z \qquad z' = \sinh(\omega)t - \cosh(\omega)z.$$

b. Determine the components of ω for an infinitesmal Lorentz boost along the z axis.

c. The differential of the proper time is $d\tau = dx \cdot dx$. Verify that $d\tau$ is a scalar. Show that $\frac{dx^\mu}{d\tau}$ is a four vector.

Covariant Equations, Covariant Tensors

Consider an equation involving two four vectors:

$$A^\mu = B^\mu$$

Since both sides of this equation transform in the same way under Lorentz transformations, this equation is true in any frame. Similarly, for more complicated tensors,

$$T^{\mu\nu\rho} = \mathcal{J}^{\mu\nu\rho}.$$

The derivative is a four vector with indices which are "downstairs" (covariant, as opposed to "contravariant" indices). To see this, it is simplest to note that

$$\frac{\partial}{\partial x^\mu} x^\mu = 4$$

is Lorentz invariant. It is customary to simplify notation and write

$$\partial_\mu = \frac{\partial}{\partial x^\mu}$$

so the equation above, for example, reads

$$\partial_\mu x^\mu = 4$$

The components of the partial derivative are:

$$\partial_\mu = (\partial_o, \vec{\nabla})$$

Examples of Covariant equations:

1. Equation of motion for a free particle:

$$\frac{d^2 x^\mu}{d\tau^2} = 0$$

2. Wave equation for a scalar field:

$$\partial^\mu \partial_\mu \phi + m^2 \phi = 0 \quad \text{or} \quad \partial^2 \phi + m^2 \phi = 0.$$

3. Lorentz gauge equation for the four-vector potential:

$$\partial^2 A^\mu = j^\mu$$

We will see more examples shortly.

Exercises:

d. The charge and current density, together form a four vector, $j^\mu = (\rho, \vec{j})$. Write the equation for current conservation in a covariant form.

It is helpful also to consider relativistic kinematics in the four-vector notation. The components of the momentum are

$$p_\mu = (E, \vec{p}).$$

$$p^2 = p_\mu p^\mu = E^2 - \vec{p}^2 = m^2.$$

More generally, the dot product of any two four momenta is Lorentz invariant. In a scattering experiment, four example, if the two incoming particles have four momenta p_1 and p_2, one can construct the invariant

$$s = (p_1 + p_2)^2.$$

This is the total center of mass energy, since in the center of mass frame, $\vec{p}_1 + \vec{p}_2 = \vec{0}$. It is easily evaluated in any frame.

Exercises

e. Evaluate s in the lab frame for an electron-proton collision.

f. Suppose the electron-proton collision is elastic, the initial electron momentum is $\vec{p}_1$ and the final momentum $\vec{p}_3$. Calling $t = (p_1 - p_3)^2$ (where p_1 and p_3 are the corresponding four-vectors), evaluate t. What is the connection of t to the momentum transfer?

Following Einstein, we can rewrite Maxwell's equations in a covariant form. The scalar and vector potentials can be grouped as a four vector,

$$A^\mu = (\phi, \vec{A})$$

Then introduce the field strength (sometimes called Faraday) tensor:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

Exercises:

g. Show that $F_{oi} = E_i$, $F_{ij} = \epsilon_{ijk} B_k$.

h. $F_{\mu\nu}$, being a tensor, transforms like the product $x_\mu x_\nu$ under Lorentz transformations. Work out the transformation of $\vec{E}$ and $\vec{B}$ under Lorentz transformations along the z axis using this fact.

i. Verify that Maxwell's equations take the form:

$$\partial_\mu F^{\mu\nu} = j^\nu.$$

Explain why this result shows that Maxwell's equations are Lorentz covariant.