Lorentz Invariance and Lorentz Group: A Brief Overview

These are things it is important you should know. You can find more background, for example, in Jackson.

Four vectors:

\[ x^\mu = (x^0, \vec{x}) = (t, \vec{x}) \quad x_\mu = (x^0, -\vec{x}) \]

\[ x_\mu = g_{\mu\nu} x^\nu \]
\[ x^\mu x_\mu = x^0^2 - \vec{x}^2 = x^\mu g_{\mu\nu} x^\nu \]

where the metric tensor, \( g_{\mu\nu} \) is

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 \\
\end{pmatrix}
\]

\( x^\mu \), with the index upstairs, is called a contravariant vector; \( x_\mu = g_{\mu\nu} x^\nu \) is called covariant. Note that \( g_{\mu\nu} \) is its own inverse. Indices are lowered with \( g_{\mu\nu} \) and raised with \( g^{\mu\nu} \) which is the inverse of \( g_{\mu\nu} \) (and, as a matrix, is the same as \( g_{\mu\nu} \). If one raises one index on \( g_{\mu\nu} \), one gets

\[ g^{\nu\mu} = g^{\mu\rho} g_{\rho\nu} = \delta^{\nu}_{\mu}. \]

Note we are using the Einstein summation convention, which says that repeated indices are summed.

**Lorentz transformations**

\[ x^{\mu'} = \Lambda^{\mu}_{\nu} x^\nu \quad x^\mu = \Lambda^\mu_{\nu} x_\nu. \]

(Indices on \( \Lambda \) are raised and lowered with the metric tensor). In general, a four vector is any quantity which transforms like \( x^\mu \) under Lorentz transformations. Examples include the four velocity, the four momentum, the vector potential and the current density of electrodynamics.

Scalars are invariant. \( x \cdot y \) is one example. Others are \( p^2 = p \cdot p, A_\mu j^\mu \), where \( A \) is the vector potential four vector (more below) and \( j \) is the electromagnetic current.
Under Lorentz transformations, the dot product of two four vectors, $x \cdot y = x^\mu y_\mu$ is preserved. In terms of $\Lambda$, this means:

$$x^\mu y^\nu = \Lambda^\mu_\nu x^\nu \Lambda_\mu^\rho y_\rho$$

so

$$\Lambda^\mu_\nu \Lambda_\mu^\rho = \Lambda_\mu^\rho \Lambda^\mu_\nu = g_\nu^\rho = \delta_\nu^\rho$$

**Infinitesimal form of the Lorentz Transformation**

$$\Lambda^\mu_\nu = \delta^\mu_\nu + \omega^\mu_\nu$$

so

$$\omega^\mu_\nu \delta^\mu_\nu + \delta^\mu_\nu \omega^\mu_\nu = 0,$$

i.e.

$$\omega_\nu^\mu + \omega_\mu^\nu = 0.$$

So, lowering indices,

$$\omega_{\nu \mu} + \omega_{\mu \nu} = 0$$

i.e. $\omega_{\mu \nu}$ is antisymmetric.

**Exercises:**

a. For a Lorentz transformation along the $z$ axis, determine the components of $\Lambda$. Note that you can write this nicely by taking:

$$t' = \cosh(\omega) t + \sinh(\omega) z \quad z' = \sinh(\omega) t - \cosh(\omega) z.$$

b. Determine the components of $\omega$ for an infinitesimal Lorentz boost along the $z$ axis.

c. The differential of the proper time is $d\tau = dx \cdot dx$. Verify that $d\tau$ is a scalar. Show that $\frac{dx^\mu}{d\tau}$ is a four vector.
Covariant Equations, Covariant Tensors

Consider an equation involving two four vectors:

\[ A^\mu = B^\mu \]

Since both sides of this equation transform in the same way under Lorentz transformations, this equation is true in any frame. Similarly, for more complicated tensors,

\[ T^{\mu\nu\rho} = J^{\mu\nu\rho}. \]

The derivative is a four vector with indices which are “downstairs” (covariant, as opposed to “contravariant” indices). To see this, it is simplest to note that

\[ \frac{\partial}{\partial x^\mu} x^\mu = 4 \]

is Lorentz invariant. It is customary to simplify notation and write

\[ \partial_\mu = \frac{\partial}{\partial x^\mu} \]

so the equation above, for example, reads

\[ \partial_\mu x^\mu = 4 \]

The components of the partial derivative are:

\[ \partial_\mu = (\partial_\nu, \nabla) \]

Examples of Covariant equations:

1. Equation of motion for a free particle:

\[ \frac{d^2 x^\mu}{d\tau^2} = 0 \]

2. Wave equation for a scalar field:

\[ \partial^\mu \partial_\mu \phi + m^2 \phi = 0 \quad \text{or} \quad \partial^2 \phi + m^2 \phi = 0. \]

3. Lorentz gauge equation for the four-vector potential:

\[ \partial^2 A^\mu = j^\mu \]

We will see more examples shortly.
Exercises:

d. The charge and current density, together form a four vector, \( j^\mu = (\rho, \vec{j}) \).

Write the equation for current conservation in a covariant form.

It is helpful also to consider relativistic kinematics in the four-vector notation. The components of the momentum are

\[
p_\mu = (E, \vec{p}).
\]

\[
p^2 = p_\mu p^\mu = E^2 - \vec{p}^2 = m^2.
\]

More generally, the dot product of any two four momenta is Lorentz invariant. In a scattering experiment, for example, if the two incoming particles have four momenta \( p_1 \) and \( p_2 \), one can construct the invariant

\[
s = (p_1 + p_2)^2.
\]

This is the total center of mass energy, since in the center of mass frame, \( \vec{p}_1 + \vec{p}_2 = \vec{0} \). It is easily evaluated in any frame.

Exercises

e. Evaluate \( s \) in the lab frame for an electron-proton collision.

f. Suppose the electron-proton collision is elastic, the initial electron momentum is \( \vec{p}_1 \) and the final momentum \( \vec{p}_3 \). Calling \( t = (p_1 - p_3)^2 \) (where \( p_1 \) and \( p_3 \) are the corresponding four-vectors), evaluate \( t \). What is the connection of \( t \) to the momentum transfer?

Following Einstein, we can rewrite Maxwell’s equations in a covariant form. The scalar and vector potentials can be grouped as a four vector,

\[
A^\mu = (\phi, \vec{A})
\]

Then introduce the field strength (sometimes called Faraday) tensor:

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.
\]

Exercises:
g. Show that $F_{oi} = E_i$, $F_{ij} = \epsilon_{ijk}B_k$.

h. $F_{\mu\nu}$, being a tensor, transforms like the product $x_\mu x_\nu$ under Lorentz transformations. Work out the transformation of $\vec{E}$ and $\vec{B}$ under Lorentz transformations along the $z$ axis using this fact.

i. Verify that Maxwell’s equations take the form:

$$\partial_\mu F^{\mu\nu} = j^\nu.$$  

Explain why this result shows that Maxwell’s equations are Lorentz covariant.