Physics 217. Quantum Field Theory. Professor Dine

Point Particle Action

It is useful to recall the classical point particle action, and how the "minimal subsitution", $\vec{p} \to \vec{p} - e\vec{A}$ comes from.

First, for a free particle, what is the action? It should be relativistically invariant, so it should be built from $d\tau$. So we take:

$$S = -m \int d\tau \tag{1}$$

where the integral is taken along the world line of the particle. In the non-relativistic limit this becomes:

$$S = -m \int dt \sqrt{\frac{dx^\mu}{dt}\frac{dx_\mu}{dt}} \approx -m \int dt (1 - \frac{1}{2} m \frac{d\vec{x}}{dt}^2) \tag{2}$$

Exercise:

1. Work out the canonical momentum (don't make the non-relativistic approximation)
2. Find the Hamiltonian. Show that $H = \sqrt{m^2 + \vec{p}^2}$.

Now couple the system to an electromagnetic field. We will take the action to be

$$S = -m \int d\tau - e \int d\tau \frac{dx_\mu(\tau)}{d\tau} A^\mu(x(\tau)) \approx \frac{m}{2} \int dt (\frac{d\vec{x}}{dt})^2 - \int dt [e\phi(\vec{x}(t)) - e\frac{d\vec{x}}{dt} \cdot \vec{A}(\vec{x}(t))] \tag{3}$$

So the canonical momentum is:

$$p_i = \frac{\partial L}{\partial \dot{x}^i} = m\dot{x}^i + eA^i(\vec{x}(t)). \tag{4}$$

You can readily work out the Hamiltonian in the non-relativistic limit. It is given by:

$$H = \frac{(\vec{p} - e\vec{A})^2}{2m} + e\phi. \tag{5}$$

(remember that the Hamiltonian is supposed to be written in terms of the canonical momenta).

Exercise:

1. In the non-relativistic limit, show that the equations of motion obtained from this lagrangian yield the Lorentz force law. The ϕ term is rather obvious, so just focus on the $\vec{A}$ term.
2. Show that
$$S = \int d\tau - \int d^4x j^\mu A_\mu \tag{6}$$
where j^μ, the electromagnetic current, is
$$j^\mu(x) = \int d\tau e \frac{dx_o^\mu}{d\tau} \delta(x^\mu - x_o^\mu(\tau)) d\tau \tag{7}$$
where x_o^μ is the trajectory of the particle. (Hint: don't work too hard. Work in the instantaneous rest frame of the particle, where $dt = d\tau$, and then use relativistic invariance).

3. Verify that in the non-relativistic limit, this reduces to the expected expression.
4. Show that j^μ is conserved.
5. Show that, as a result, S is gauge invariant, i.e. that it is unchanged if $A_\mu \to A_\mu + \partial_\mu \Lambda$.