Applications of Wick's theorem: calculation of correlation functions

In correlation functions, we will be able to keep only terms where all products of fields are replaced by propagators, or "contracted"

Ex:

$$T\langle|\phi(x_1)\phi(x_2)\phi(x_3)\cdots\phi(x_n)|0\rangle = 0 \text{ if } n \text{ odd}$$

Ex:

$$T\langle 0|\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0\rangle$$

$$= D_F(x_1-x_2)\,D_F(x_3-x_4)$$

$$+ D_F(x_1-x_3)\,D_F(x_2-x_4) + D_F(x_1-x_4)\,D_F(x_2-x_3)$$

Some notation:

$$T\langle 0|\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0\rangle$$

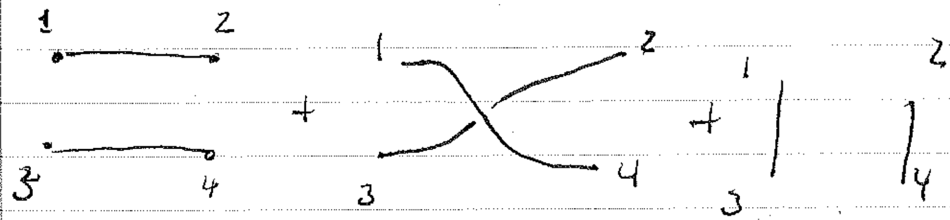
$$= \langle 0|\underbrace{\phi(x_1)\phi(x_2)}\,\underbrace{\phi(x_3)\phi(x_4)}|0\rangle + \langle 0|\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0\rangle$$

$$+ \langle 0|\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0\rangle$$

$$= D_F(x_1-x_2)\,D_F(x_3-x_4) + D_F(x_1-x_3)\,D_F(x_2-x_4)$$

$$+ D_F(x_1-x_4)\,D_F(x_2-x_3)$$

We can represent these ~~terms~~ diagramatically (Feynman diagrams - almost)



Each line represents a propogator; points are X_1, X_2, X_3, X_4

Let's consider the perturbation expansion of the propagator (2-pt. function)

$$\langle 0 | T(\phi(x)\phi(y) \exp(-i \tfrac{\lambda}{4!} \int d^4z\, \phi^4(z)) | 0 \rangle$$

$$= \langle 0 | T(\phi(x)\phi(y)) | 0 \rangle + T \langle 0 | \phi(x)\phi(y) \frac{(-i\lambda)}{4!} \int d^4z\, \phi^4(z) | 0 \rangle$$

$$+ \ldots$$

$$= D_F(x-y) - \frac{i\lambda}{4!} \phi(x)\phi(y) \int d^4z\, \phi(z)\phi(z)\phi(z)\phi(z)$$

(3 ways; identical. multiply by 3)

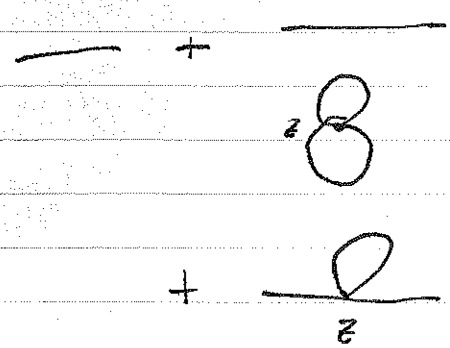
$$+ -\frac{i\lambda}{4!} \phi(x)\phi(y) \int \phi(z)\phi(z)\phi(z)\phi(z)\, d^4z$$

4x3 ways

$$= D_F(x-y) - \frac{i\lambda}{8} D_F(x-y) \int D_F^2(0)\, d^4z$$

$$- \frac{i\lambda}{2} \int d^4z\, D_F(x-z)\, D_F(y-z)\, D_F(z-z)$$

Diagramatic representation:



("disconnected graph" we will interpret shortly).

These expressions are all explicit (we know D_F). We will discuss their evaluation shortly. This is usually most straightforward in momentum space.

As one considers higher orders of perturbation theory (higher orders of λ), things get progressively more complicated.

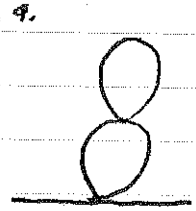
Let's consider one more order

$$T\langle 0|\,\phi(x)\phi(y)\,\frac{1}{2!}\left(-\frac{i\lambda}{4!}\right)\int d^4z_1\,\phi\phi\phi\phi$$

$$\times\left(-\frac{i\lambda}{4!}\right)\int d^4z_2\,\phi\phi\phi\phi\,|0\rangle$$

Now many possible contractions; must be careful about combinatorics

E.g.



$$2\,\frac{1}{2!}\left(\frac{-i\lambda}{4!}\right)^2\langle 0|\,\phi(x)\int \underbrace{\phi(z_1)}_{4\text{ ways}}\phi(z_1)\,\phi(z_1)\,\overbrace{\phi(z_1)\int d^4z_2\,\phi(z_2)\,\phi(z_2)}^{3\text{ ways}}\phi(z_2)\phi(z_2)\times\phi(y)|0\rangle$$

from interchang z_1, z_2

4 ways to contract

4 ways

3 remaing ways

$$=\frac{(-i\lambda)^2}{(4!)^2}\,(4\times 3)^2\int d^4z_1\,d^4z_2\,D_F(x-z_1)\,D_F(z_1-y)$$

$$\times D_F(z_1-z_2)^2\,D_F(0)$$

Other diagrams:

Connected:

Disconnected:

Let us consider the momentum space version of these rules.

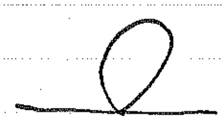
Study:

$$\int d^4x \, e^{ip\cdot x} \langle 0| T(\phi(x)\,\phi(0)\,\exp(\quad)|0\rangle$$

Lowest order: $\displaystyle \int d^4x \, e^{ip\cdot x} \int \frac{d^4p'}{(2\pi)^4} e^{-ip'\cdot x} \frac{i}{p'^2 - m^2 + i\epsilon}$

$$= \frac{i}{p^2 - m^2 + i\epsilon}$$

Next order:

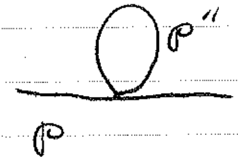


$$= \frac{-i\lambda}{4!} \int d^4x \, e^{ip\cdot x} \int D_F(x-z)\, D_F(0)\, D_F(z) \times 4 \times 3 \, d^4z$$

$$= \frac{-i\lambda}{2}\int d^4x\, d^4z \int \frac{d^4p'\, d^4p''\, d^4q}{(2\pi)^{12}}\, e^{ip\cdot x}\, e^{-ip'\cdot(x-z)}\, e^{-ip''\cdot 0}\, e^{-iqz}$$

$$\times \frac{i}{p'^2-m^2+i\epsilon}\,\frac{i}{p''^2-m^2+i\epsilon}\,\frac{i}{q^2-m^2+i\epsilon}$$

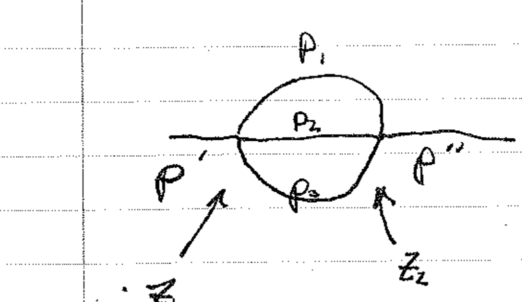
$$= \frac{-i\lambda}{2}\,\frac{i}{p^2-m^2+i\epsilon}\,\frac{i}{p^2-m^2+i\epsilon}\int \frac{d^4p''}{(2\pi)^4(p''^2-m^2+i\epsilon)}$$



$\frac{i}{p^2-m^2+i\epsilon}$ for each "external" line. $\frac{i}{p''^2-m^2+i\epsilon}$ for "internal."

Momentum conserved at (each) vertex. p'' undetermined, so have to integrate. $(-i\lambda)\times$ (combinatoric factor) at vertex.

One more:



P', P'', P_i

labels for momentum integrals

Integrals over locations of vertices, z_1, z_2, give δ-functions.

$$\int d^4x \int d^4z_1 d^4z_2 \; e^{ip\cdot x} \, e^{-ip'\cdot(x-z_1)} \, e^{-ip_1\cdot(z_1-z_2)} \, e^{-ip_2\cdot(z_1-z_2)}$$
$$\times e^{-ip_3\cdot(z_1-z_2)} \, e^{-ip''\cdot(z_2-0)}$$

$$= (2\pi)^4 \delta(p-p') \, (2\pi)^4 \delta(p''-p_1-p_2-p_3) \, \delta(p-p_1-p_2-p_3)$$

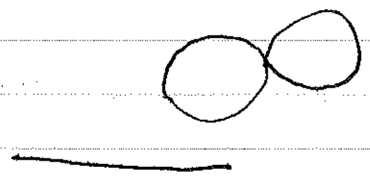
Momentum conservation at each vertex. Two momenta (say p_2, p_3) still integrated (not fixed)

With combinatories gives

$$\frac{(-i\lambda)^2}{(4!)^2 \, 2!} \times 2 \times 4! \times 4 \times 3 \int \frac{d^4p_1}{(2\pi)^4} \frac{d^4p_2}{(2\pi)^4} \frac{i}{p_1^2 - m^2 + i\epsilon} \, \frac{i}{p_2^2 - m^2 + i\epsilon}$$
$$\times \frac{i}{(p_1+p_2)^2 - m^2 + i\epsilon} \times \frac{i}{(p^2 - m^2 + i\epsilon)^2}.$$

Cancellation of the disconnected diagrams

What's going on with things like



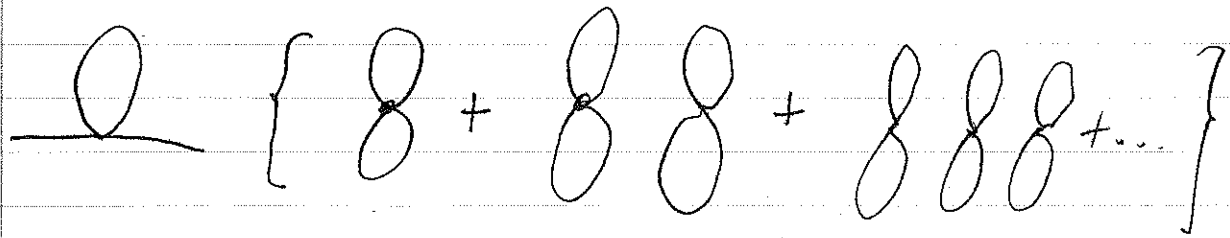
$$D_F(x-y)\left(\frac{-i\lambda}{4!}\right)\times 3\times\int d^4z\, D_F(0)^2$$

Weird; proportional to space-time volume, $VT \to \infty$.

In fact, these graphs exponentiate. Can think of them as representing $e^{-iE_0 VT}$

E_0: vacuum energy density

Let's see how this works for diagrams of this type



$$\frac{1}{(n+1)!}\,\phi(x)\,\phi(y)\int \frac{d^4y_1\,(-i\lambda)}{4!}\,\phi^4(y_1)\int\frac{d^4y_2}{4!}(-i\lambda)\,\phi^4(y_2)$$

...

Choose vertex to contract $\phi(x)$, $\phi(y)$ with.

$n+1$ ways to do this.

In rest, contract $\underbrace{\phi(y_i)\,\phi(y_i)}\,\underbrace{\phi(y_i)\,\phi(y_i)}$

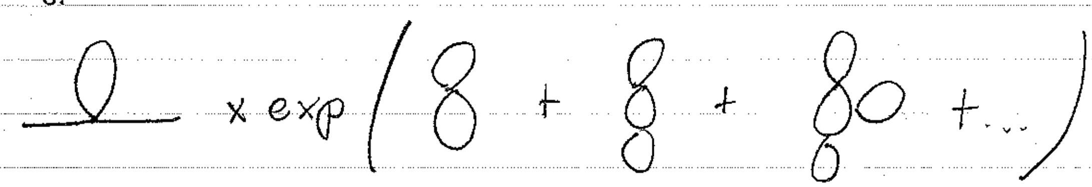
(3 ways).

So obtain

$$\phi(x)\,\phi(y)\left(\frac{-i\lambda}{4!}\right)\int d^4z\;\phi(z)\phi(z)\,\phi(z)\phi(z)$$

$$\times\left(\sum \frac{1}{n!}\,\frac{(-i\lambda)^n}{(4!)^n}\left(\int d^4y_1\,\underbrace{\phi(y_1)\,\phi(y_1)}\,\underbrace{\phi(y_1)\,\phi(y_1)}\times 3\right)^n\right.$$

Clearly generalizes to all of the disconnected graphs, i.e.



But the denominator of our expression for the Green's function, which so far we've been ignoring, has precisely the same form:

$$\langle 0| T \exp\left(-i\int d^4x\, \mathcal{H}_I(x)\right)|0\rangle$$

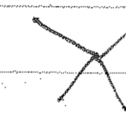
It is for this reason we can throw away the vacuum graphs.

We can summarize this by a set of Feynman rules.

(1) Draw all diagrams connected to the external lines, with n vertices at order n.

(2) Label each line with a momentum so that momentum is conserved at each vertex.

(3) Associate a factor $(-i\lambda)$ with each vertex.



$(-i\lambda)$

(4) Associate $\dfrac{i}{p^2-m^2+i\epsilon}$ with each propagator

$\dfrac{i}{p^2-m^2+i\epsilon}$

(5) Include appropriate symmetry factor

S-matrix

Now we calculate a cross section, using our formulas.

We want to obtain the matrix element $\mathcal{M}$, in

$$d\sigma = \frac{1}{2E_A 2E_B |v_A - v_B|} \int_{\Delta p_i} \prod \left(\frac{d^3 p_i}{(2\pi)^3} \frac{1}{2E_{p_i}} \right)$$

$$\times \left| \mathcal{M}(k_A + k_B \to \{p_i\}) \right|^2 (2\pi)^4 \delta^{(4)}(k_A + k_B - \Sigma p_i)$$

Goal: reduce the computation of $\mathcal{M}$ to a problem of Feynman diagrams.

Subtleties:

(i) In quantum mechanics, one puts in the $i\epsilon$'s by imagining that the interaction turns off in the far past and future.

In field theory, the initial, final states have "self interactions." We will ignore this for now, but will encounter it later.

(ii) As defined, the amplitude has $e^{-iE_0 T}$.

We get rid of this by dividing by $\langle \Omega | \Omega \rangle$, as in correlation functions. This will again get rid of disconnected parts.

So we study:

$$\langle p_1 \cdots p_n | S | k_A k_B \rangle = \lim_{T \to \infty(1-i\epsilon)} \langle p_1 \cdots p_n | T \exp\left(-i \int_{-T}^{T} dt \, H_I(t)\right) \times | k_A k_B \rangle$$

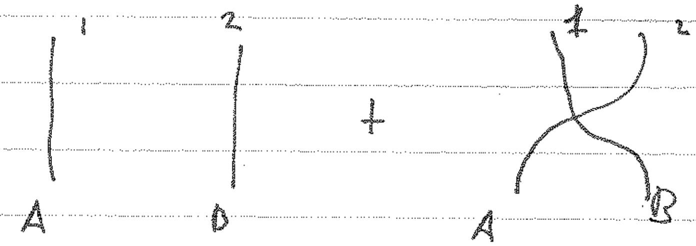
where we keep only connected parts.

Let's study the first few terms in the expansion for the case $2 \to 2$. The leading (1) term is:

$$_{out}\langle p_1 p_2 | k_A k_B \rangle_{in} = \sqrt{2E_1 2E_2 2E_A 2E_B}\, \langle 0 | a_1 a_2 a_A^\dagger a_B^\dagger | 0 \rangle$$

$$= 2E_A 2E_B (2\pi)^6 \left[\delta^{(3)}(k_A - p_1)\, \delta^{(3)}(\vec{k}_B - \vec{p}_2) + (1 \leftrightarrow 2)\right]$$

This corresponds to no scattering. Diagramatically,



(note Bose statistics here).

Now lets take the $\mathcal{O}(\lambda)$ term:

$$\langle p_1 p_2 | T\left(-i\frac{\lambda}{4!}\int d^4x\, \phi_I^4(x)\right) | k_A k_B \rangle$$

Use Wick's theorm:

$$T\,\phi_I^4 = N(\phi\phi\phi\phi) + 4\times 3\, \underbrace{\phi\phi}\phi\phi + 3\,\underbrace{\phi\phi}\,\underbrace{\phi\phi}$$

Consider the first term, with no contractions. Move a's ($\phi^{(+)}$) to right, $\phi^{(-)}$ to left, to annihilate vacuum. E.g.

$$[\phi^{(+)}(x), a^\dagger(k_A)] = (2\pi)^3 \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}\, e^{-ip\cdot x}\, \delta^{(3)}(p - k_A)$$

$$-\frac{i\lambda}{4!}\int d^4x\, e^{-i(q_3+q_4-q_1-q_2)\cdot x}\, \frac{d^3q_1 \cdots d^3q_4}{\sqrt{2E_{q_1}}\cdots\sqrt{2E_{q_4}}} \left[\delta(q_1 - k_A)\delta(q_2 - k_B) \times \delta(q_3 - p_1)\delta(q_3 - p_2) + \ldots\right]$$

$$\times \sqrt{E_{k_A} E_{k_B} E_{p_1} E_{p_2}}$$

So, in the present case, we obtain

$$-i\lambda\,(2\pi)^4\,\delta(k_A+k_B-p_1-p_2)$$

So the amplitude is very simple. The cross section is:

$$\frac{1}{|v_A-v_B|}\int \frac{d^3p_1\,d^3p_2}{2E_1 2E_2 2E_A 2E_B\,(2\pi)^6}\,(2\pi)^4\delta(k_A+k_B-p_1-p_2)\,|\lambda|^2$$

To do the integrals: $\int d^3p_2$ from $\delta^{(3)}$
CM $E_i = E$

$$= \lambda^2\int \frac{d^3p_1}{(2\pi)^2(2E)^4}\,\frac{\delta(2E-2E_p)}{|v_A-v_B|}$$

$$E_p = \sqrt{p^2+m^2} \qquad dE_p = dp\,\frac{p}{E} \qquad p/E = v$$

So

$$= \frac{\lambda^2}{(2\pi)^2}\int\frac{d\Omega}{16E^4}\,\cancel{dp}\,dE\left(\frac{E}{p}\right)p^2\,\frac{1}{2}\,\frac{\delta(E-E_p)}{2v}$$

cm $v_A = -v_B$

$$= \frac{\lambda^2}{4\pi^2}\int\frac{d\Omega}{64E^2}$$

$$\frac{d\sigma}{d\Omega} = \frac{\lambda^2}{64\pi^2 E_{cm}^2}$$

Let us give a diagrammatic interpretation of what we have done.

We can introduce the notion of contraction of operators

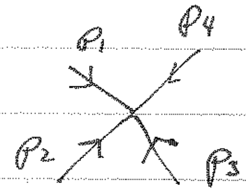
with an external state.

$$\langle p_1\, p_2 \,|\, N(\phi\,\phi\,\phi\,\phi) \,|\, k_A\, k_B \rangle$$

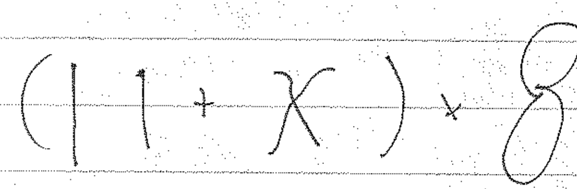
The rule associated with contraction with an

external state: $\underline{1}$.

Vertex:

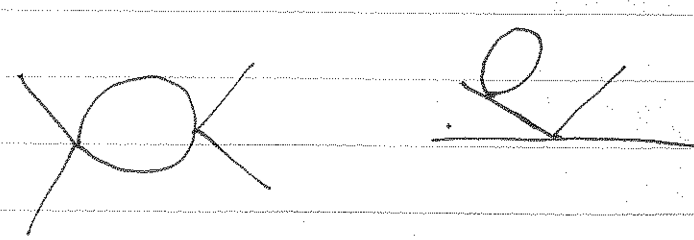


Other diagrams:



: disconnected

Next order:



We can summarize the rules as follows.

With n external lines,

(i) for each propagator, include a factor

$$\frac{i}{p^2 - m^2 + i\epsilon}$$

(ii) for each vertex: $(-i\lambda)(2\pi)^4 \delta^{(4)}(\Sigma p_i)$

(iii) For each external line, $\xleftarrow[p]{\quad} = 1.$

(iv) Divide by symmetry factor.

Instead of including momentum δ-functions, can state rules as:

At order $(\lambda)^n$, draw all distinct diagrams with n vertices. Label external lines with their momenta, internal lines so as to conserve momentum, integrate over all unconstrained momenta.