Feynman Diagrams for Fermions and Bosons: Higgs Decay and QED

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Higgs decay to $f\,\bar{f}$

This is a simple and important application of these techniques. We'll consider a fermion, which we will call ψ; most important will be b and τ.

We have

$$\mathcal{L} = y\phi\bar{\psi}\psi. \tag{1}$$

Here $y = m_f/v$, where v is related to the Fermi constant, and is approximately 250 GeV.

We are interested in $\mathcal{M}(\phi \to f\bar{f})$.

$$\langle 0|a(p_1, s_1)b(p_2, s_2)\int d^4zy\phi(z)\bar{\psi}(z)a^\dagger(q)|0\rangle. \tag{2}$$

(We have not indicated the factors of $\sqrt{2E}$ for each external state).

Now we contract the fields with the external states. Each gives us a factor of $\frac{1}{\sqrt{2E}}$, canceling $\sqrt{2E}$ from the external states. So do $(2\pi)^3$ factors. The exponentials give $(2\pi)^4\delta(q - p_1 - p_2)$ so we have

$$\mathcal{M} = iy\bar{u}(p_1, s_1)v(p_2, s_2). \tag{3}$$

Now we square; we assume the experiment doesn't distinguish the different spins. So we square the amplitude and sum over spins. Note

$$(\bar{u}(p_1, s_1)v(p_2, s_2))^* = \bar{v}(p_2, s_2)u(p_1, s_1). \tag{4}$$

So

$$\sum_{s_1, s_2} |\mathcal{M}|^2 = \mathrm{Tr}(\not{p}_1 + m)(\not{p}_2 - m) \tag{5}$$

The trace is

$$4(p_1 \cdot p_2 - m^2) \tag{6}$$

The Lifetime

$$\Gamma = \frac{1}{2M_H} \int \frac{d^3p_1 d^3p_2}{(2\pi)^6} \frac{1}{\sqrt{2E_1 2E_2}} \tag{7}$$

$$(2\pi)^4 \delta(q_1 - p_1 - p_2) y^2 4(p_1 \cdot p_2 - m^2).$$

To simplify things, neglect the fermion mass compared to M_H. Note also that $2E_1 = M_H$, and $p_1^2 = \frac{M_H^2}{4}$. Also $p_1 \cdot p_2 = \frac{1}{2} M_H^2$. So we have

$$\Gamma = \frac{y^2 M_H}{8\pi}. \tag{8}$$

We'll be anxiously awaiting results from LHC on this process for b's and τ's.

Quantum Electrodynamics

Now the interaction lagrangian is:

$$\mathcal{L} = -e\bar{\psi}\gamma^{\mu}\psi A_{\mu}. \tag{9}$$

Using the “contraction" idea, we can write the Feynman rules for the *S*-matrix in quantum electrodynamics.

Applications: Coulomb Scattering

We consider Coulomb scattering of a particle (initial and final momenta p and p' and positron (initial and final momenta k and k') The basic amplitude is

$$\mathcal{M} = i^2 e^2 \bar{u}(p', a')\gamma^\mu u(p, s) \frac{-ig^{\mu\nu}}{q^2} \bar{v}(k, \tilde{s})\gamma^\nu v(k', \tilde{s}') \qquad (10)$$

plus an annihilation diagram (we will deal with annihilation shortly; at small impact parameters the term above dominates. Now consider the non-relativistic limit. Up to terms of order $v^i = \frac{p^i}{m}$, the spinors are very simple, and only the γ^0 term contributes, giving

$$\mathcal{M} = -\frac{e^2}{\vec{q}^2} \delta_{s,s'} \delta_{\tilde{s},\tilde{s}'}. \qquad (11)$$

The Fourier transform of this amplitude is the Coulomb potential.

Applications: $e^+e^- \to \mu^+\mu^-$

This is an important process experimentally (with muons replaced by quarks it is an important part of our understanding of QCD). The amplitude is

$$\mathcal{M} = -i^3 \bar{v}(p')\gamma^\mu u(p)\bar{u}(k)\gamma_\mu v(k')\frac{1}{s} \tag{12}$$

where we have suppressed the spins and $s = (p + p')^2$ is the total center of mass energy squared. Note that in the first factor, the spinors are those for the electron; in the second for the muon. Now we average over initial spins (unpolarized beams) and sum over final spins (spin-independent measurement). Now when we square, we will encounter

$$(\bar{v}(p')\gamma^\nu u(p))^* = \bar{u}(p)\gamma^\nu v(p') \tag{13}$$

(follows from $\gamma^{\nu\dagger} = \gamma^0\gamma^\nu\gamma^0$), and similarly for the second factor.

This gives:

$$\frac{1}{4}\sum_{spins}\mathcal{M}^2 = \mathrm{Tr}[(\not{p}+m_e)\gamma^\mu(\not{p}'-m_e)\gamma^\nu]\mathrm{Tr}[(\not{k}+m_\mu)\gamma^\mu(\not{k}'-m_\mu)\gamma^\nu] \tag{14}$$

Trace formulas

To evaluate these expressions, we can use:

$$\mathrm{Tr}(1) = 4 \qquad \mathrm{Tr}(\gamma^\mu \gamma^\nu) = 4g^{\mu\nu} \tag{15}$$

$$\mathrm{Tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) = 4(g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}) \tag{16}$$

$$\mathrm{Tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_5) = 4i\epsilon^{\mu\nu\rho\sigma}. \tag{17}$$

The trace of an odd number of γ matrices vanishes.

This yields (simplifying things by neglecting the masses):

$$\frac{1}{4}|\mathcal{M}|^2 = 8[k \cdot p' k' \cdot p + k \cdot p k' \cdot p'] \frac{e^4}{s^2}. \tag{18}$$

We can easily compute the cross section in the center of mass frame. If θ is the angle between $\vec{k}$ and $\vec{p}'$,

$$p \cdot k = E^2(1 - \cos\theta) = p' \cdot k' \quad p' \cdot k = E^2(1 + \cos\theta) = p' \cdot k \tag{19}$$

And:

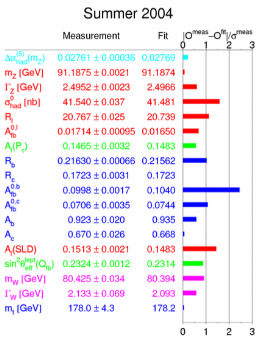
$$d\sigma = \frac{1}{2} \times 16 \int \frac{d^3k d^3k'}{(2\pi)^6 (2E)^4} (2\pi)^4 \delta(p + p' - k - k') E^4 (1 + \cos^2\theta) \frac{e^4}{s^2} \tag{20}$$

Doing the integrals over the δ-functions, using $s = 4E^2 = E_{cm}^2$ (the $\vec{k}'$ integral just enforces the usual momentum conditions of the center of mass frame)

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_{cm}^2}(1 + \cos^2\theta) \tag{21}$$

$$\sigma_{tot} = \int d\Omega \frac{d\sigma}{d\Omega} = \frac{4\pi\alpha^2}{3E_{cm}^2}. \tag{22}$$

Two applications. Leading computation same as above.
Measured rate, Z width.


Summer 2004
Measurement
Fit
|O^meas−O^fit|/σ^meas
0 1 2 3
Δα^(5)_had(m_Z) 0.02761 ± 0.00036 0.02769
m_Z [GeV] 91.1875 ± 0.0021 91.1874
Γ_Z [GeV] 2.4952 ± 0.0023 2.4966
σ^0_had [nb] 41.540 ± 0.037 41.481
R_l 20.767 ± 0.025 20.739
A^0,l_fb 0.01714 ± 0.00095 0.01650
A_l(P_τ) 0.1465 ± 0.0032 0.1483
R_b 0.21630 ± 0.00066 0.21562
R_c 0.1723 ± 0.0031 0.1723
A^0,b_fb 0.0998 ± 0.0017 0.1040
A^0,c_fb 0.0706 ± 0.0035 0.0744
A_b 0.923 ± 0.020 0.935
A_c 0.670 ± 0.026 0.668
A_l(SLD) 0.1513 ± 0.0021 0.1483
sin²θ^lept_eff(Q_fb) 0.2324 ± 0.0012 0.2314
m_W [GeV] 80.425 ± 0.034 80.394
Γ_W [GeV] 2.133 ± 0.069 2.093
m_t [GeV] 178.0 ± 4.3 178.2
0 1 2 3

Compton Scattering

This is slightly more complicated than the electron-muon scattering examples because we now have two diagrams and interference. At this point we are adept at writing down the scattering amplitudes upon examination of the diagrams. Calling the initial electron and photon momenta p and k, and the final momenta p' and k', and the initial and final photon polarizations $\epsilon(k)$ and $\epsilon(k')$, we have

$$\mathcal{M} = (-ie)^2 \tag{23}$$

$$\times \bar{u}(p')\left[\gamma^\mu i\frac{\not{p}+\not{k}+m}{(p+k)^2-m^2}\gamma^\nu - \gamma^\nu i\frac{\not{p}-\not{k}'+m}{(p-k')^2-m^2}\gamma^\mu\right]u(p)\epsilon(k)^\mu\epsilon(k')^\nu.$$

Before squaring, it is useful to simplify this expression. In the denominators, we can use $k^2 = k'^2 = 0$, $p^2 = p'^2 = m^2$, while in the numerators we can anticommute the $\not{p}$ factors with the γ matrices and use $\not{p}u(p) = mu(p)$, to write:

$$\mathcal{M} = (-ie)^2\bar{u}(p')\left[\frac{2p^\nu\gamma^\mu + \gamma^\mu\not{k}\gamma^\nu}{2p\cdot k} + \frac{2p^\mu\gamma^\nu + \gamma^\nu\not{k}'\gamma^\mu}{(-2p\cdot k')}\right]u(p)\epsilon(k)^\mu\epsilon(k')^\nu. \tag{24}$$

Now we consider the spin-averaged and spin-summed expression. As in the simple cases we have considered up to now, the effect of taking the absolute square leads us to simple expressions. It is important to introduce additional dummy indices for the sums over the polarization vectors, and to use the rule for the sum over polarizations we have derived above. This gives

$$\frac{1}{4}\sum_{s,s';\lambda,\lambda'}|M|^2 = \frac{A}{(p\cdot k)^2} + \frac{B+C}{(p\cdot k)(-p\cdot k')} + \frac{D}{p\cdot (k')^2.} \tag{25}$$

Here

$$A = \mathrm{Tr}\left[(\not{p}' + m)(2p^\nu\gamma^\mu + \gamma^\mu \not{k}\gamma^\nu)(\not{p} + m)(2p_\nu\gamma_\mu + \gamma_\mu \not{k}\gamma_\nu)\right] \tag{26}$$

$$B = \mathrm{Tr}\left[(\not{p}' + m)(2p^\nu\gamma^\mu + \gamma^\mu \not{k}\gamma^\nu)(\not{p} + m)(2p_m u\gamma_\nu - \gamma_\nu \not{k}\gamma_\mu)\right] \tag{27}$$

C and D are quite similar; in fact, it is a simple exercise to show that $B = C$, $A(k) = D(k')$.

Here we have traces of up to eight gamma matrices, but the identities we have proven are adequate to evaluate all of them. Consider, for example, A. The term with 8 γ matrices simplifies immediately due to the following identity:

$$\gamma^\nu \gamma_\nu = -2 \tag{28}$$

which follows from

$$a_\rho \gamma^\nu \gamma^\rho \gamma_\nu = a_\rho (2\gamma^\nu g^\rho_\nu - \gamma^\nu \gamma_\nu \gamma^\rho). \tag{29}$$

Then (the braces below indicate traces)

$$[\not{p}' \gamma^\mu \not{k} \gamma^\nu \not{p} \gamma_\nu \not{k} \gamma_\mu] = [-2 \not{p}' \gamma^\mu \not{k} \not{p} \not{k} \gamma_\mu] \tag{30}$$

$$= [4 \not{p}' \not{k} \not{p} \not{k}]$$

which can be evaluated using our earlier identities.

One finds

$$A = 16(4m^4 - 2m^2 p \cdot p' + 4m^2 p \cdot k - 2m^2 p' \cdot k + 2p \cdot kp' \cdot k). \quad (31)$$

and working through all four terms:

$$\frac{1}{4}\sum |\mathcal{M}|^2 \quad (32)$$

$$= 2e^4 \left[\frac{p \cdot k'}{p \cdot k} + \frac{p \cdot k}{p \cdot k'} + 2m^2 \left(\frac{1}{p \cdot k} - \frac{1}{p \cdot k'} \right) + m^4 \left(\frac{1}{p \cdot k} - \frac{1}{p \cdot k'} \right)^2 \right].$$

Now we can work out the cross section in various frames.

Compton Scattering in the Lab Frame

It is helpful to be methodical and to write the four vectors in detail

$$k = (\omega, \omega \hat{z}) \quad p = (m, \vec{0}) \quad k' = (\omega', \omega' \sin\theta, 0, \omega' \cos\theta).$$

(Note that this defines the z axis).
Then four momentum conservation allows us to solve for ω':

$$p'^2 = m^2 = (p + k - k')^2 = p^2 + 2p \cdot (k - k') - 2k \cdot k'. \tag{33}$$

Evaluating the invariants in terms of the lab frame ω, ω', and θ, this is:

$$0 = 2m(\omega - \omega') - 2\omega\omega'(1 - \cos\theta). \tag{34}$$

Solving for ω':

$$\omega' = \frac{\omega}{1 + \frac{\omega}{m}(1 - \cos\theta)}. \tag{35}$$

The invariants appearing in $|\mathcal{M}|^2$ are simple:

$$p \cdot k = m\omega; p \cdot k' = m\omega'. \tag{36}$$

Finally, we need to evaluate the phase space integral. For this we need an expression for $E_{p'}$. Starting with

$$\vec{p}' = \vec{k} - \vec{k}' \Rightarrow \vec{p}'^2 = \omega^2 + \omega'^2 - 2\omega\omega' \cos\theta \tag{37}$$

we have

$$E_{p'} = \sqrt{m^2 + \omega^2 + \omega'^2 - 2\omega\omega' \cos\theta} \tag{38}$$

In the energy conserving δ function, we have $f = \omega' + E_{p'} - \omega - m$, so

$$\frac{\partial f}{\partial \omega'} = \frac{m + \omega - \omega \cos\theta}{E_{p'}} \tag{39}$$

Klein-Nishina Expression for the Compton Cross Section

So

$$d\sigma = \frac{d^3k' d^3p'}{(2\pi)^6(2\omega')2E'_p}(2\pi)^4\delta^{(4)}(k'+p'-k-p)|\mathcal{M}|^2 \qquad (40)$$

$$= \frac{\omega'^2 d\omega' d\Omega_{k'}}{(2\pi)^2 4\omega' E_{p'}}\delta(\omega' + E'_p - \omega - m)|\mathcal{M}|^2.$$

Using our expressions above

$$d\sigma = \frac{1}{8\pi}\int \frac{d\cos\theta}{2m2\omega}\frac{\omega'}{m(1+\frac{\omega}{m}(1-\cos\theta))}\frac{1}{4}\sum|\mathcal{M}|^2. \qquad (41)$$

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{m^2}\left(\frac{\omega'}{\omega}\right)^2\left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - \sin^2\theta\right]. \quad (42)$$

At low frequencies:

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{m^2}(1 + \cos^2\theta). \quad (43)$$

$$\sigma_{tot} = \frac{8\pi\alpha^2}{3m^2}. \quad (44)$$

This is the same as the Thompson formula we derive in E and M.

The High Energy Limit

Here we will uncover an interesting feature. Work in the center of mass frame; take

$$k = (\omega, 0, 0, \omega);\ p = (E, -\omega\hat{z})\ p' = (E, -\omega \sin \sin\theta, 0, -\omega\cos\theta).$$

We have all of the ingredients we need to compute the cross section:

$$\frac{d\sigma}{d\cos\theta} \approx \frac{2\pi\alpha^2}{2m^2 + s(1+\cos\theta)} \tag{45}$$

where $s = m^2 + 2p \cdot k$. The total cross section is:

$$\sigma_{tot} \approx \frac{2\pi\alpha^2}{s} \ln(s/m^2). \tag{46}$$

Colinear Singularities

Why the singularity as $m \to 0$. Should be able to see a problem if set $m = 0$ from the start problem comes when $2p \cdot k$ or $2p \cdot k' = 0$. Corresponds to p along k or k'. The precise form of the singularity requires understanding the behavior of the spinors $u(p)$ (see Peskin and Schroder).