# Loop Corrections: Radiative Corrections, Renormalization and All That 

# Physics 217 2012, Quantum Field Theory 

Michael Dine
Department of Physics
University of California, Santa Cruz

Nov 2012

## Loop Corrections in $\phi^{4}$ Theory

At tree level, we had

$$
\begin{equation*}
\mathcal{M}\left(p_{1}+p_{2} \rightarrow k_{1}+k_{2}\right)=\lambda \tag{1}
\end{equation*}
$$

from the first diagram below. At next order in $\lambda$, we have the diagrams on the next slide, plus corrections to the external lines.

$$
\begin{gathered}
\mathcal{M}^{(2)}=\mathcal{M}(s)+\mathcal{M}(t)+\mathcal{M}(u) \\
\mathcal{M}(t)=\lambda^{2} \int \frac{d^{4} q}{(2 \pi)^{4}} \frac{1}{\left(p_{1}-k_{1}+q\right)^{2}-m^{2}+i \epsilon} \\
\times \frac{1}{q^{2}-m^{2}+i \epsilon}
\end{gathered}
$$

$\mathcal{M}(s)$ is obtained from the above by replacing $p_{1}-k_{1}$ with $p_{1}+p_{2} ; \mathcal{M}(u)$ by replacing with $p_{1}-k_{2}$.

We could start to evaluate this integral by doing the $q_{0}$ integral, picking up the residues of the various poles, and then doing the $\vec{q}$ integral. The result would be quite messy, essentially intractable. Instead, we first simplify the expression using a trick due to Feynman. We start by noting:

$$
\begin{equation*}
\frac{1}{A B}=\int_{0}^{1} \frac{d \alpha}{(A \alpha+B(1-\alpha))^{2}} \tag{4}
\end{equation*}
$$

This generalizes to (prove by induction)

$$
\begin{equation*}
\frac{1}{\prod_{i=1}^{n} A_{i}}=\int \frac{d \alpha_{1} \ldots d \alpha_{n} \delta\left(1-\sum \alpha_{i}\right)}{\left(\sum A_{i} \alpha_{i}\right)^{n}} . \tag{5}
\end{equation*}
$$

This us allows us to write

$$
\begin{equation*}
\mathcal{M}(t)=\int \frac{d^{4} q}{(2 \pi)^{4}} d \alpha \frac{1}{\left(q^{2}+2 q \cdot\left(p_{1}-k_{1}\right) \alpha+t \alpha-m^{2}+i \epsilon\right)^{2}} \tag{6}
\end{equation*}
$$

$t$ is the usual Mandelstam variable,

$$
\begin{equation*}
t=\left(k_{1}-p_{1}\right)^{2} \tag{7}
\end{equation*}
$$

Shifting $q \rightarrow q-\left(p_{1}-k_{1}\right) \alpha$ eliminates the cross term in the denominator, leaving:

$$
\begin{equation*}
\mathcal{M}(t)=\int \frac{d^{4} q}{(2 \pi)^{4}} d \alpha \frac{1}{\left(q^{2}+t \alpha(1-\alpha)-m^{2}+i \epsilon\right)^{2}} \tag{8}
\end{equation*}
$$

Now we might proceed by doing the $q^{0}$ integral. There are poles at

$$
\begin{equation*}
q^{0}= \pm \sqrt{\vec{q}^{2}+m^{2}-t \alpha(1-\alpha)} \mp \epsilon . \tag{9}
\end{equation*}
$$

We can, for example, close above. We need to remember the rule for the residue of a second order pole.

## Wick rotation

But there is a simpler way to proceed. Note that $t<0$. So, as indicated in the figure below, we can consider the $q^{0}$ integral as an integral in the complex plane, and deform the contour so as to run along the imaginary axis, avoiding both poles. This decoration is referred to as a "Wick rotation", and leaves us with an ordinary Euclidean integral (in four dimensions)

$$
\begin{equation*}
\mathcal{M}(t)=\int \frac{d^{4} q}{(2 \pi)^{4}} d \alpha \frac{1}{\left(q^{2}+|t| \alpha(1-\alpha)+m^{2}\right)^{2}} \tag{10}
\end{equation*}
$$

Now we encounter another challenge: this integral is divergent for large $|q|$. If we just cut off the integral at $q=\Lambda$, the result will be proportional to $\log (\Lambda)$.

The integral, in other words, is sensitive to physics at arbitrarily high energy scale, physics we don't know. This is what is referred to as an "ultraviolet divergence". For scales high compared to the momentum of interest, the result becomes essentially independent of the momentum; in coordinate space, this means (thinking of Green's functions) that it looks like a $\delta$ function, i.e. it is local. In other words, it looks like a (correction to) a local vertex, the original four scalar interaction.

The effect of high energy physics is to correct the original, "bare" vertex. The physical ("renormalized") coupling is the sum of these two couplings,

$$
\begin{equation*}
\lambda_{r e n}=\lambda_{0}\left(1+c \lambda_{0} \log (\Lambda)\right) \tag{11}
\end{equation*}
$$

for some constant $c$.

## Regularization

Simply cutting off the integral is rather disturbing, and it is not clear, in any case, how we would extend this procedure to higher orders. Instead, we introduce a trick, due to 't Hooft and Veltman, known as dimensional regularization (other standard regularization procedures, which we will not have time to explore, are Pauli-Villars and lattice regularization; both are distinctly more cumbersome, but are more effective for certain problems and have some conceptual advantages).
The idea is to evaluate the Feynman integrals in $d$ dimensions. The resulting expressions, as we will see, are analytic functions of $d$, and make sense for $d=4-\epsilon$. The singularity in the four dimensional integral will reflect itself as a pole in the amplitude as a function of $\epsilon$.

We need to evaluate integrals of the form

$$
\begin{equation*}
I_{n}=\int \frac{d^{d} q}{\left(q^{2}+\Delta\right)^{n}} . \tag{12}
\end{equation*}
$$

We can break up the integral as an integral over the $d$ dimensional solid angle and and integral over $q$ :

$$
\begin{equation*}
I_{n}=\int d \Omega_{d} \int_{0}^{\infty} \frac{d q q^{d-1}}{\left(q^{2}+\Delta\right)^{n}} . \tag{13}
\end{equation*}
$$

We'll do each separately.

## The $d$ dimensional solid angle

Here we proceed by the following simple trick, familiar from the usual evaluation of Gaussian integrals:

$$
\begin{equation*}
\int d^{d} q e^{-\vec{q}^{2}}=\int q_{1} \ldots d q_{n} e^{-\sum q_{i}^{2}}=\sqrt{\pi}^{d / 2} . \tag{14}
\end{equation*}
$$

Alternatively, this can be evaluated as

$$
\begin{gather*}
\int d \Omega_{d} \int d q q^{d-1} e^{-q^{2}}  \tag{15}\\
=\frac{1}{2} \int d \Omega_{d} \int_{0}^{\infty} d t t^{d / 2-1} e^{-t} \\
=\frac{1}{2} \int d \Omega_{d} \Gamma\left(\frac{d}{2}\right)
\end{gather*}
$$

So

$$
\begin{equation*}
\int d \Omega_{d}=\frac{2 \pi^{d / 2}}{\Gamma\left(\frac{d}{2}\right)} . \tag{16}
\end{equation*}
$$

Noting that $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$, and therefore $\Gamma\left(\frac{3}{2}\right)=\frac{\sqrt{\pi}}{2}$, you can check various special cases ( $d=2,3$ ).

We still need

$$
J_{n}=\int d q \frac{q^{d-1}}{\left(q^{2}+\Delta\right)^{n}}
$$

This can be done by tricks as in the text, or you can just use Mathematica (I checked that this one is a little hard for Wolfram Alpha):

$$
J_{n}=\frac{\Gamma\left(\frac{d}{2}\right) \Gamma\left(n-\frac{d}{2}\right)}{\Gamma(n)} .
$$

Putting this all together,

$$
\begin{equation*}
\int \frac{d^{d} q}{(2 \pi)^{d}} \frac{1}{\left(q^{2}+\Delta\right)^{n}}=\frac{1}{(4 \pi)^{\frac{d}{2}}} \frac{\Gamma\left(n-\frac{d}{2}\right)}{\Gamma(n)} \Delta^{\frac{d}{2}-n} \tag{17}
\end{equation*}
$$

We can use this to evaluate

$$
\begin{equation*}
\mathcal{M}(t)=\frac{\lambda^{2}}{(4 \pi)^{\frac{d}{2}}} \Gamma\left(\frac{\epsilon}{2}\right) \int d \alpha\left(m^{2}+|t| \alpha(1-\alpha)\right)^{\frac{d}{2}-2} \tag{18}
\end{equation*}
$$

For very small $\epsilon$, this term is singular $\left(\Gamma\left(\frac{\epsilon}{2}\right) \approx \frac{2}{\epsilon}\right)$. So

$$
\begin{equation*}
\mathcal{M}(t) \approx \frac{\lambda^{2}}{16 \pi^{2}} \frac{2}{\epsilon} \tag{19}
\end{equation*}
$$

Note that $\frac{2}{\epsilon} \leftrightarrow \log \left(\Lambda^{2}\right)$.

So the renormalized coupling is (there is a similar divergence/infinity from each of the three diagrams):

$$
\begin{equation*}
\lambda_{\text {ren }}=\lambda_{0}\left(1+3 \frac{\lambda_{0}}{16 \pi^{2}} \frac{2}{\epsilon}\right) \tag{20}
\end{equation*}
$$

## Other corrections in scalar field theory

Let's consider the correction to the propagator in a $\phi^{3}$ field theory: $\mathcal{L}_{1}=\frac{\Gamma}{3!} \phi^{3}$. There is a one loop correction to the two-point function (two point Green's function), indicated in the figure. We study the "truncated" Green's function, with the external legs (propagators) removed. This object we call $-i \Sigma(p)$ :

$$
\begin{equation*}
-i \Sigma(p)=\frac{\Gamma^{2}}{2} \int \frac{d^{d} q}{(2 \pi)^{d}} \frac{1}{q^{2}-m^{2}+i \epsilon} \frac{1}{(q+p)^{2}-m^{2}+i \epsilon} \tag{21}
\end{equation*}
$$

Combining denominators in the way which is becoming familiar,

$$
\begin{equation*}
-i \Sigma(p)=\frac{\Gamma^{2}}{2} \int \frac{d^{d} q}{(2 \pi)^{d}} d \alpha \frac{1}{\left(q^{2}+p^{2} \alpha(1-\alpha)-m^{2}+i \epsilon\right)^{2}} \tag{22}
\end{equation*}
$$

Wick rotating (taking $p^{2}<0$ ), and then performing the integral using our integral table:

$$
\begin{equation*}
\Sigma(p)=-\frac{\Gamma^{2}}{2} \int d \alpha \frac{\Gamma(2-d / 2)}{(4 \pi)^{d / 2}}\left(m^{2}-p^{2} \alpha(1-\alpha)\right)^{d / 2-2} . \tag{23}
\end{equation*}
$$

Expanding, and setting $p^{2}=m^{2}$ (in general, it turns out we are interested in this Green's function for all $p^{2}$, but we can start with this)

$$
\begin{equation*}
\Sigma(p)=-\frac{\Gamma^{2}}{32 \pi^{2}} \frac{2}{\epsilon}+\text { constant } . \tag{24}
\end{equation*}
$$

## Correction to the Mass

Starting with $\Sigma$, we can consider the corrections to the propagator. We can build up the propagator in powers of $\Sigma$ :

$$
\begin{gather*}
G(p)=\frac{i}{p^{2}-m^{2}}\left[-i \Sigma(p) \frac{i}{p^{2}-m^{2}}\right.  \tag{25}\\
\left.+(-i \Sigma(p)) \frac{i}{p^{2}-m^{2}}(-i \Sigma(p)) \frac{i}{p^{2}-m^{2}}+\ldots\right]
\end{gather*}
$$

This is a geometric series, which sums to

$$
\begin{equation*}
G=\frac{i}{p^{2}-m^{2}-\Sigma(p)} \tag{26}
\end{equation*}
$$

So we can think of $\Sigma\left(p^{2}=m^{2}\right)$ as a shift in the physical mass. It is divergent; but $m_{r}^{2}=m_{0}^{2}-\Gamma^{2} \frac{2}{\epsilon}$ we can define to be the physical (renormalized) mass, i.e. the parameter which is measured in experiments.

## Physical Meaning of Wave Function and Mass Renormalization: The Spectral Representation

We will focus on scalar field theories, to avoid writing lots of indices, and consider, in the interacting theory, the Green's function:

$$
\begin{equation*}
G(x-y)=T\langle\Omega| \phi(x) \phi(y)|\Omega\rangle . \tag{27}
\end{equation*}
$$

Let's consider one particular time ordering, $x^{0}>y^{0}$, and introduce a complete set of states. which we take to be energy eigenvalues.
These states can be labeled by their total energy-momentum, $p$, and some other quantum numbers, $n$. In other words:

$$
\begin{equation*}
G(x-y)=\int \frac{d^{3} p}{2 E(p)}\langle\Omega| \phi(x)|n, p\rangle\langle n, p| \phi(y)|\Omega\rangle \tag{28}
\end{equation*}
$$

Now we use translation invariance to rewrite this as:

$$
\begin{equation*}
\left.G(x-y)=\int \frac{d^{3} p}{2 E(p)}|\langle\Omega| \phi(0)| p, n\right\rangle\left.\right|^{2} e^{-i p \cdot(x-y)} \tag{29}
\end{equation*}
$$

Now we separate off states of definite mass $\left(\sqrt{E(p)^{2}-p^{2}}=M^{2}\right)$. We define

$$
\begin{equation*}
\left.\rho\left(M^{2}\right)=\delta\left(p^{2}-M^{2}\right)|\langle p, n| \phi(0)| \Omega\right\rangle\left.\right|^{2} . \tag{30}
\end{equation*}
$$

Then we have, including the other time ordering, and noting the connection to the free propagator:

$$
\begin{equation*}
G(x-y)=\int d M^{2} \rho\left(M^{2}\right) D_{F}(x-y ; M) \tag{31}
\end{equation*}
$$

One can immediately Fourier transform this expression. In simple field theories, $\rho\left(M^{2}\right)$ includes a $\delta$ function (at the mass of the meson) and a continuum (e.g. starting at $9 M^{2}$ in the case of the $\phi^{4}$ theory).

One writes:

$$
\begin{equation*}
\rho\left(M^{2}\right)=Z \delta\left(M^{2}-m^{2}\right)+f_{\text {cont }}\left(M^{2}\right) \tag{32}
\end{equation*}
$$

$m^{2}$ is the actual mass of the physical state, by our construction.
This is known as the "spectral representation", or the "Kallen-Lehman representation".
So we confirm our interpretation of $m_{0}^{2}+\delta m^{2}$ as the physical mass. We will discuss the factor $Z$ when we discuss loop corrections in quantum electrodynamics.

## The LSZ Formula

So far, we have dealt with the $S$ matrix in a rather ad-hoc way. There is a more systematic approach, which relates the $S$ matrix to Green's functions. This also illustrates the role of truncated Green's functions. The basic idea is that for an $S$ matrix, say, for $k_{1}+k_{2} \rightarrow k_{3}+\ldots k_{n}$, one studies the Green's function: $G^{n+2}\left(k_{1}, \ldots, k_{n}\right)$. In the limit that the momenta go in shell, this behaves as
$G^{n+2}\left(k_{1}, \ldots, k_{n}\right) \rightarrow \sqrt{Z^{n+2}} \prod \frac{1}{k_{i}^{2}-m^{2}+i \epsilon} \mathcal{M}\left(k_{1}+k_{2} \rightarrow k_{3}+\ldots k_{n}\right)$.
(33)

It is easy to see that this is true of the simple examples we have studied. Let's prove it in general (we will be somewhat schematic here; Peskin and Schroder give a more detailed proof).

LSZ has other virtues. Most important, it is not a statement based on perturbation theory. It applies to any operator with matrix elements between the ground state and the single particle state of interest. So it can be used, for example, in strongly coupled theories like QCD.

First, we review the idea of in and out states; this is already a notion which appears in quantum mechanical discussions of scattering (though often not in your first quantum course).

Define the operator:

$$
\begin{equation*}
S=U(\infty,-\infty) \tag{34}
\end{equation*}
$$

( $U$ is the full time development operator). Then

$$
\begin{gather*}
\left\langle k_{1}, k_{2}, \ldots k_{n}\right| S\left|p_{1}, \ldots p_{n}\right\rangle  \tag{35}\\
=\left\langle k_{1}, k_{2}, \ldots k_{n}\right| e^{-i H(\infty-0)} e^{-i H(0-(-\infty))}\left|p_{1}, \ldots p_{n}\right\rangle \\
\equiv \text { out }\left\langle k_{1}, \ldots k_{n} \mid p_{1} \ldots p_{n}\right\rangle_{\text {in }} .
\end{gather*}
$$

For LSZ, we want to connect Green's functions, $\langle\Omega| \phi\left(x_{1}\right) \ldots \phi\left(x_{n}\right)|\Omega\rangle$, Fourier transformed, to this object. For example, Fourier transforming in $x_{1} \equiv x$, we want to study:

$$
\begin{equation*}
T \int d^{4} x e^{-i p \cdot x}\langle\Omega| \phi(x) \ldots \phi\left(x_{n}\right)|\Omega\rangle . \tag{36}
\end{equation*}
$$

Poles can only arise as one approaches infinity in the integrand. If $p_{0}>0$, one needs $x^{0} \rightarrow-\infty$, and to pick up $e^{i E x^{0}}$ from the creation of a state of energy $E \approx p^{0}$ from the vacuum. So take $x_{0} \rightarrow-\infty, \vec{x} \rightarrow \infty$. Then the time ordering is easy; $\phi(x)$ stands to the right, and we can introduce a complete set of states:

$$
\begin{equation*}
\sum_{\lambda} \int \frac{d^{3} q}{\left(2 \pi^{3}\right) 2 \omega(q)} \int d^{4} x e^{-i p \cdot x}\langle\Omega| \phi\left(x_{2}\right) \ldots \phi\left(x_{n}\right)\left|\lambda_{q}\right\rangle\left\langle\lambda_{q}\right| \phi(x)|\Omega\rangle . \tag{37}
\end{equation*}
$$

Here the sum indicates labels of states other than the momentum. We can study the matrix element

$$
\begin{gathered}
\left\langle\lambda_{q}\right| \phi(x)|\Omega\rangle=\left\langle\lambda_{q}\right| e^{i \vec{P} \cdot \vec{x}} e^{-i P_{0} x^{0}} e^{-i \vec{P} \cdot \vec{x}} \\
e^{i P_{0} x^{0}} \phi(x) e^{i \vec{P} \cdot \vec{x}} e^{-i P_{0} x^{0}} e^{-i \vec{P} \cdot \vec{x}} e^{i P_{0} x^{0}}|\Omega\rangle \\
=\left\langle\lambda_{q}\right| \phi(0)|\Omega\rangle e^{i q \cdot x_{1}}
\end{gathered}
$$

Now the integral over $\vec{p}$ gives a $\delta$ function, $2 \pi^{3} \delta(\vec{p}-\vec{q})$. The time integral diverges as $p_{0} \rightarrow q_{0}$. Including a suitable $i \epsilon$ in the exponent to render the integral convergent (more precisely, one can think of this as adding a small imaginary part to $p_{0}$; the $S$ matrix turns out to be analytic in momenta), one can do the $p_{0}$ integral. Combining the important factors:

$$
\begin{equation*}
\frac{1}{2 \omega(p)} \frac{1}{p_{0}-E(q)+i \epsilon}=-\frac{1}{p^{2}+m^{2}-i \epsilon} \tag{39}
\end{equation*}
$$

implementing the delta function.

So far, this is quite similar to the manipulations used to derive the Kallen-Lehman representation. Now one repeats this process for the other fields. Here we have to be a bit more careful about what we take as the complete set of states. We start with asymptotic states which are well separated in space (wave packets) in the far past (or far future). Then most of the integral comes from the region where time is large and negative (or positive) and the states are still separated. The effect of the time evolution operator ( $e^{i P^{0} x^{0}}$ ) is to yield the in and out states, with a phase like that above. Repeating iteratively, we obtain the full LSZ expression.

Note, in particular, that Fourier transforming with respect to $e^{-i p_{i} \cdot x_{i}}$, with $p_{i}^{0}$ positive, the singular part of the integration corresponds to $x_{0}^{i} \rightarrow-\infty$; for $e^{i k_{i} \cdot x_{i}}$, with $k_{i}^{0}$ positive, the singular region corresponds to $x_{i}^{0} \rightarrow+\infty$. The former are incoming states, the latter outgoing states.

The inputs to the LSZ formula are unitarity, and the assumption of a gap in the spectrum. We have seen that we can understand the formula readily at the level of Feynman graphs. But the result is general. It is useful in theories like QCD, where, instead of $\phi$, one studies composite operators of quarks and gluons, which are assumed to have amplitudes to create physical (color neutral) states from vacuum. It is also a useful setup for thinking about the $S$ matrix in the path integral formalism, where Green's functions are the natural objects.

At one loop, there are three types of corrections which lead to divergences: the fermion self energy, the photon self energy (vacuum polarization), and the vertex correction. The fermion self energy is simplest, so we begin with this.

## Fermion Self Energy

This one is the easiest in some ways. We will see that there is a correction to the fermion mass, and an overall constant correction to the propagator. We might expect a linearly divergent correction to the mass. This follows by analogy to Lorentz's calculation of the self-energy of the electron, and also from dimensional analysis. But let's check. We'll work with the fermion Green's function. We'll drop the external lines; the result is called the "one-particle irreducible graph." We'll call it $-i \Sigma$, the "fermion self energy".

$$
\begin{equation*}
-i \Sigma(p)=(-i)^{2}(i)(-i) \int \frac{d^{4} k}{(2 \pi)^{4}} \gamma^{\mu} \frac{\not p+\not k+m}{(p-k)^{2}-m^{2}} \gamma_{\mu} \frac{1}{k^{2}} \tag{40}
\end{equation*}
$$

Doing this integral would seem to require introducing angles in the four dimensional space. But there is a much better trick, introduced by Feynman. Using our Feynman parameter trick we can rewrite this.

$$
\begin{equation*}
-i \Sigma(p)=2 \int \frac{d^{4} k}{(2 \pi)^{4}} \int d \alpha \frac{p+\not k-2 m}{\left[p^{2} \alpha-2 p \cdot k \alpha+k^{2}\right]^{2}} \tag{41}
\end{equation*}
$$

Now we can make one further simplification - and this is the critical one. The change of variables $k \rightarrow k-\alpha p$ gets rid of the nasty cross term in the denominator, leaving us with

$$
\begin{equation*}
-i \Sigma(p)=2 \int \frac{d^{4} k}{(2 \pi)^{4}} \int d \alpha \frac{p(1-\alpha)+\not K-2 m}{\left[k^{2}+p^{2} \alpha(1-\alpha)\right]^{2}} \tag{42}
\end{equation*}
$$

We can Wick rotate as we have done up to now:

$$
\begin{equation*}
\Sigma(p)=-2 \int \frac{d^{4} k}{(2 \pi)^{4}} \int d \alpha \frac{p(1-\alpha)+\not k-2 m}{\left[k^{2}+\alpha(1-\alpha)(-p)^{2}\right]^{2}} . \tag{43}
\end{equation*}
$$

The integral is now an ordinary integral without peculiar singularities anywhere (if $p^{2}<0$; for $p^{2}>0$, we can obtain the result by analytic continuation).

First simplification: the integral over the $k$ term in the numerator is odd, so gives zero.
First complication: the integral is not well-defined. It diverges (logarithmically) for large $k$.

Using our results for dimensional regularization, we have:

$$
\begin{gather*}
\Sigma \approx-2 \frac{e^{2}}{16 \pi^{2}} \Gamma(\epsilon / 2) \int_{0}^{1} d \alpha(\not p(1-\alpha)-2 m)  \tag{44}\\
=-\frac{2}{\epsilon} \frac{e^{2}}{16 \pi^{2}}((\not p-m)+3 m) .
\end{gather*}
$$

We can use this to correct the propagator. Calling

$$
\begin{equation*}
\Sigma(p)=\left(1-Z^{-1}\right)\left(\not p-m_{0}\right)+Z^{-1} \delta m \tag{45}
\end{equation*}
$$

we have

$$
\begin{equation*}
S_{F}(p)=i \frac{Z}{p p-(m+\delta m)} . \tag{46}
\end{equation*}
$$

So we see that there is a shift in the normalization of the propagator, and also of the physical mass. This correction to the mass is logarithmically divergent (Weiskopf).

Collecting our results:

$$
\begin{align*}
& Z^{-1}=1+\frac{2}{\epsilon} \frac{e^{2}}{16 \pi^{2}}  \tag{47}\\
& \delta m=3 m_{0} \frac{2}{\epsilon} \frac{e^{2}}{16 \pi^{2}} \tag{48}
\end{align*}
$$

Note in the $S$ matrix we will be interested in powers of

$$
\begin{equation*}
Z^{\frac{1}{2}}=1-\frac{2}{\epsilon} \frac{e^{2}}{16 \pi^{2}} \frac{1}{2} \tag{49}
\end{equation*}
$$

## The Vacuum Polarization

In $\delta m$, we have our first real example of the renormalization of a parameter. The observed, physical mass is $m+\delta m$. $\delta m$ is "infinite" (depends on the cutoff, $1 \epsilon \sim \log (\Lambda)$, but we only care about the observable quantity, the physical mass, and this is a parameter of the theory in any case (not something we can predict). We will see that the electromagnetic coupling is also renormalized.

For this, we consider the corrections to the photon propagator, focussing on the one loop expressions. This is equivalent to

$$
\begin{equation*}
T\langle\Omega| j^{\mu}(x) j^{\nu}(y)|\Omega\rangle \equiv-i \Pi^{\mu \nu}(q) \tag{50}
\end{equation*}
$$

(after Fourier transform).
The first thing to note is that since this involves conserved currents, we have

$$
\begin{equation*}
q_{\mu} \Pi^{\mu \nu}=0 \tag{51}
\end{equation*}
$$

We can write

$$
\begin{equation*}
\Pi^{\mu \nu}=\left(g^{\mu \nu} q^{2}-q^{\mu} q^{\nu}\right) \Pi\left(q^{2}\right) \tag{52}
\end{equation*}
$$

This makes the calculation easier; it is not hard to read off the $q^{\mu} q^{\nu}$ piece.
To simplify the analysis, we will take $q^{2} \gg m^{2}$. Then writing down the full diagram, it is not hard to pull out the $q^{\mu} q^{\nu}$ pieces.

Introducing Feynman parameters, combining denominators, and shifting the $k$ integral in the usual way:
$-i \Pi^{\mu \nu}=-e^{2} \int \frac{d^{4} k}{(2 \pi)^{4}} d \alpha \frac{\operatorname{Tr}\left((K+\not q(1-\alpha)+m) \gamma^{\mu}(K-\not q \alpha+m) \gamma^{\nu}\right)}{\left[k^{2}+q^{2} \alpha(1-\alpha)-m^{2}\right]^{2}}$.
(53)

The $q^{\mu} q^{\nu}$ part can only arise from the numerator piece involving

$$
\begin{equation*}
\operatorname{Tr}\left(\not q(1-\alpha) \gamma^{\mu} \not q \alpha \gamma^{\nu}\right)=8 q^{\mu} q^{\nu} \alpha(1-\alpha) \tag{54}
\end{equation*}
$$

So we have

$$
\begin{equation*}
\Pi\left(q^{2}\right)=-e^{2} \int \frac{d^{4} k}{(2 \pi)^{4}} d \alpha \frac{8 \alpha(1-\alpha)}{\left[k^{2}+q^{2} \alpha(1-\alpha)-m^{2}\right]^{2}} \tag{55}
\end{equation*}
$$

Now we consider $q^{2}<0$ (this is natural here, e.g. for scattering in the field of a nucleus). Then we can use our integral table to obtain (for $q^{2} \ll m^{2}$ ):

$$
\begin{equation*}
\Pi=\frac{4}{3} \frac{e^{2}}{16 \pi^{2}}\left(\frac{2}{\epsilon}\right)\left(1-2 \epsilon / 2 \log \left(\mu^{2} / m^{2}\right)\right) \tag{56}
\end{equation*}
$$

Finally, we need to think about one more diagram at one loop, the vertex. This introduces some new features. We will see another infinity, which cancels against the infinite wave function renormalization in the fermion self-energy. Then we will encounter an infrared divergence, which we will have to explain (we can't "renormalize" away). Finally, we will see a correction to the electron magnetic moment from its Dirac value, the famous $g-2$.

$$
\begin{equation*}
\Gamma^{\mu}\left(p, p^{\prime}, q\right)=-e^{3} \int \frac{d^{4} k}{(2 \pi)^{4}} \frac{\bar{u}\left(p^{\prime}\right) \gamma^{\nu}\left(\not p^{\prime}+\not k+m\right) \gamma^{\mu}(\not p+\not k+m) \gamma_{\nu} u(p)}{\left[\left(p^{\prime}+k\right)^{2}-m^{2}\right]\left[(p+k)^{2}-m^{2}\right] k^{2}} \tag{57}
\end{equation*}
$$

Let's look for the ultraviolet divergent part of the vertex. This comes from the term with most factors of $k$ in the numerator, So we have:

$$
\begin{equation*}
-e^{3} \int \frac{d^{4} k}{(2 \pi)^{4}} \frac{\bar{u}\left(p^{\prime}\right) \gamma^{\nu} \not K \gamma^{\mu} k \gamma_{\nu} u(p)}{\left[k^{2}+2 p^{\prime} \cdot k \alpha_{1}+2 p \cdot k \alpha_{2}-m^{2} \alpha_{12}\right]^{3}} . \tag{58}
\end{equation*}
$$

Under the integral,

$$
\begin{equation*}
k_{\rho} k_{\sigma} \gamma^{\rho} \gamma^{\mu} \gamma^{\sigma} \rightarrow-\frac{2}{4} k^{2} \gamma^{\mu} . \tag{59}
\end{equation*}
$$

The $k$ integral can be done shifting as usual, and using our integral table. The result is

$$
\begin{equation*}
i i e \bar{u}\left(p^{\prime}\right) \gamma^{\mu} u(p) \frac{1}{2}\left(\frac{e^{2}}{16 \pi^{2}} \frac{2}{\epsilon}\right) \tag{60}
\end{equation*}
$$

This is also a renormalization of the electron charge, but it cancels against a similar infinity from the self energy (Ward identity).

## Renormalization of the electric charge (e):

Consider Coulomb scattering. The infinite part of the vertex correction cancels that of the fermion self energy. This is a consequence of a "Ward identity", itself a consequence of gauge invariance. The amplitude is then proportional to:

$$
\begin{equation*}
\frac{e^{2}}{q^{2}}\left(1-\frac{4}{3} \frac{e^{2}}{16 \pi^{2}} \frac{2}{\epsilon}\left(1-2 \epsilon / 2 \log \left(\mu^{2} / m^{2}\right)\right)\right) \tag{61}
\end{equation*}
$$

Since what we call the electron charge* is the coefficient of $\frac{e^{2}}{q^{2}}$, we can simply define the "renormalized" charge

$$
\begin{equation*}
e_{R}^{2}=e^{2}\left(1-\frac{4}{3} \frac{e^{2}}{16 \pi^{2}} \frac{2}{\epsilon}\left(1-2 \epsilon / 2 \log \left(\mu^{2} / m^{2}\right)\right)\right) \tag{62}
\end{equation*}
$$

## Infrared Divergences

Now we consider the behavior of the vertex at low (virtual) photon momentum. We can neglect factors of $k$ in the numerator, and use the Dirac equation to write as:

$$
\begin{equation*}
\Gamma^{\mu}=-e^{3} \int \frac{d^{4} k}{(2 \pi)^{4}} \frac{\left(4 p \cdot p^{\prime}\right) \gamma^{\mu}}{k^{2}(2 p \cdot k)\left(2 p^{\prime} \cdot k\right)} \tag{63}
\end{equation*}
$$

Doing the $k^{0}$ integral (restoring the $i \epsilon$ ) gives:

$$
\begin{equation*}
-e^{3} \frac{2 \pi i}{(2 \pi)^{4}} \int \frac{d^{3} k}{2|k|} \frac{\left(4 p \cdot p^{\prime}\right) \gamma^{\mu}}{(2 p \cdot k)\left(2 p^{\prime} \cdot k\right)} \tag{64}
\end{equation*}
$$

## Cancelation by soft photon emission

Consider the interference of the tree graph and one loop vertex correction, and compare with the interference diagram involving photon emission before and after the virtual photon exchange. The latter is

$$
\begin{equation*}
\bar{u}\left(p^{\prime}\right) \gamma^{\nu} \frac{p^{\prime}+\not K+m}{2 p^{\prime} \cdot k} \gamma^{\mu} u(p) \bar{u}(p) \frac{\gamma^{\nu}(\not p-\not k+m) \gamma^{\rho}}{-2 p \cdot k} u(p) . \tag{65}
\end{equation*}
$$

For low $k$, using the Dirac equation, and integrating over the photon phase space (and summing over the photon polarizations) the correction to the cross section is proportional to the original squared amplitude, time the factor

$$
\begin{equation*}
-e^{2} \int \frac{d^{3} k}{2|\vec{k}|(2 \pi)^{3}} \frac{4 p \cdot p^{\prime}}{(2 p \cdot k)\left(-2 p^{\prime} \cdot k\right)} \tag{66}
\end{equation*}
$$

these have the same form up to a sign.

Now no experiment can resolve photons of arbitrarily small $\vec{k}$. So we can introduce an energy resolution, $E_{r}$. The actual divergence then cancels between the two diagrams, and we are left with a result proportional to $\log \left(E_{r} / E\right)$, where $E$ is a typical energy scale in the process. For high energies, we also get a log of the mass, as we saw in Compton scattering, from the integral over angles, The result is known as a "Sudakov double logarithm". There are actually such logs in every order of perturbation theory, and it is possible to add up these large terms (they exponentiate; see Peskin and Schroeder).

## The magnetic moment

For the magnetic moment, we look for a coupling of the form

$$
\begin{equation*}
F_{2}\left(q^{2}\right) q_{\mu} \sigma^{\mu \nu} \tag{67}
\end{equation*}
$$

Taking $q$ to have spatial components, and taking $\mu=j, \nu=k$, this is

$$
\begin{equation*}
\partial_{i} A^{i} \sigma^{k}=\vec{\sigma} \cdot \vec{B} . \tag{68}
\end{equation*}
$$

It is not hard to isolate this coupling.

Starting with our expression for the vertex:

$$
\begin{gather*}
-i \Gamma^{\mu}\left(p, p^{\prime}, q\right)=-e^{3} \int \frac{d^{4} k}{(2 \pi)^{4}}  \tag{69}\\
\times \frac{\bar{u}\left(p^{\prime}\right) \gamma^{\nu}\left(\not p^{\prime}+\not k+m\right) \gamma^{\mu}(\not p+\not k+m) \gamma_{\nu} u(p)}{\left[\left(p^{\prime}+k\right)^{2}-m^{2}\right]\left[(p+k)^{2}-m^{2}\right] k^{2}} .
\end{gather*}
$$

Take $p^{2}=p^{\prime 2}=m^{2}$, and introduce Feynman parameters. The denominator becomes

$$
\begin{equation*}
k^{2}+2 k \cdot\left(\alpha_{1} p+\alpha_{2} p^{\prime}\right) \tag{70}
\end{equation*}
$$

. Shift $k \rightarrow k-\alpha_{2} p^{\prime}-\alpha_{1} p$. Then the integral becomes:

$$
\begin{gather*}
\Gamma^{\mu}\left(p, p^{\prime}, q\right)=-2 e^{3} \int \frac{d^{4} k}{(2 \pi)^{4}} d \alpha_{1} d \alpha_{2} d \alpha_{3} \delta\left(1-\alpha_{123}\right)  \tag{71}\\
\frac{\bar{u}\left(p^{\prime}\right) \gamma^{\nu}\left(\not p^{\prime}\left(1-\alpha_{12}\right)+\not k+\not q \alpha_{1}+m\right) \gamma^{\mu}\left(\not p\left(1-\alpha_{12}\right)+\not k-\not q \alpha_{2}+m\right) \gamma_{\nu} u(p)}{\left[k^{2}-\left(p \alpha_{1}+p^{\prime} \alpha_{2}\right)^{2}\right]^{3}} .
\end{gather*}
$$

We can simplify by rewriting $p^{\prime}=p+q$ when $p^{\prime}$ is near $u(p)$, and similarly $p=p^{\prime}-1$ when $p$ is near $\bar{u}\left(p^{\prime}\right)$, and then moving the factors of $\not p$ to the right and $\not p^{\prime}$ to the left and using the Dirac equation:

Terms in the numerator containing only $k$ cannot contribute to $F_{2}$ (they can contribute to $F_{1}$, the coefficient of $\gamma^{\mu}$ ). We can rewrite the numerator using the Dirac equation:

$$
\begin{gather*}
\bar{u}\left(p^{\prime}\right)\left(2 p^{\prime \nu}\left(1-\alpha_{12}\right)+\gamma^{\nu} m \alpha_{12}+\gamma^{\rho} \not q \alpha_{1}\right) \gamma^{\mu}  \tag{72}\\
\left(2 p^{\rho}\left(1-\alpha_{12}\right)+m \gamma^{\rho} \alpha_{12}-\not q \gamma^{\rho} \alpha_{2}\right) u(p) .
\end{gather*}
$$

There are nine terms in the product. Many don't contribute to $F_{2}$. E.g.
(1) $4 p \cdot p^{\prime}\left(1-\alpha_{12}\right)^{2} \gamma^{\mu}: F_{1}$ only.
(2)

$$
\begin{gathered}
2 m \gamma^{\mu} \not \text { p }^{\prime} \alpha_{12}\left(1-\alpha_{12}\right)_{2} m \not p \alpha_{12}\left(1-\alpha_{12}\right) \gamma^{\mu} \\
=2 m \gamma^{\mu}(\not p+\not p) \alpha_{3}\left(1-\alpha_{3}\right)+2 m\left(\not p^{\prime}-\not q\right) \gamma^{\mu} \alpha_{3}\left(1-\alpha_{3}\right) \\
=2 m \alpha_{3}\left(1-\alpha_{3}\right)\left[\gamma^{\mu}, \not q\right]+F_{1} \text { term. }
\end{gathered}
$$

Combining the rest, the result is:

$$
\begin{equation*}
\Gamma^{\mu}=\cdots-2 e^{3} \int \frac{d^{4} k d \alpha_{1} d \alpha_{2} d \alpha_{3} \delta\left(1-\alpha_{123}\right) m \alpha_{3}\left(1-\alpha_{3}\right)\left[\gamma_{\mu}, q\right]}{(2 \pi)^{4}\left[k^{2}-m^{2}\left(1-\alpha_{3}\right)\right]^{3}} . \tag{73}
\end{equation*}
$$

(In the denominator we have set $q^{2}=0$.) The $k$ integral yields:

$$
\begin{gather*}
\Gamma^{\mu}=-i e^{3} \int \frac{d \alpha_{1} d \alpha_{2} d \alpha_{3} \delta\left(1-\alpha_{123}\right) 4 \alpha_{3}\left(1-\alpha_{3}\right)}{16 \pi^{2}} \frac{i q_{\nu} \sigma^{\nu \mu}}{2 m\left(1-\alpha_{3}\right)^{2}}  \tag{74}\\
=-i \frac{\alpha}{\pi} \frac{i q_{\nu} \sigma^{\nu \mu}}{2 m} .
\end{gather*}
$$

This yields

$$
\begin{equation*}
\frac{g-2}{2}=\frac{\alpha}{2 \pi}=0.0011614 \tag{75}
\end{equation*}
$$

vs. measured 0.0011596 .

