Physics 218. Quantum Field Theory. Professor Dine

Canonical Quantization of the Electromagnetic Field

1 A Tension

In a gauge theory like electromagnetism, there is a tension between two basic principles: Lorentz invariance and unitarity (unitarity is the statement that in quantum mechanics, time evolution is described by a unitary operator, as a result of which probability is conserved). If we choose Lorentz gauge:

$$\partial_\mu A^\mu = 0 \tag{1}$$

and follow our usual quantization procedure, we will be lead to write:

$$A^\mu(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}(a^\mu(p)e^{ip\cdot x} + a^{\dagger\ \mu}e^{-ip\cdot x}) \quad [a^\mu(p), a^{\nu\ \dagger}(p)] = g^{\mu\nu}(2\pi)^3\delta(\vec p - \vec p^\prime). \tag{2}$$

The problem is with the a^o commutator. The states $a^{o\dagger}|0>$ have *negative norm* (check!). Doesn't sound good for quantum mechanics. It turns out that the states with negative norm are never produced in scattering processes, but proving this is a bit involved. An alternative approach gives up manifest Lorentz invariance. One chooses the Coulomb (or "transverse" or "radiation") gauge:

$$\vec\nabla \cdot \vec A = 0 \tag{3}$$

Note that in writing this condition, we are making a choice of Lorentz frame. The expansion of the gauge field is now:

$$A^i(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}(a(p,\lambda)\epsilon^i(p,\lambda)e^{ip\cdot x} + a^\dagger(p,\lambda)\epsilon^{i\ *}(p,\lambda)e^{-ip\cdot x}). \tag{4}$$

From the gauge condition,

$$\vec p \cdot \vec\epsilon(p,\lambda) = 0. \tag{5}$$

The commutation relations of the a's are just what you might expect:

$$[a(p,\lambda), a^\dagger(p^\prime,\lambda^\prime)] = \delta_{\lambda,\lambda^\prime}(2\pi)^3\delta(\vec p - \vec p^\prime). \tag{6}$$

From these expressions we can work out the propagator. In this computation, analogous to what we saw in the Dirac case, one encounters:

$$\sum_\lambda \epsilon^i(p,\lambda)\epsilon^{*j}(p,\lambda) = P_{ij}(p) = (\delta_{ij} - \frac{p^ip^j}{\vec p^2}). \tag{7}$$

Then (**Exercise: check**):

$$T < A^i(x)A^j(y) >= \int \frac{d^4p}{(2\pi)^4}e^{ip\cdot x}\frac{iP_{ij}(\vec p)}{-p^2 + i\epsilon} \tag{8}$$

It is also natural to define a propagator for the scalar potential, remembering that propagators are just Green's functions. In momentum space, this is just

$$< A_o A_o >= i \frac{1}{\vec{p}^2}. \tag{9}$$

Not surprisingly, these propagators don't look very Lorentz invariant. But we can fix this by noting that the full propagator can be written (in momentum space, using the "west coast metric"):

$$D^{\mu\nu} = -\frac{g^{\mu\nu}}{p^2 + i\epsilon} - \frac{p^\mu p^\nu}{\vec{p}^2(p^2 + i\epsilon)} + \frac{\eta^\mu p^\nu + \eta^\nu p^\mu}{(p^2 + i\epsilon)} \tag{10}$$

where $\eta = (\frac{p^o}{\vec{p}^2}, 0, 0, 0)$ is a fixed four vector. **Exercise: Check this. Don't worry about the $i\epsilon$'s.**

Now in electrodynamics, we have seen that A^μ couples to j^μ, a conserved current. So p^μ always multiplies $j^\mu(p)$, and thus these terms vanish by current conservation. We will actually see how this works in scattering amplitudes later. As a result, we can use the covariant propagator. Note that this is the propagator one might have written in Lorentz gauge by analogy with the propagator for a scalar field.

The fact that in the end one can write manifestly Lorentz invariant Feynman rules means that the non-Lorentz invariant gauge choice doesn't matter in the end. It is possible to prove that, in Coulomb gauge, there are a nice set of operators which generate Lorentz invariance. But this is rather involved and, for the moment, not particularly instructive. In the path integral approach, one can take care of these problems, at least as far as developing perturbation theory is concerned, quite efficiently; this is particularly important for non-abelian gauge fields, which we will encounter in 218.