Physics 218. Quantum Field Theory. Professor Dine

Non-Relativistic Limit of the Dirac Equation

1 Lowest non-trivial order in v^2/c^2: The Pauli Lagrangian

We will proceed in a very straightforward way. First, it will be helpful to have an explicit representation of the Dirac matrices, or more specifically of Dirac's matrices, somewhat different than the one in your text:

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \tag{1}$$

The Dirac equation takes the form:

$$(i\not{D} - m)\psi = 0, \tag{2}$$

where

$$D_\mu = \partial_\mu + eiA_\mu \tag{3}$$

Here we will ignore the dynamics of A_μ, treating it as a fixed classical background.

It is helpful to multiply the Dirac equation by γ^0 $\vec{\alpha} = \vec{\gamma}\gamma_0$ and $\beta = \gamma^0$. Defining matrices

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \tag{4}$$

Then the equation takes the form:

$$(i\frac{\partial}{\partial t} - eA^0)\psi = i\vec{\alpha}\cdot\vec{D}\psi. \tag{5}$$

Now we want to define a *wave function* for a single electron in this background field. By analogy to the single particle wave function for a free quantum field of definite momentum:

$$\Psi = \langle 0|\phi|\vec{k}\rangle \tag{6}$$

we define here:

$$\Psi(x) = \langle 0|\psi|\Psi\rangle \tag{7}$$

This object satisfies the Dirac equation as written above. Historically, this is the object which Dirac first studied.

We will write the Dirac wave function in terms of two two-component objects, ϕ and χ, but these are no longer helicity components:

$$\begin{pmatrix} \phi \\ \chi \end{pmatrix} \tag{8}$$

We will be interested in positive energy solutions, in which case ϕ are the large components, χ the small components in the non-relativistic limit.

Another simplification arises from assuming that all fields are time-independent. For the solution of the Dirac equation in the presence of a static nucleus, this is adequate. More generally, we will allow for static magnetic fields described in terms of time-independent vector potentials.

1.1 Equation for ϕ

There are two issues we need to face in this analysis. First, we need to eliminate χ in favor of ϕ in the Dirac equation. Second, we need to determine the identification of ϕ with the Schrodinger wave function.

In our basis, note that the equations for ϕ and χ are:

$$((p^0 - eA^0 - m)\phi - \vec{\sigma} \cdot (\vec{p} - e\vec{A})\chi = 0 \tag{9}$$

and

$$((p^0 - eA^0 + m)\chi - \vec{\sigma} \cdot (\vec{p} - e\vec{A})\phi = 0. \tag{10}$$

We can solve for χ in terms of ϕ. We will first work to first order in fields. However, for the hydrogen atom problem, powers of A^0 are of order powers of $\vec{p}^2$, so there we will set $\vec{A} = 0$ and work systematically order by order both in p^2 and A^0. In the present approximation we write:

$$\chi = \frac{1}{p^0 - eA^0 + m}\vec{\sigma} \cdot (\vec{p} - e\vec{A})\phi \tag{11}$$

$$= \frac{1}{p^0 + m}\vec{\sigma} \cdot (\vec{p} - e\vec{A})\phi + \frac{1}{(p^0 + m)^2}eA^0\vec{\sigma} \cdot \vec{p}\phi$$

Now substitute back in the equation for ϕ:

$$(p^0 - eA^0 - m)\phi + \frac{1}{p^0 + m}\vec{\sigma} \cdot (\vec{p} - e\vec{A})\vec{\sigma} \cdot (\vec{p} - e\vec{A})\phi + \vec{\sigma} \cdot \vec{p}\frac{eA^0}{(p^0 + m)^2}\vec{\sigma} \cdot \vec{p}\phi + \tag{12}$$

Using the identity $\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k$, we can rewrite this expression as:

$$(p^0 + eA^0 - m)\phi + \frac{1}{p^0 + m}\left(\vec{p}^2 + e(\vec{p} \cdot \vec{A} + \vec{A} \cdot \vec{p}) + i\epsilon_{ijk}(p^iA^j + A^ip^j)\sigma^k\phi - \frac{1}{(p^0 + m)^2}\vec{\sigma} \cdot \vec{p}eA^0\vec{\sigma} \cdot \vec{p}\phi\right]. \tag{13}$$

(It will be convenient to leave the last term in this form). Now the term $i\epsilon_{ijk}(p^iA^j + A^ip^j)$, would vanish, except that p_i and A_j don't commute, and we obtain, from this term, $\epsilon_{ijk}\partial_iA^j\sigma^k = \vec{B} \cdot \vec{\sigma}$. The term involving A^0 can be rewritten as:

$$\frac{1}{(p^0 + m)^2}\vec{\sigma} \cdot \vec{p}eA^0\vec{\sigma} \cdot \vec{p} \tag{14}$$

$$= -\vec{\sigma} \cdot \vec{p}\vec{\sigma} \cdot \vec{p}\frac{eA^0}{(p^0 + m)^2} - \frac{i\hbar\vec{\sigma} \cdot \vec{p}\partial_jA^0\sigma^j}{(p^0 + m)^2}$$

$$= -\vec{p}^2\frac{3A^0}{(p^0 + m)^2} + \frac{e\hbar}{(p^0 + m)^2}(i\vec{p} \cdot \vec{E} + \vec{\sigma} \cdot (\vec{E} \times \vec{p}))$$

1.2 Normalization of ϕ

Following Baym, we argue that it is $\int d^3x\psi^\dagger\psi$ which is the preserved probability, where ψ is the usual four point function. At lowest non-trivial order, eliminating χ in terms of ϕ, we have

$$\int d^3x(\phi^\dagger\phi + \chi^\dagger\chi) = \int d^3x\phi^\dagger(1 + \frac{\vec{p}^2}{4m^2})\phi \tag{15}$$

so we identify the (two-component) Schrodinger wave function with

$$\psi_S = (1 + \frac{\vec{p}^2}{8m^2})\phi \qquad \phi = (1 - \frac{\vec{p}^2}{8m^2})\psi_S. \tag{16}$$

So the leading terms in the Dirac equation become:

$$(p^0 - eA^0 - m)\psi_S - (p^0 - m - eA^0)\frac{\vec{p}^2}{8m^2}\psi_S. \tag{17}$$

In the second term, we would like to bring the expression in parenthesis next to ψ_S, so that we can use the lowest order equation for ψ. In other words

$$(p^0 - m - eA^0)\frac{\vec{p}^2}{8m^2}\psi_S = \frac{p^2}{8m^2}(p^0 - m - eA^0)\psi_S + \frac{\nabla^2}{8m^2}(eA^0) - i\frac{\nabla_i eA^0 p_i}{4m^2}\psi_S \tag{18}$$

So the peculiar term (the last above) cancels the term we found earlier.

1.3 The Full Non-Relativistic Expression

Putting all of this together, the full equation is:

$$i\frac{\partial \Psi}{\partial t} = \left[m + \frac{1}{2m}(\vec{p} - e\vec{A})^2 - \frac{p^4}{8m^3c^2}\right]\Psi - \left[\frac{e}{2m}\vec{\sigma}\cdot\vec{B} + \frac{3}{4m^2}\vec{\sigma}\cdot(\vec{E}\times\vec{p})\right]\Psi + [eA^0 + \frac{1}{8m^2}e\vec{\nabla}^2A^0]\Psi \tag{19}$$

Using Posson's equation, the $\vec{\nabla}^2$ term can be replaced with a Delta function term, known as the Darwin term.