Physics 218. Advanced Quantum Field Theory. Professor Dine

Winter, 2009. Homework Set 1. SOLUTIONS.

Problem numbers refer to your textbook.

1. Verify eqn. 10 on the handout on Canonical Quantization of the Electromagnetic Field (correcting errors as necessary, esp. making sure conventions agree with Srednicki)

   Solution:

$$D^{\mu\nu} = \frac{-g^{\mu\nu}}{p^2 + i\epsilon} - \frac{p^\mu p^\nu}{\vec{p}^2(p^2 + i\epsilon)} + \frac{\eta^\mu p^\nu + \eta^\nu p^\mu}{(p^2 + i\epsilon)} \tag{1}$$

   Simply go component by component. For D^{00}

$$D^{00} = -\frac{1}{p^2} - \frac{p^{0\ 2}}{\vec{p}^2 p^2} + \frac{2p^{0\ 2}}{p^2 \vec{p}^2} \tag{2}$$

   while for D^{i0} we get zero, and for

$$D^{ij} = \frac{\delta^{ij}}{p^2} - \frac{p^i p^j}{\vec{p}^2 p^2} \tag{3}$$

   as expected.

2. Sketch how to modify Srednicki's derivation of the Kallen-Lehman representation in chapter 13 to apply to fermions. Don't write pages and pages; just focus on the main differences, esp. around eqn. 13.8, 13.9 (matrix elements of the fermions, esp. to one particle states). The Lorentz structure is different, but the goal is to write an eqn. like 13.17, where the first term is the ordinary single particle propagator and the second is the contribution of the multiparticle continuum. If necessary, make restrictive assumptions about the spectrum (e.g. that the only single-particle fermionic states have spin 1/2), but try to spell them out explicitly.

   Solution:

   First, a few remarks about the derivation in Sredniki. It should be stressed that the derivation applies:

   (a) To any scalar operator, including composite operators.

   (b) To interacting field theories, including strongly interacting theories, for which perturbation theory is not useful.

   So let's briefly reformulate his analysis for a (local) bosonic operator, $\mathcal{O}(x)$. We study the two point correlation function

$$\Delta(x, y) = T\langle\Omega|\mathcal{O}(x)\mathcal{O}(y)|\Omega\rangle. \tag{4}$$

   Take a particular time ordering, $x_0 > y_0$. Then introduce a complete set of states. Divide the states into three groups:

(a) The vacuum, Ω.

(b) Single particle states, assumed to have mass m.

(c) Multiparticle states, with some threshold for total mass-squared, M^2 (in simple theories, this will be $4m^2$).

To simplify, we'll assume $\langle|\mathcal{O}|\Omega\rangle = 0$. We'll label the single particle states simply by their momentum, $\vec{q}$. The multiparticle states we will label as $|\lambda_q, q\rangle$. We will need matrix elements of the form:

$$\langle\Omega|\mathcal{O}(x)|\vec{q}\rangle = \langle\Omega|\mathcal{O}(0)|\vec{q}\rangle e^{iq\cdot x} \equiv Ze^{iq\cdot x}. \tag{5}$$

Here we have used translation invariance, and I am using my metric. A similar formula holds for the states λ_q. Let's examine, first, the contribution of the single particle states to $\Delta(x-y)$. Then we have

$$\Delta(x-y) = \theta(x^0 - y^0)\int \frac{d^3k}{(2\pi^3)2E(q)} Ze^{iq\cdot x}. \tag{6}$$

Considering the other time ordering, we get for the contribution of the single particle states:

$$\Delta(x-y) = iZ\int d^4x \frac{1}{p^2 - m^2 + i\epsilon} e^{iq\cdot x}. \tag{7}$$

I.e. it has exactly the form of a single particle propagator, but with weight Z.

The contribution of the multiparticle states is identical, except that we must integrate over the quantity λ, and the matrix element is now some function of λ. This integral leads to the additional contribution to the spectral function associated with the continuum; there is no pole in this piece.

Now let's consider what happens with the correlation function for fermions. Again, we can consider single particle states and multiparticle states, and we begin with the former. First, we consider the single particle states (this time we don't have to worry about the vacuum at least!) We are interested in

$$S_{\alpha,\beta} = \langle\Omega|T(\psi_\alpha(x)\bar{\psi}_\beta(y))|\Omega\rangle \tag{8}$$

so, introducing a complete set of states, we need the single particle matrix elements

$$\langle\Omega|\bar{\psi}|q,s\rangle_\pm \quad \langle\Omega|\bar{\psi}|q,s\rangle_\pm \tag{9}$$

where we need to include spin, and where the subscript $\pm$ indicates particles or antiparticles. We will sharpen this distinction in a moment.

Note, implicit in our discussion, is the fact that the single particle states must carry spin 1/2. This follows from the Wigner-Eckart theorem. What we mean by particles and antiparticles requires some thought (again, we don't just want to limit our considerations to perturbation theory). Just as in the scalar case, we want to find a general form for these matrix elements. Translation invariance will again allow us to pull out factors of $e^{iq\cdot x}$. But we need to understand how to deal with spin, and the various spinor indices. For this, we can use Lorentz invariance. We start with $\vec{q} = 0$. Then we write, for example:

$$\langle\Omega|\psi(0)_\alpha|s\rangle \equiv \sqrt{Z}\begin{pmatrix}\xi_s\\0\\0\end{pmatrix}. \tag{10}$$

This defines the spin states (I am using the basis where γ_0 is at rest, so the spinors at rest are very simple).[1]

[1] You can legitimately ask, why assume the form of eqn. 11; why not write

$$\langle\Omega|\psi(0)_\alpha|s\rangle \equiv \begin{pmatrix}\xi_s\\a\lambda_s\end{pmatrix}, \tag{11}$$

Then we insert the operator which generates Lorentz transformations, U, similarly to the way we inserted earlier the translation operator (always in quantum mechanics, symmetries are realized by unitary operators). We use

$$U\psi U^\dagger = S\psi \tag{13}$$

where S is the *ordinary* (i.e. not an operator) matrix which multiplies spinors in Lorentz transformations. So from this we learn that the matrix element in 11 is transformed just like u's, i.e. we can take it to be $\sqrt{Z}u(\vec{q},s)_\alpha$. In this way, for the time-ordering with ψ to the left, we get what we obtained in the scalar case, multiplied by

$$\sum_s u\bar{u} = \not{q} + m. \tag{14}$$

Similarly, for the other time ordering, we get

$$\sum_s v\bar{v} = \not{q} - m. \tag{15}$$

These are exactly the structures which emerge in the *free field* calculation of the propagator, so we obtain

$$S(p) = \frac{iZ}{\not{p} - m} + \text{continuum piece.} \tag{16}$$

I.e. we obtain the same structure as in the scalar case.

The main issue here is the matrix element of the fermions between vacuum and the single-particle states.

3. Modify the derivation we performed in class of the LSZ formula for scalar fields to obtain a formula appropriate for fermions. The modifications required are minor; the main issue is the structure of the matrix elements for the fermions as in the previous problem.

Solution: The derivation is very similar to the bosonic case, described in the earlier handouts, and uses features of our derivation of the fermionic Kallen-Lehman representation above. We start with a Green's function involving several fermions, and Fourier transform in each of the variables in turn. For example, we study:

$$\int d^4x_4 T\langle\Omega|\psi(x_1)\psi(x_2)\bar{\psi}(x_3)\bar{\psi}(x_4)|\Omega\rangle e^{-ip\cdot x}. \tag{17}$$

Poles can only come from regions of the integration where the coordinates tend to ∞. Consider the limit $x_4^0 \to -\infty$ Then the time ordering is simple. We introduce a complete set of states. As we are now becoming accustomed to recognizing, only the single particle states contribute to the poles, so we only have to include them in the complete set. So we need to study:

$$\int d^4x_4 \int \frac{d^3q}{(2\pi)^3 2E(q)} \langle\Omega|\psi(x_1)\psi(x_2)\bar{\psi}(x_3)|q,s\rangle\langle q,s|\bar{\psi}(x_4)|\Omega\rangle e^{-ip\cdot x} \tag{18}$$

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where λ is another two component spinor, and a is a constant. In this case, doing the Lorentz group analysis below, one gets, for the general state

$$\langle\Omega|\psi(0)|q,s\rangle = \sqrt{Z}(u(q,s) + av(q,s)) \tag{12}$$

and similarly for the other matrix elements in this computation. However, the a terms are irrelevant to the singular behavior (pole) of the Green's function; rather than giving $/p + m$, for example, for the first time ordering, they give $/p - m$, and the reverse for the second time ordering. These factors cancel the pole. If someone has a better argument for ignoring a, I will be eager to hear it; this is the best I could come up with!

$$= \int d^4x_4 \int \frac{d^3q}{(2\pi)^3 2E(q)} \langle\Omega|\psi(x_1)\psi(x_2)\bar{\psi}(x_3)|q,s\rangle u_\alpha(q,s) e^{-ip\cdot x+iq\cdot x}.$$

Now manipulating this expression, just as in the scalar case, gives a pole, $\frac{1}{p^2-m^2}$. What about the spinor; we would like to turn this into $\not{p}+m$, so as to obtain a fermion propagator. We can do this using our explicit expression for the spinors:

$$\bar{u}(q,s) = \bar{\chi}(\not{p}+m)N(q). \tag{19}$$

(Note: had we tried to put v here, we would have canceled the pole). We can get rid of χ and the pole by multiplying by $(\not{q}-m)u(q,s')$ on the right. This gives

$$\bar{\chi}(\not{q}+m)(\not{q}-m)(\not{q}+m)\chi N(q)^2 = (q^2-m^2)\bar{\chi}2m(\not{q}+m)\chi N(q)^2). \tag{20}$$

This cancels the pole, and leaves $\delta_{s,s'}$.

Now repeat this process for the other fields. Each ψ yields either a factor $\bar{u}$ times a fermion propagator for a final state fermion, and v times a fermion propator for an initial state antifermion; the role is revered for $\bar{\psi}$. As for the scalar case, the final result multiplies the S matrix. As in that case, there are the usual subtleties. We should really take the states to be normalizable wave packets, for example. But as familiar in non-relativistic scattering theory, this is not important in (virtually) every situation.

The result can be summarized by saying: to obtain the S matrix, construct the Green's function (Fourier transformed) and take the limit as each of the external momenta goes on shell. Remove the external legs, and multiply by suitable wave function factors. In this form, the same result holds for photons in QED.

A discussion along these lines of the LSZ formula in the bosonic case appears in Peskin and Schroeder, pp. 222-230.

4. Do the exercise on the spin sums in the handout on the Dirac field.

5. Derive the Feynman rules for *Green's functions* in Yukawa theory and QED. From your understanding of LSZ, sketch the derivation of the Feynman rules for the S-matrix in QED.

Solution: This is very similar to what we did in class for electrodynamics. It is convenient to work with the operator expressions, and use Wick's theorem. If we are calculating a Green's function, for example, like:

$$\frac{T\langle\Omega|\psi(x_1)\psi(x_2)\dots\psi(x_n)\dots\bar{\psi}(y_1)\dots\bar{\psi}(y_n)|\Omega\rangle}{\langle\Omega|\Omega\rangle} \tag{21}$$

we need to bring down powers of the interaction lagrangian, $ig\bar{\psi}\psi\phi$. In n'th order, we have

$$\frac{1}{N!}T\int d^4z_1\dots d^4z_n\langle\Omega|\psi(x_1)\psi(x_2)\dots\psi(x_n)\dots\bar{\psi}(y_1)\dots\bar{\psi}(y_n)(ig\bar{\psi}(z_1)\psi(z_1)\phi(z_1))\dots(ig\bar{\psi}(z_N)\psi(z_N)\phi(z_N))|\Omega\rangle. \tag{22}$$

Now start contracting. Each external line must be contracted with another external line, giving a fermion propagator, or with a fermion in one of the vertices (ψ with $\bar{\psi}$ in each case). The boson lines in the various vertices must be contracted with each other. So one gets an ig at each vertex, and associates the usual

$$\frac{i}{\not{p}-m} \tag{23}$$

with each fermion line. The $1/N!$ factor is canceled by the number of ways of relabeling the vertices; in general, there are no other interesting combinatoric factors.

Some sample scattering amplitudes are worked out in the handout on the website.

6. 47.1-47.3