Physics 218. Advanced Quantum Field Theory. Professor Dine

Winter, 2009. Homework Set 4. SOLUTIONS

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1. Calculate the vacuum polarization at one loop order in *two dimensional* QED with *massless* fermions. You can take the shortcut of calculating the coefficient of $q_\mu q_\nu$ in the vacuum polarization tensor. This should be finite (why?). Show that $\Pi(q^2)$ has a *pole* as $q^2 \to 0$. Construct the full propagator, and show that the "photon" has a mass. Make sure you understand why your expression for the mass is dimensionally correct. Try to convince yourself that your result is exact, i.e. higher order corrections to Π vanish. A trick for doing this is to study Π^μ_μ. Look at the two loop graphs and check that they vanish if you contract the indices (you'll want to use the fact that $\gamma^\mu\gamma^\nu\gamma_\mu = 0$ in two dimensions. The result is, in fact, exact. You have solved a quantum field theory! (The model was first studied by Julian Schwinger around 1960, and is known as the massless Schwinger model).

   Solution: The first part of this problem is completely straightforward and did not cause trouble. This is just a repeat of the exercise of the previous problem set, except that the integrals are done in two dimensions rather than four. The result is:

   $$\Pi(q^2) = \frac{e^2}{\pi q^2} \tag{1}$$

   so that the propagator is:

   $$D^{\mu\nu} = -\frac{i g^{\mu\nu}}{q^2 - \frac{e^2}{\pi}}. \tag{2}$$

   So the "photon" has a mass e^2/π. A few comments are in order:

   (a) In two dimensions, you should check if you have not already, that e has dimensions of mass. So this formula makes sense dimensionally.

   (b) While the result is finite, the one loop diagram actually has an ultraviolet divergence. If you calculate the full vacuum polarization tensor, and you don't worry about the regulator (i.e. you simply cut off the Euclidean momentum integral at $p^2 = \Lambda^2$, you'll find that the result is not transverse; there is an extra $g^{\mu\nu}$ piece which goes as $e^2 g^{\mu\nu}\ln(\Lambda)$. This is not gauge invariant. Pauli Villars is a gauge invariant regulator (the Pauli Villars field, despite its weird properties, couples in a gauge invariant way); it gives a similar contribution to the polarization, which cancels. Dimensional regularization also is gauge invariant and cures this problem. Beyond one loop, interestingly, there is no UV divergence.

   (c) Exactness of the result: This theory was first solved, by operator methods, in about 1960 by Schwinger. His analysis is interesting to study. But one can show that, for the vacuum polarization, this is the only non-vanishing diagram. The trick is to take the trace of Π:

   $$\Pi^\mu_\mu = q^2 \Pi \tag{3}$$

   (in two dimensions). At one loop, one would seem to be able to prove that the result is zero. Just look at the Feynman diagram, and use

   $$\gamma^\mu \not{p} \gamma_\mu = 0 \tag{4}$$

in two dimensions. The reason the result is not zero follows from our remark above, that the diagram is divergent. It must be regulated. In Pauli-Villars, there are additional masses in the numerator factors in the Dirac trace. In dimensional regularization,

$$\Pi^\mu_\mu = (1-\epsilon)q^2\Pi \tag{5}$$

and the loop integral gives a $1/\epsilon$, which cancels (you might want to check). Examining higher order diagrams, which are finite, repeated use of eqns. 3,4 gives zero for Π.

2. Fill in the details of our calculation of $g-2$, being careful about factors of two and signs.

   Solution: This problem is straightforward, and did not cause people difficulty.