Physics 218. Advanced Quantum Field Theory. Professor Dine

Winter, 2009. Homework Set 5. Due Wed, March 4.

Problem numbers refer to your textbook.

1. As a model for the Z_0 boson, consider a massive vector with a coupling to a massless spinor,

$$\mathcal{L}_I = gZ^\mu \bar{\psi}\gamma_\mu(1-\gamma_5)\psi. \tag{1}$$

Calculate the lifetime of Z at lowest order in perturbation theory.

Solution: This problem has the basic features of the real calculation, as we will see. To get the full answer, one needs simply to be careful about the couplings of the Z to the various fields in terms of the hypercharges of the underlying theory. To evaluate, one needs to introduce polarization vectors for the Z, and square and sum over final state spins and average over the (three) initial polarizations. In the sum over polarizations, most of you wrote:

$$\sum \epsilon^\mu \epsilon^\nu = -g^{\mu\nu}. \tag{2}$$

This is correct, but it worth giving it some thought. If one goes to the rest frame of the vector, there are three polarizations, so $P_{ij} = \delta_{ij}$. We can write this in a covariant form as

$$P_{\mu\nu} = -g_{\mu\nu} + \frac{k^\mu k^\nu}{M_Z^2}) \tag{3}$$

where we have used the on-shell condition for k^μ. Now one can use current conservation, again, to drop the second term. It is easy to see, using the kinematic arguments below, that if included it does not contribute.

In any case,

$$\frac{1}{3}\sum |\mathcal{M}|^2 = -g^{\mu\nu}\mathrm{Tr}(\not{p}\gamma_\mu(1-\gamma_5)\ \not{p}'\gamma^\nu(1-\gamma_5)) \tag{4}$$
$$= -g^{\mu\nu}\mathrm{Tr}(\not{p}\gamma_\mu\ \not{p}'\gamma_\nu(1-\gamma_5))$$

Now note that, because of momentum conservation, there are only two independent momenta. So there is no way to make an invariant out of $\epsilon_{\mu\nu\rho\sigma}$, so we can drop the γ_5 term.

The rest of the evaluation is simple, when you note that squaring the energy-momentum conservation relation relates all invariants to M_Z^2:

$$p + p' = q;\ 2p\cdot p' = M_Z^2;\ q\cdot p = -\frac{1}{2}M_Z^2$$

All that is left to do is to include the two body phase space.

2. For the model of the previous problem, suppose that the field, ψ, carries electric charge. Discuss the corrections to the Z lifetime to order e^2. Don't actually compute them, but discuss the types of divergences which occur at order e^2. Interpret the ultraviolet divergences and explain what resolves the various infrared divergences. Is the lifetime finite in the next order?

Solution: The relevant diagrams are shown below.

The vertex and self energy are ultraviolet divergent. As we have seen elsewhere, these divergences cancel. If they did not, we would interpret them as renormalizing the $Z\psi\psi$ coupling. The fact that they do is related to current conservation at the vertex, as we will hopefully discuss next quarter. The vertex diagram is infrared divergent. Exactly as in our other discussions, this divergence cancels against the divergence in the real emission. One is left over with a "double logarithmic" correction which, as you can see in the lectures of Michael Peskin on the course website, are quite substantial.

3. Pions as Goldstone bosons in the strong interactions. As a model for the pions, define a field, M, which is a two by two matrix. Take the symmetry to be $SU(2)_L \times SU(2)_R$, where

$$M \to U_L M U_R \tag{5}$$

where U_L and U_R are (distinct) $SU(2)$ matrices. Show that the lagrangian:

$$\mathcal{L} = \text{Tr } \partial_\mu M^\dagger \partial^\mu M + \frac{\mu^2}{2} \text{Tr } M^\dagger M - \frac{\lambda}{4} \text{Tr } (M^\dagger M)^2 \tag{6}$$

is invariant under the symmetry.

Solution: (A correction to the kinetic term from the original version of the set has been included above.) The invariance of the lagrangian follows immediately from the cyclic property of the trace.

Show that at the minimum of the potential, M has the form

$$M = \sigma_0 \tag{7}$$

(i.e. it is proportional to the unit matrix). Argue that an $SU(2)$ subgroup of the original symmetry group is preserved; this can be identified with ordinary isospin. Writing

$$M = \sigma_0 + \delta\sigma + \vec{\pi}(x) \cdot \vec{\sigma} \tag{8}$$

show that the $\vec{\pi}$ fields are massless, and that they form a triplet of isospin.

Solution: M is a constant in the vacuum. If it is proportional to the unit matrix, it preserves the subgroup of transformations with $U_L = U_R^\dagger$. These transformations form an $SU(2)$ subgroup of the original symmetry. Under these, the fields π^a transform as a vector. To see this, study infinitesmal transformations, $U_L = 1 + i\alpha^a \frac{\sigma^a}{2}$. Then

$$\delta M = i\alpha^b[\sigma^b/2, \sigma^c\pi^c] = -\epsilon_{abc}\sigma^a\alpha^b\pi^c \tag{9}$$

or

$$\delta\pi^a = -\epsilon_{abc}\alpha^b\pi^c \tag{10}$$

which is the transformation law of a vector in $SU(2)$. This transformation law guarantees that the point where M is proportional to the unit matrix is a stationary point; if we Taylor series expand the potential about that point, the symmetry requires that there are no linear terms in π^a, so $\frac{\partial V}{\partial \pi^a} = 0$. To verify explicitly that, as required by Goldstone's theorem, the π^a fields are massless, first plug the vacuum solution into the potential, and find the minimum:

$$\sigma_0^2 = \frac{\mu^2}{\lambda}. \tag{11}$$

Then plug in the full form of M:'

$$M = \sigma_0 + \pi^a\sigma^a \tag{12}$$

and expand to order π^{a2}. The coefficient of $\pi^{a2} = -2\mu^2 + 2\mu^2 = 0$.

One can see, by the way, that M can be made proportional to the unit matrix in a variety of ways. In particular, one can diagonalize M by a symmetry transformation, so $M = \sigma_0(a + ib\sigma_3)$. where $a^2 + b^2 = 1$. Then, taking, say $U_R = 1$, $U_L = \cos\theta + i\sigma_3 \sin\theta$, with $\tan\theta = -b/a$, brings the matrix to the desired form.