Physics 218. Advanced Quantum Field Theory. Professor Dine

Winter, 2009. Homework Set 6. Due Mon., March 16.

Problem numbers refer to your textbook.

1. Derive the Feynman rules for non-Abelian gauge theories (this is just an exercise; obviously you can find this in your text, Peskin and Schroeder, or in a sketchy form, in your class notes. You can use either the path integral form, or (in my view simpler) just use Wick's theorem, which follows from either the path integral or the operator approach).

   Solution:

   This was outlined in some detail in class and much is worked out in your textbook and/or Peskin and Schroeder. This did not cause difficulties.

2. Verify our calculation of the asymptotic freedom of unbroken, non-Abelian gauge theories. You can consider, as we did in class, the force between two heavy quarks, or consider alternative approaches.

   Solution:

   Again, this was outlined in some detail in class and in the lecture notes I placed on the web, and did not generally cause difficulties.

3. Before the discovery of "neutral currents" in the early 1970's (processes mediated by gauge bosons) an interesting alternative to the $SU(2) \times U(1)$ model was a theory written down by Georgi and Glashow (and known as the Georgi-Glashow model). In this theory, the gauge group is simply $SU(2)$, and the Higgs fields form a triplet of the gauge group, ϕ_a (real; you might also want to write this as $\vec{\phi}$).

   (a) Write a potential which leads to symmetry breaking (i.e. so that the theory lies in the Higgs phase). Write the form of the solution. Show that the unbroken gauge symmetry is $U(1)$.

   Solution:

   Thinking of ϕ as a vector, $\vec{\phi}$, we can write:

   $$V = -\frac{1}{2}\mu^2\vec{\phi}^2 + \frac{\lambda}{4}(\vec{\phi}^2)^2 \qquad (1)$$

   By a symmetry transformation, we can bring $\vec{\phi}$ so it "points" in the three direction. It is convenient, rather than a Cartesian basis, to use a "spherical" basis for $\vec{\phi}$, analogous to $Y_{1,m}$ for the spherical harmonics. Then it is natural to write:

   $$\vec{\phi} = \begin{pmatrix} \phi_+ \\ \phi_0 \\ \phi_- \end{pmatrix}.$$

The expectation value of $\phi_0 = \frac{v}{\sqrt{2}}$ is given by

$$v^2 = 2\frac{\mu^2}{\lambda}.$$

This expectation value leaves unbroken a $U(1)$. In the analogy with rotations, this is just rotations about the z axis. Under these rotations, $\phi_\pm$ transform with phase $e^{\pm i\alpha}$, analogous to $Y_{1,\pm 1}$, and can be identified as fields of charge ± 1. The gauge bosons decompose in exactly the same way. The charged gauge bosons are massive.

(b) Determine the spectrum of the theory. Thinking of the $U(1)$ as electromagnetism, show that the states have definite charge.

(c) Write the Feynman rules for this theory in the 't Hooft-Feynman gauge

Solution:

The main features worth noting are that the gauge boson propagators for the two massive charged fields are:

$$D_{\mu\nu} = -\frac{g_{\mu\nu}}{q^2 - M_W^2 + i\epsilon} \tag{2}$$

for the charged bosons, and similarly for the charged fields $\phi_\pm$. The photon couples to the charged $W^\pm$ through covariant derivative terms, as for any charged particle. These are the only three gauge boson couplings. The four gauge boson couplings arise from the covariant derivative squared.

While the Georgi-Glashow theory does not describe nature, it is an interesting model. One of its most interesting features (which we will talk about in Physics 222) is that the *classsical* solutions of this theory have finite energy, static solutions which describe *magnetic monopoles.* It is the simplest field theory with this feature.

4. Write the fermionic couplings, in some detail, for a two generation version of the standard model (i.e. $e, \mu, \nu_\mu, \nu_\tau, u, c, d, s$, couplings in particular to W and Z, and to Higgs). Verify that the KM mixing involves a single angle, known as the Cabibo angle. If it is not familiar to you, check in the particle data group listings the value of this angle; look over the general form of the CKM matrix for three generations, and familiarize yourself with the experimental numbers.

Solution:

As in class, we can first diagonalize the u quark Yukawa couplings. The u quark mass matrix then looks like:

$$M_u = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix}. \tag{3}$$

By phase redefinitions of $\bar{u}$ and $\bar{d}$ we can make m_u and m_d real.

In contrast to the transformations which diagonalized m_u, where we rotated the full $SU(2)$ multiplets, $Q_f, f = 1, 2$, so as to preserve the $SU(2)$, we can not diagonalize m_d by a transformation which preserves $SU(2)$. In general, this matrix is diagonalized by separate $U(2)$ transformations of the fields u, d and $\bar{u}, \bar{d}$. But things are actually simpler. Before diagonalization

$$M_d = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{4}$$

We can first make $a, \ldots, d$ real by separate phase rotations of $\bar{s}, \bar{d}, Q_1, Q_2$. Note, in particular, we have four phases at our disposal, enough to remove all four phases. The rotations of the

Q's reintroduce phases in M_U, but these can be compensated by rotations of $\bar{u}, \bar{c}$. So we have to diagonalize a real 2×2 matrix. This can be achieved by a pair of *orthogonal* matrices. The matrix acting on $\bar{d}, \bar{s}$ can be absorbed into the definition of these fields, without affecting the $SU(2)$ symmetry. We cannot do the same for the left-handed transformation. So the original "weak eigenstates" are related to the "mass eigenstates" by an $O(2)$ matrix:

$$V = M_u = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{5}$$

So in the charged currents of the standard model, we make the replacements:

$$d' = \cos\theta d - \sin\theta s \qquad s' = \sin\theta d + \cos\theta s \tag{6}$$

Here the charged currents were originally written in terms of the primed fields; the unprimed fields are the mass eigenstates. In other words, the currents involve:

$$u_f \sigma^\mu V_{ff'} d^*_f. \tag{7}$$

The angle θ is known as the Cabbibo angle. For three generations, $V_{ff'}$ is known as the Kobiyashi-Maskawa matrix.