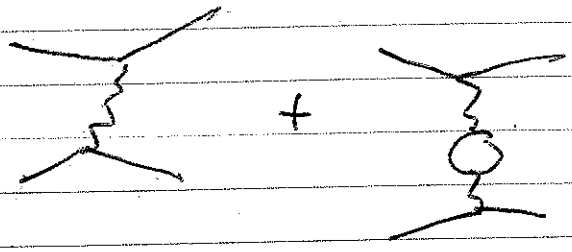
Renormalization of e

Call e_0 parameter in $\mathcal{L}$

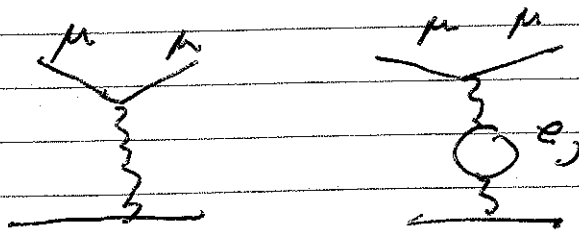
E.g. Coulomb scattering, $\vec{q} \to 0$



$$= \frac{e_0^2}{\vec{q}^2}\left[1 - \frac{e_0^2}{16\pi^2}\,\frac{4}{3}\,\frac{2}{\epsilon}\right]$$

$$e_R^2 \qquad e_R^2 = 1 - \frac{e_0^2}{16\pi^2}\,\frac{4}{3}\,\frac{2}{\epsilon}$$

Suppose e, μ:



$m_\mu^2 >> \vec{q}^2 >> m_e^2$

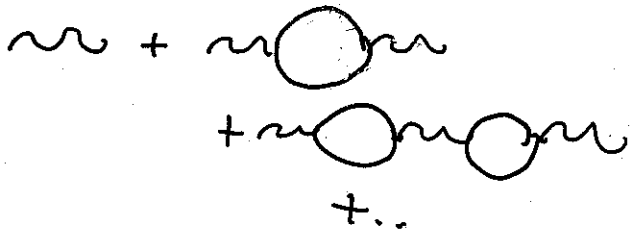
$$\Rightarrow \frac{e_0^2}{\vec{q}^2}\left[1 - \frac{e_0^2}{16\pi^2}\,\frac{4}{3}\,\frac{2}{\epsilon} \times 2 - \frac{e_0^2}{16\pi^2}\,\frac{4}{3}\ln\left(\mu^2/Q^2\right)\right]$$

$$\simeq e_R^2/q^2\left[1 - \frac{e_R^2}{16\pi^2}\,\frac{4}{3}\ln\left(\mu^2/Q^2\right)\right]$$

QED Renormalization: Vertex and Self Energy

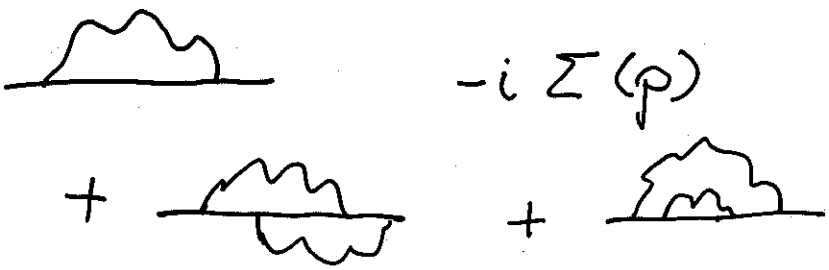
We have seen, for photon

$$D_{\mu\nu} = \frac{-i g_{\mu\nu}}{q^2(1-\Pi(q^2))}$$



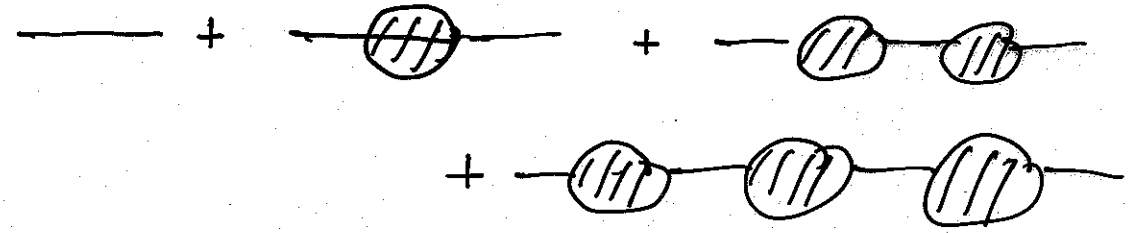
[$\Pi(q^2)$: $\Pi(q^2) = (1-Z^{-1}) + q^2\Pi'(0) + \ldots$ "1PI"]

For fermion propagator



$-i\Sigma(p)$ "self energy" 1PI

$$\Sigma(p) = A(p^2)\not{p} + B'(p^2) \equiv A(p^2)(\not{p}-m_0) + B(p^2)$$
$$= (\not{p}-m_0)\left(A + B(\not{p}-m_0)^{-1}\right)$$

$S_F =$ 

$$= \frac{i}{\not{p} - m_0} \left(1 + (A + B(\not{p} - m_0)^{-1}) + (\quad)^2 + \ldots\right)$$

$$= \frac{i}{\not{p} - m_0} \frac{1}{(1 - A(p^2) - B(p^2)(\not{p} - m_0)^{-1})}$$

$$= \frac{i}{[\not{p} - m_0](1 - A(p^2)) - B(p^2)}$$

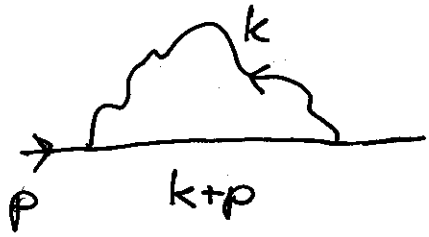
$$= \frac{i\,(1 - A(p^2))^{-1}}{\not{p} - m(p^2)}$$

$$m(p^2) = m_0 + B(p^2)(1 - A(p^2))$$

As $p^2 \to m^2$, $(1 - A(p^2))^{-1} = Z$

$$m^2 = m_0 + B(m^2)$$

All of this is easy to understand at one loop:



$$-i\Sigma(p) = (-ie^*)^2(-i)(i)\int \frac{d^dk}{(2\pi)^d}\,\frac{\gamma^\mu(\not p+\not k+m)\gamma_\mu}{k^2[(k+p)^2-m^2]}$$

$$= -e^2\int dx\,\frac{d^dk}{(2\pi)^d}\,\frac{\gamma^\mu(\not p(1-x)+\not k+m)\gamma_\mu}{[k^2+p^2x(1-x)-m^2x]^2}$$

$$= -e^2\int \frac{d^dk}{(2\pi)^d}\,\frac{(-\not k-2\not p(1-x)+4m)}{[k^2+p^2x(1-x)-m^2x]^2}$$

$$k^0 \to ik^0;\quad p^2 = -p^2$$

$$= -ie^2\int \frac{d^dk\,dx}{(2\pi)^d}\,\frac{[-2\not p(1-x)+4m]}{[k^2+p^2x(1-x)+m^2x]^2}$$

$$\Sigma = \frac{e^2}{16\pi^2}\,\Gamma(2-d/2)\int dx\,[p^2x(1-x)+m^2x]^{d/2-2} \times[-2\not p(1-x)+4m]$$

We can already read off A, B, to $O(e^2)$, from our definitions above.

Let's focus on $1/\epsilon$ poles:

$$\Sigma = \frac{\alpha}{4\pi}\frac{2}{\epsilon}\int_0^1 dx\,[-2\not{p}(1-x)+4m]$$

$$= -\frac{\alpha}{4\pi}\frac{2}{\epsilon}\not{p} + \frac{\alpha}{4\pi}\frac{8m}{\epsilon}$$

$$= +\frac{\alpha}{4\pi}\frac{2}{\epsilon}\left(-(\not{p}-m)+3m\right)$$

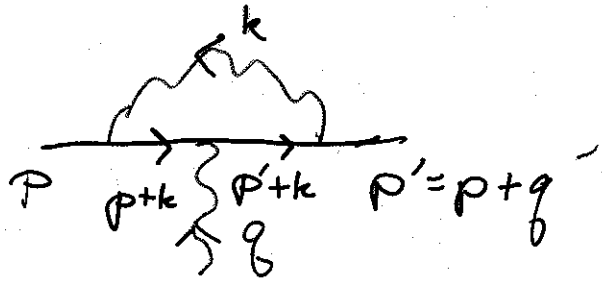
$$A = -\frac{e^2}{16\pi^2}\frac{2}{\epsilon}$$

$$B = \frac{e^2}{16\pi^2}\frac{2}{\epsilon}3m_0$$

$$\delta m = \frac{e^2}{16\pi^2}\frac{2}{\epsilon}3m_0 \cdot \left(= \frac{e^2}{16\pi^2}3m_0\ln\left(\Lambda^2/m^2\right)\right)$$

Mass shift proportional to bare mass;
logarithmic in cutoff.

Vertex correction: UV divergent part



$$-i\Gamma^\mu = -\int \frac{d^d k}{(2\pi)^d}\, \gamma^\nu \frac{\not p + \not k + m}{[(p+k)^2 - m^2]}\, \gamma^\mu \frac{(\not p' + \not k + m)}{[(p+k)^2 - m^2]}\, \gamma_\nu \frac{1}{k^2}$$

UV divergence: from two factors of k in numerator only, so

$$-i\Gamma^\mu \cong -e^2 \int \frac{d^d k}{(2\pi)^d}\, \frac{\gamma^\nu \not k \gamma^\mu \not k \gamma_\nu}{[k^2+\Delta]^3}$$

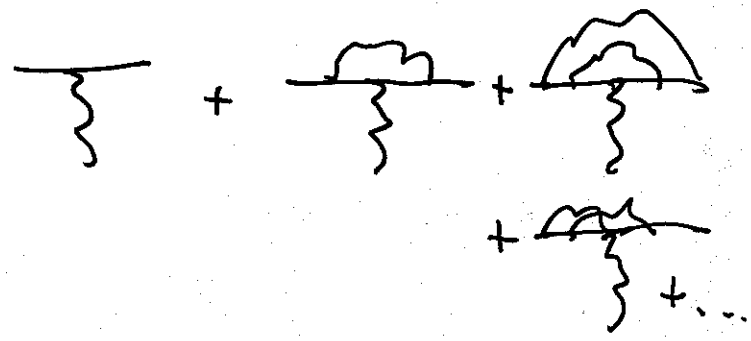
$$= -ie^2 \int \frac{d^d k}{(2\pi)^d}\, \frac{\gamma^\nu \gamma^\rho \gamma^\mu \gamma_\sigma \gamma_\nu k_\rho k_\sigma}{[k^2+\Delta]^3}$$

$$= -\frac{ie^2}{4} \int \frac{d^d k}{(2\pi)^d}\, \frac{\gamma^\nu \gamma_\rho \gamma^\mu \gamma^\rho \gamma^\nu k^2}{[k^2+\Delta]^3}$$

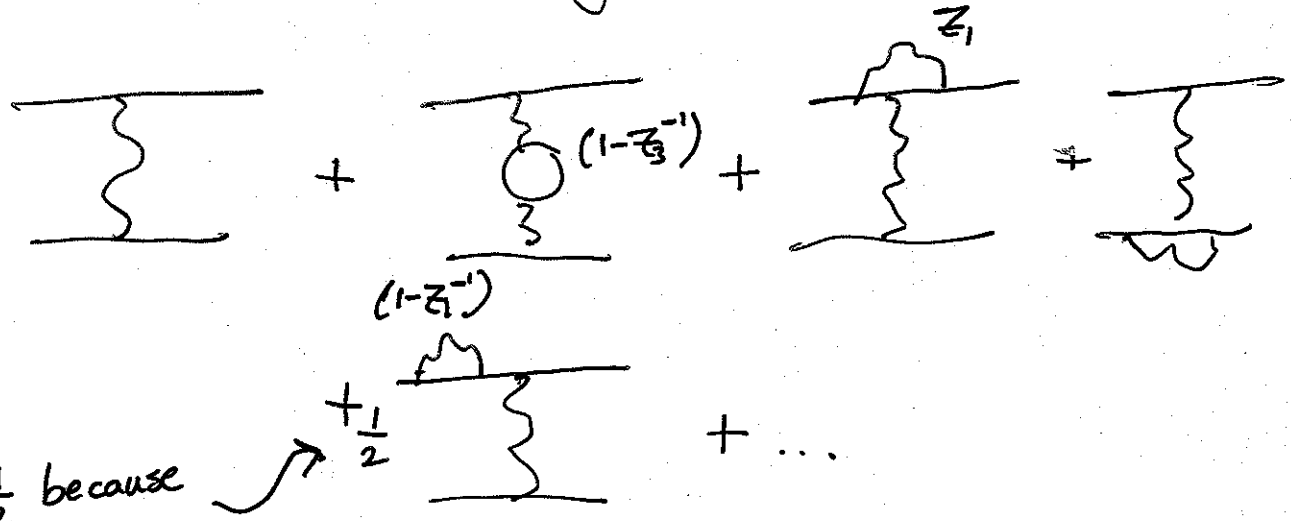
$$= -ie^2 \int \frac{d^d k}{(2\pi)^d}\, \frac{k^2}{[k^2+\Delta]^3}\, \gamma^\mu = \frac{2}{\epsilon} \frac{e^2}{16\pi^2} (-i\gamma^\mu)$$

Features: only UV divergence appears in same structure as "bare" vertex.

Redefine $\Gamma^\mu =$



Renormalized coupling:



$\frac{1}{2}$ because of $\sqrt{Z}$ in S matrix

$$\propto (-ie_0)^2 \left(\frac{-i}{q^2}\right) (\bar{u}\gamma^\mu u)(\bar{u}\gamma_\mu u)$$

$$\times \left[Z_1^2 Z_2^2 Z_3\right]$$

$$e_0^2 = Z_1^2 Z_2^2 Z_3 = Z_3 \quad \text{(to order we work)}$$

Now if you look back, you'll see, to order we work, $Z_1 = Z_2$. Turns out to be exact; a consequence of gauge invariance (Ward identity) (proof shortly)

QED: <u>all</u> divergences can be absorbed into redefinition of charge, electron mass.

More on vertex function:

$$\bar{u}(p')\Gamma^\mu(p,p',q)u(p)$$ ("on shell fermions")

Only invariant: q^2; $q^2 = (p-p')^2 = 2m^2 - 2p\cdot p'$

$$q\cdot p = p\cdot(p-p')$$

So $\Gamma^\mu = \Gamma^\mu(q)$

$$q_\mu \Gamma^\mu = 0$$

$$\Gamma^\mu = \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu} q_\nu F_2(q^2)}{2m}$$

$F_1(0)$: corrects electric charge.

$F_2(0)$: magnetic moment

Non-rel. limit $\sigma^{oi} \cong 0$

$$\frac{i\sigma^{ij}}{2m} q_j A_i(q) F_2(0)$$

$$i\frac{\sigma^{ij}}{2m} (i\partial_j A_i)$$

$$\sigma^{ij} = \epsilon^{ijk}\sigma^k$$

$$\Rightarrow -\vec{\sigma}\cdot\vec{B}$$

$F_2(0) = 0$ at tree level.

$\Rightarrow$ correction to Dirac value.

Finite.

g-2 fast

$$-i\Gamma^\mu = -2e^2 \int \frac{d^4k}{(2\pi)^4} d\alpha_1 d\alpha_2 d\alpha_3 \, \delta(1-\Sigma\alpha_i)$$

$$\times \frac{\bar{u}(p')\gamma^\rho (\not{k} + \not{p}'(1-\alpha_{12}) + \not{q}\alpha_1 + m)\gamma^\mu (\not{k} + \not{p}(1-\alpha_{12}) - \not{q}\alpha_3 + m)\gamma_\rho u(p)}{[k^2 - (p\alpha_1 + p'\alpha_2)^2]^3}$$

Now terms in numerator containing only k contribute only to F_1. Rewrite numerator using Dirac eqn:

$$\bar{u}(p')\left(2p^\rho(1-\alpha_{12}) + \gamma^\rho m\alpha_{12} + \gamma^\rho \not{q}\alpha_1\right)\gamma^\mu \left(2p^\rho(1-\alpha_{12}) + m\gamma^\rho\alpha_{12} - \not{q}\gamma^\rho\alpha_2\right)u(p)$$

Nine terms in product; many don't contribute to F_2

① $4p\cdot p'(1-\alpha_{12})^2\gamma^\mu$: F_1 only

② $2m\gamma^\mu \not{p}'\alpha_{12}(1-\alpha_{12}) + 2m\not{p}\alpha_{12}(1-\alpha_{12})$

$= 2m\gamma^\mu(\not{p}+\not{q})\alpha_3(1-\alpha_3) + 2m(\not{p}'-\not{q})\gamma^\mu\alpha_3(1-\alpha_3)$

$= 2m\alpha_3(1-\alpha_3)[\gamma^\mu, \not{q}] + F_1$ term

Just a few more to go (homework!)