Physics 218. Quantum Field Theory. Professor Dine

Aspects of the Dirac Field

1 Anticommutation Relations and the Exclusion Principle

Suppose we have operators a, $a^\dagger$ which obey the anticommutation relations:

$$\{a, a^\dagger\} = 1; \{a, a\} = 0; \{a^\dagger, a^\dagger\} = 0$$

Then construct the "number operator"

$$N = a^\dagger a$$

and the state $|0\rangle$ by the condition

$$a|0\rangle = 0$$

(a is a destruction operator). Then

$$Na^\dagger|0\rangle = a^\dagger a a^\dagger|0\rangle.$$

Using the anticommutation relations to move a to the right,

$$= -a^\dagger a^\dagger a|0\rangle + a^\dagger|0\rangle.$$

In other words

$$Na^\dagger|0\rangle = |0\rangle$$

so $a^\dagger$ creates a one particle state. But since $(a^\dagger)^2 = 0$ (from the anticommutation relations) there is no two particle state.

So anticommutation relations of this kind build in the exclusion principle; only the zero and one-particle states are allowed. For the Dirac Hamiltonian in momentum space, taking the volume to be finite:

$$H = \sum_{\vec p, s} (a^\dagger(p,s)a(p,s) + b^\dagger(p,s)b(p,s)) + \infty.$$

So we see that we can have states with zero or one electron and zero or one positron for each momentum and spin.

2 Quick calculation of spin sums, Normalizations, etc.

In your text, various relations for spinors, including orthogonality relations and spin sums, are worked out by looking at explicit solutions. We can short circuit these calculations in a variety of ways. Here is one:

Two things slightly different than your text:

1. Dirac matrices: It will be helpful to have an explicit representation of the Dirac matrices, or more specifically of Dirac's matrices, somewhat different than the one in your text:

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \vec\gamma = \begin{pmatrix} 0 & \vec\sigma \\ -\vec\sigma & 0 \end{pmatrix} \tag{1}$$

2. Sign conventions: I'll use the west coast metric in this handout; compared to Srednicki, $\not p \rightarrow - \not p$

If χ is a constant spinor,

$$u(p) = N(\not p + m)\chi$$

solves the Dirac equation. Now take χ to be a solution of the Dirac equation with $\vec{p} = 0$. We can work, for this discussion, in any basis, so let's choose our original basis, where the $\vec{p} = 0$ spinors are particularly simple, and take the two linearly-independent spinors to be

$$\chi_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}; \chi_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Let's first get the normalizations straight. We will require:

$$\bar{u}(p)u(p) = 2m.$$

With our solution the left hand side is

$$N^2\chi^\dagger(\not p^\dagger + m)\gamma^o(\not p + m)\chi$$

$$= N^2\chi^\dagger\gamma^o\gamma^o(\not p^\dagger + m)\gamma^o(/p + m)u(p)$$

$$= N^2\chi^\dagger\gamma^o(\not p + m)(\not p + m)\chi$$

$$= N^2\chi^\dagger 2m(\not p + m)\chi$$

From the explicit form of the Dirac matrices and the χ's, $\chi^\dagger \not p\chi = E$. So

$$N^2 = \frac{1}{(E + m)}.$$

With this we can do the spin sums. First note that for the χ's, looking at their explicit form:

$$\sum_s \chi\chi^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \frac{1}{2}(1 + \gamma^o).$$

So now

$$\sum_s u(p, s)\bar{u}(p, s) = \sum_s u(p, s)u^\dagger(p, s)\gamma^o$$

$$= \frac{1}{2}N^2(\not p + m)(1 + \gamma^o)(\not p^\dagger + m)\gamma^o$$

$$= \frac{1}{2}N^2(\not p + m)(1 + \gamma^o)\gamma^o\gamma^o(\not p^\dagger + m)\gamma^o$$

$$= \frac{1}{2}N^2(\not p + m)(1 + \gamma^o)(\not p + m)\gamma^o$$

$$= \frac{1}{2}N^2[(\not p + m)(\not p - (\not p - m) + 2p^o](/p + m)\gamma^o$$

$$= \frac{1}{2}N^2 2(m + p^o)(\not p + m)$$

$$= (\not p + m).$$

Finally, we can compute the inner products:

$$u^\dagger(p, s)u(p, s') = N^2\chi^\dagger(\not p^\dagger + m)(\not p + m)\chi$$

$$= N^2\chi^\dagger\gamma^o(\not p + m)\gamma^o(\not p + m)\chi$$
$$= N^2\chi^\dagger\gamma^o(p^2 - m^2 + 2p^o(\not p + m)\gamma^o\chi$$
$$= 2E\delta_{s,s'}$$

Exercise:

Work out the corresponding relations for the negative energy spinors, $v(p,s)$, including the spin sums, normalization, and orthogonality relations.

$$\sum_s u_\alpha(p,s)\bar{u}_\beta(p,s) = (\not p + m)_{\alpha\beta} \qquad \sum_s v_\alpha(p,s)\bar{v}_\beta(p,s) = (\not p - m)_{\alpha\beta}$$

$$u^\dagger_\alpha(p,s)u_\alpha(p,s') = 2E_p\delta_{ss'} = v^\dagger_\alpha(p,s)v_\alpha(p,s')$$
$$\bar{u}_\alpha(p,s)u_\alpha(p,s') = 2m\delta_{ss'} = \bar{v}_\alpha(p,s)v_\alpha(p,s')$$

The Discrete Symmetries P and C

Parity: The Dirac lagrangian is unchanged if we make the replacement:

$$\psi(\vec{x},t) \to \gamma^o\psi(-\vec{x},t) \tag{2}$$

Let's see what effect this has on the creation and annihilation operators, a, b, etc.

$$\psi_p(\vec{x},t) = \gamma^o\psi(-\vec{x},t) = \int \frac{d^3p}{(2\pi)^3\sqrt{E_p}}(a(\vec{p},s)\gamma^o u(\vec{p},s)e^{-ip^ox^o - i\vec{p}\cdot\vec{x}} + b^\dagger(\vec{p},s)\gamma^o v(\vec{p},s)e^{ip^ox^o + i\vec{p}\cdot\vec{x}}). \tag{3}$$

We can easily determine what γ^o does to u and v using our explicit expressions (ignoring the normalization factor, which is unimportant for this discussion):

$$\gamma^o(\not p + m)\chi = (p^o\gamma^o + \vec{p}\cdot\vec{\gamma} + m)\gamma^o\chi = u(-\vec{p},s)\gamma^o v(\vec{p},s) = v(-\vec{p},s) \tag{4}$$

So making the change of variables $\vec{p} \to -\vec{p}$ in our expression for ψ_p, gives

$$\psi_p(\vec{x},t) = \gamma^o\psi(-\vec{x},t) = \int \frac{d^3p}{(2\pi)^3\sqrt{E_p}}(a(-\vec{p},s)u(\vec{p},s)e^{-i\cdot x} + b^\dagger(\vec{p},s)v(-\vec{p},s)e^{ip\cdot x}) \tag{5}$$

Charge Conjugation: Now we can do the same thing for C. Here:

$$\psi_c(x) = \gamma^2\psi^*(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{E_p}}(a^\dagger(\vec{p},s)\gamma^2 u^*(\vec{p},s)e^{ip\cdot x} + b(\vec{p},s)\gamma^2 v^*(\vec{p},s)e^{-ip\cdot x}). \tag{6}$$

Now we consider the action of γ^2 on u^*, v^*:

$$\gamma^2 u^* = \gamma^2(\not p^* + m)\chi^* = \gamma^2(p^o\gamma^o - p^1\gamma^1 + p^2\gamma^2 - p^3\gamma^3 + m)\chi = (-\not p + m)\gamma^2\chi. \tag{7}$$

Here we have not been ashamed to use the explicit properties of the γ matrices; γ^1 and γ^3 are real, while γ^2 is imaginary; the first two anticommute with γ^2 while the third commutes. Now we use the explicit form of γ^2 to see that it takes the positive energy χ to the negative energy χ, with the opposite spin. So, indeed, we have that

$$a_c(p,s) = b(p,-s) \qquad b_c(p,s) = a(p,-s) \tag{8}$$

i.e. it reverses particles and antiparticles and flips the spin.