1. **Fadeev-Popov Ghosts**

   a. Derive the transformation law for the gauge field $A_\mu^a$, under an infinitesimal gauge transformation $\omega^a$, and show that it can be elegantly expressed in terms of the covariant derivative of $\omega^a$, thought of as a scalar field.

   **Solution:** Let’s start from scratch. Working with matrix-valued fields, and take the transformation law of the gauge field to be:
   \[
   A_\mu \rightarrow UA_\mu U^\dagger - i\partial_\mu UU^\dagger. \tag{1}
   \]
   Then, for a field $\psi \rightarrow U\psi$,
   \[
   D_\mu \psi = (\partial_\mu - iA_\mu)\psi \rightarrow U(\partial_\mu \psi - iA_\mu \psi) \tag{2}
   + \partial_\mu U\psi - \partial_\mu U\psi.
   \]
   Consider the infinitesimal transformation,
   \[
   U = 1 + i\omega^a T^a \tag{3}
   \]
   Then expanding the transformation law for $A_\mu$ to first order in $\omega$, and writing $A_\mu = A_\mu^a T^a$, etc.,
   \[
   \delta A_\mu = i[\omega^b T^b, A_\mu^c T^c] - i \times i\partial_\mu \omega^a T^a \tag{4}
   = 0 f^{abc} \omega^b A_\mu^c T^a + \partial_\mu \omega^a T^a.
   \]
   In other words
   \[
   \delta A_\mu^a = \partial_\mu \omega^a - f^{abc} \omega^b A_\mu^c. \tag{5}
   \]
   Compare this with (remember that the generators in the adjoint representation are $T^a_{bc} = -if^{abc}$)
   \[
   (D_\mu \omega)^a = \partial_\mu \omega^a - (-i)(-if^{bac} A_\mu^b)\omega^c \tag{6}
   \]
   \[
   = \partial_\mu \omega^a + f^{abc} A_\mu^b \omega^c.
   \]
   b. Implement the Fadeev-Popov procedure; derive the ghost lagrangian (you don’t have to write hundreds of pages for this; just give a brief summary).

2. **Feynman rules:** derive the three gauge boson coupling and the ghost propagator and interaction terms.

   **Solution:** The basic elements of the Fadeev-Popov procedure are pretty simple. One needs to study the integral over gauge transformations of
   \[
   \delta(\partial_\mu (A_\mu^a + \delta A_\mu^a)) \tag{7}
   \]
where $A^{\mu a}$ already satisfies the gauge condition (as always with $\delta$ functions, it is only necessary to work near the point where the $\delta$ function condition is satisfied). So, using our expression above for the gauge variation, $\delta A$, we have

$$\delta(\partial^2 \omega^a + f^{abc} A^{\mu b} \partial_\mu \omega^a).$$

(8)

So we need

$$\det(\partial^2 + f^{abc} A^{\mu b} \partial_\mu) = \int dc \, dc^\dagger \exp \left(i \int d^4 x c^\dagger (\partial^2 + f^{abc} A^{\mu b} \partial_\mu)c \right).$$

(9)

[Here, as in much of these notes, I am scaling $A$ so that $1/g^2$ appears in front of the gauge boson kinetic term, and there are no factors of $g$ in the couplings; one can rescale to obtain this more standard form.]

3. **Higgs phenomenon**: consider an $O(3)$ (equivalent to $SU(2)$) gauge theory with Higgs fields in the adjoint representation. If we call the fields $\vec{\phi}$, take

$$V(\vec{\phi}) = -\frac{1}{2} \mu^2 |\vec{\phi}|^2 + \frac{\lambda}{4} |\vec{\phi}|^4.$$ 

(10)

Determine the pattern of symmetry breaking, and the mass of the gauge bosons (there is an unbroken gauge symmetry). Determine the charges of the gauge bosons under the remaining symmetry.

**Solutions:** By a gauge transformation, we can bring $\phi^a$ to the form

$$\phi^a = \frac{v}{\sqrt{2}} \delta^{a3}.$$ 

(11)

$v^2$ satisfies

$$\frac{v^2}{2} = \frac{\mu^2}{\lambda}.$$ 

(12)

We can work out the gauge boson masses by going to unitary gauge. They then arise from

$$\begin{align*}
(D_{\mu} \phi)^{a2} &= e^{abc} A^{\mu b} \phi^c e^{ade} A^{\mu e} \phi^3 \\
&= (\delta^{bd} \delta^{ce} - \delta^{be} \delta^{cd}) A^{\mu b} A^{\mu d} \phi^c \phi^e \\
&= A^{\mu b} A^{\mu b} \phi^c \phi^c - A^{\mu c} A^{\mu c} \phi^b \phi^c \\
&= \frac{v^2}{2} (A^{\mu 1} A^{\mu 1} + A^{\mu 2} A^{\mu 2}).
\end{align*}$$

(13)

Noting that the kinetic terms for the gauge bosons behave as

$$\frac{1}{2g^2} (\partial_\mu A^{\mu}_{\nu})^2$$

(14)

we see that the gauge bosons $A^{\mu 1}, A^{\mu 2}$ have mass $g^2 v^2$, while $A^{\mu 3}$ is massless.

The masslessness of $A^{\mu 3}$ arises because $\phi$ is invariant under rotations about the 3 axis (in the isospin space). So this symmetry is unbroken. Correspondingly, the generator $T^3$ is unbroken. In Georgi and Glashow’s original implementation of this model, this generator was identified with electric charge. The fields

$$W^\pm_{\mu} = \frac{1}{\sqrt{2}} (A^{\mu 1}_\mu \pm i A^{\mu 2}_\mu)$$

(15)
together form a complex field. They are charged under the $U(1)$. You can check this, or think by analogy to spherical harmonics, where

$$Y_{1\pm1} \propto (x \pm iy)$$

(16)

transform by a phase under rotations about the $z$ axis. (To check it explicitly, you will first want to consider infinitesimal transformations).

Note that

$$(A^1_\mu T^1 + A^2_\mu T^2) = \sqrt{2}(W^+\mu T^- + W^-\mu T^+)$$

(17)

with

$$T^\pm = \frac{1}{2}(T^1 \pm iT^2)$$

(18)

are the usual raising and lowering operators (matrices) which enter in the theory of angular momentum.