# Physics 222. Quantum Field Theory 3. Professor Dine

## Spring, 2011. Homework Set 1. **Solutions**.

1. **Fadeev-Popov Ghosts**

   a. Derive the transformation law for the gauge field $A^a_\mu$, under an infinitesimal gauge transformation $\omega^a$, and show that it can be elegantly expressed in terms of the covariant derivative of $\omega^a$, thought of as a scalar field.

   **Solution:** Let's start from scratch. Working with matrix-valued fields, and take the transformation law of the gauge field to be:

$$A_\mu \to U A_\mu U^\dagger - i\partial_\mu U U^\dagger. \tag{1}$$

   Then, for a field $\psi \to U\psi$,

$$D_\mu\psi = (\partial_\mu - iA_\mu)\psi \to U(\partial_\mu\psi - iA_\mu\psi) \tag{2}$$

$$+ \partial_\mu U\psi - \partial_\mu U\psi.$$

   Consider the infinitesimal transformation,

$$U = 1 + i\omega^a T^a \tag{3}$$

   Then expanding the transformation law for $A_\mu$ to first order in $\omega$, and writing $A_\mu = A^a_\mu T^a$, etc.,

$$\delta A_\mu = i[\omega^b T^b, A^c_\mu T^c] - i \times i\partial_\mu\omega^a T^a \tag{4}$$

$$= 0f^{abc}\omega^b A^c_\mu T^a + \partial_\mu\omega^a T^a.$$

   In other words

$$\delta A^a_\mu = \partial_\mu\omega^a - f^{abc}\omega^b A^c_\mu. \tag{5}$$

   Compare this with (remember that the generators in the adjoint representation are $T^a_{bc} = -if^{abc}$)

$$(D_\mu\omega)^a = \partial_\mu\omega^a - (-i)(-if^{bac}A^b_\mu)\omega^c \tag{6}$$

$$- = \partial_\mu\omega^a + f^{abc}A^b_\mu\omega^c.$$

   b. Implement the Fadeev-Popov procedure; derive the ghost lagrangian (you don't have to write hundreds of pages for this; just give a brief summary.

2. **Feynman rules:** derive the three gauge boson coupling and the ghost propagator and interaction terms.

   **Solution**: The basic elements of the Fadeev-Popov procedure are pretty simple. One needs to study the integral over gauge transformations of

$$\delta(\partial_\mu(A^{\mu a} + \delta A^{\mu a}) \tag{7}$$

where $A^{\mu a}$ already satisfies the gauge condition (as always with $\delta$ functions, it is only necessary to work near the point where the $\delta$ function condition is satisfied). So, using our expression above for the gauge variation, $\delta A$, we have

$$\delta(\partial^2\omega^a + f^{abc}A^{\mu b}\partial_\mu\omega^a). \tag{8}$$

So we need

$$\det(\partial^2 + f^{abc}A^{\mu b}\partial_\mu) = \int\ dc\ dc^\dagger\ exp\left(i\int d^4xc^\dagger(\partial^2 + f^{abc}A^b_\mu\partial_\mu)c\right). \tag{9}$$

[Here, as in much of these notes, I am scaling $A$ so that $1/g^2$ appears in front of the gauge boson kinetic term, and there are no factors of $g$ in the couplings; one can rescale to obtain this more standard form.]

3. **Higgs phenomenon**: consider an $O(3)$ (equivalent to $SU(2)$) gauge theory with Higgs fields in the adjoint representation. If we call the fields $\vec{\phi}$, take

$$V(\vec{\phi}) = -\frac{1}{2}\mu^2|\vec{\phi}|^2 + \frac{\lambda}{4}|\vec{\phi}|^4. \tag{10}$$

Determine the pattern of symmetry breaking, and the mass of the gauge bosons (there is an unbroken gauge symmetry). Determine the charges of the gauge bosons under the remaining symmetry.

**Solutions:** By a gauge transformation, we can bring $\phi^a$ to the form

$$\phi^a = \frac{v}{\sqrt{2}}\delta^{a3}. \tag{11}$$

$v^2$ satisfies

$$\frac{v^2}{2} = \frac{\mu^2}{\lambda}. \tag{12}$$

We can work out the gauge boson masses by going to unitary gauge. They then arise from

$$\begin{aligned}(D_\mu\phi)^{a2} &= \epsilon^{abc}A^b_\mu\phi^c\epsilon^{ade}A^d_\mu\phi^3 \qquad (13)\\ &= (\delta^{bd}\delta^{ce} - \delta^{be}\delta^{cd})A^b_\mu A^d_\mu\phi^c\phi^e\\ &= A^b_\mu A^{\mu b}\phi^c\phi^c - A^b_\mu A^{\mu c}\phi^b\phi^c\\ &= \frac{v^2}{2}(A^{\mu 1}A^1_\mu + A^{\mu 2}A^2_\mu).\end{aligned}$$

Noting that the kinetic terms for the gauge bosons behave as

$$\frac{1}{2g^2}(\partial_\mu A^i_\nu)^2 \tag{14}$$

we see that the gauge bosons $A^1_\mu, A^2_\mu$ have mass $g^2v^2$, while $A^3_\mu$ is massless.

The masslessness of $A^3_\mu$ arises because $\phi$ is invariant under rotations about the 3 axis (in the isospin space). So this symmetry is unbroken. Correspondingly, the generator $T^3$ is unbroken. In Georgi and Glashow's original implementation of this model, this generator was identified with electric charge. The fields

$$W^\pm_\mu = \frac{1}{\sqrt{2}}(A^1_\mu \pm iA^2_\mu) \tag{15}$$

together form a complex field. They are charged under the $U(1)$. You can check this, or think by analogy to spherical harmonics, where

$$Y_{1\pm1} \propto (x \pm iy) \tag{16}$$

transform by a phase under rotations about the $z$ axis. (To check it explicitly, you will first want to consider infinitesimal transformations).

Note that

$$(A^1_\mu T^1 + A^2_\mu T^2) = \sqrt{2}(W^+_\mu T^- + W^-_\mu T^+) \tag{17}$$

with

$$T^\pm = \frac{1}{2}(T^1 \pm iT^2) \tag{18}$$

are the usual raising and lowering operators (matrices) which enter in the theory of angular momentum.