Physics 222. Quantum Field Theory 3. Professor Dine

Spring, 2011. Homework Set 4. **SOLUTIONS**.

1. Verify, in $SU(5)$ that the combination $\bar{5}+10$ is free of anomalies. You may want to do this by considering subgroups.

   Solution:

   Here it is just necessary to check that the triangle graphs cancel. A simple way to proceed is to consider subgroups. We have seen that the $\bar{5}$ and 10 contain one generation of the Standard Model, and that the Standard Model is anomaly free. It is worth checking again; for three external non-abelian gauge fields, the contributions of the 10 and $\bar{5}$ vanish separately; for one $U(1)$ and two $SU(3)$ gauge bosons, one has the contribution of $\bar{u}$ and $\bar{d}$, proportional to $-4/3+2/3$, and that of Q, proportional to $2\times 1/3$ (since an $SU(2)$ doublet). For one $U(1)$ and two $SU(2)$ gauge bosons, one has $1/3\times 3$ for Q, and -1 for L. For three $U(1)$ gauge bosons, one has a result proportional to (for a single generation, i.e. a single $\bar{5}$ and 10:

   $$8-2-(64/9)+8/9+2/9=0. \tag{1}$$

   Actually, as stated in Peskin and Schroder, it is enough to check the cancelation for any one generator; there is only one group invariant involved. Since you are all group theory mavens, from Professor Haber's course, I will leave the proof to you. But at least note:

   a. The object appearing in the anomaly is Tr $T^aT^bT^c \equiv d^{abc}$. Note that this is completely symmetric in a, b and c.

   b. This is an invariant tensor. This follows by noting that under $SU(N)$ transformations, $T^a \rightarrow UT^aU^\dagger$ (why?).

   To complete the proof, one needs to show that this tensor is irreducible.

   Finally, it is worth giving at least one quick derivation of the anomaly, for a $U(1)$ theory. This appears at the end of these notes.

2. Verify that an adjoint field of the form

   $$\Phi = v\text{diag}(2,2,2,-3,-3)$$

   minimizes the potential for the adjoint, and determine v, if

   $$V(\Phi) = -\mu^2\text{Tr}\Phi^2 + \frac{\lambda}{2}(\text{Tr}\Phi^2)^2.$$

   To do this, you must first show that there is a stationary point of this form, and then consider the curvature about this point (i.e. the masses of the excitations).

   Solution: This can be done a number of ways, but particularly simple is to take

   $$\Phi = v\frac{\lambda^{24}}{2} \tag{2}$$

$$\lambda^{24} = \frac{1}{\sqrt{60}}\mathrm{diag}(2, 2, 2, -3, -3)$$

i.e. λ^{24} is normalized like the Gell-Mann matrices. To make the equations particularly simple-looking, I'll modify the potential slightly:

$$V(\Phi) = -\mu^2 \mathrm{Tr}\Phi^2 + \lambda(\mathrm{Tr}\Phi^2)^2.$$

Then substituting the expression above for Φ we have

$$V = -\frac{1}{2}\mu^2 v^2 + \frac{\lambda}{2}v^4 \tag{3}$$

so

$$v^2 = \frac{\mu^2}{\lambda}. \tag{4}$$

This solution is *guaranteed* to be a stationary point of the potential due to the symmetries. The unbroken symmetry is $SU(3) \times SU(2) \times U(1)$. So writing

$$\Phi = \Phi_0 + \phi^a T^a \tag{5}$$

and expanding the potential in powers of ϕ^a, only T^{24} can appear linearly, as it is the only generator neutral under the unbroken symmetries (i.e. the only generator which commutes with all of the unbroken symmetry generators). We have arranged that this does not occur, through our choice of v. So all of the ϕ's appear, at least quadratically.

To establish whether this is a local minimum, we need to evaluate the quadratic (and possibly higher) coefficients. Taking Φ in the form above, with T^a the properly normalized generators, we have, for all but $a = 24$,

$$V = -\frac{\mu^2}{2}\phi^a\phi^a + \frac{\lambda}{4} \times 2v^2\phi^a\phi^a + \mathcal{O}(\phi^2). \tag{6}$$

To evaluate this, we have noted that, with the exception of $a = 24$, $\mathrm{Tr}T^aT^{24} = 0$. This expression vanishes. We can go on and evaluate the quartic terms, and establish that this is actually a minimum. In fact, this is rather trivial, and we just get $\frac{\lambda}{4}(\phi^a\phi^a)^2$.

The vanishing of the masses is in a sense an accident. There is another term we could have added to the potential:

$$\delta V = \lambda' \mathrm{Tr}\Phi^4 \tag{7}$$

This modifies the expression for v, and also for the masses, which no longer vanish.

3. Compute the coupling constant unification, i.e. the mass where the $SU(2)$ and $SU(3)$ couplings unify, starting with their values at M_Z; from which compute the $U(1)$ coupling at low energies.

   Solution: The main ingredients here are:

   a. The equation for the evolution of the couplings:

   $$\alpha_i(M_Z)^{-1} = \alpha_i(M)^{-1} + \frac{b_0^{(i)}}{4\pi}\log(M_Z/M). \tag{8}$$

   b. One wants to remember that the properly normalized hypercharge generator, within $SU(5)$ (the generator T^{24} of the previous exercise) is related by a factor of $\frac{1}{2}\sqrt{\frac{3}{5}}$.

   The rest is just the solution of linear equations for M ($\log(M)$) and one of the three α's, in terms of M_Z and two of the α's.

1 The Chiral Anomaly

Before considering real QCD, consider a non-abelian gauge theory theory, with only a single flavor of quark. Before making any field redefinitions, the lagrangian takes the form:

$$\mathcal{L} = -\frac{1}{4g^2} F^2_{\mu\nu} + \bar{q} D^\mu \sigma_\mu \bar{q}^* + q D^\mu \sigma_\mu q^* m \bar{q} q + m^* \bar{q}^* q^*. \tag{9}$$

The lagrangian, here, is written in terms of two-component fermions (see appendix**??**). The fermion mass need not be real,

$$m = |m| e^{i\theta}. \tag{10}$$

In this chapter, it will sometimes be convenient to work with four component fermions, and it is valuable to make contact with this language in any case. In terms of these:

$$\mathcal{L} = \mathrm{Re\ m}\ \bar{q} q + \mathrm{Im\ m}\ \bar{q} \gamma_5 q. \tag{11}$$

In order to bring the mass term to the conventional form, with no γ_5's, one could try to redefine the fermions; switching back to the two component notation:

$$q \to e^{-i\theta/2} q \qquad \bar{q} \to e^{-i\theta/2} \bar{q}. \tag{12}$$

But in field theory, transformations of this kind are potentially fraught with difficulties because of the infinite number of degrees of freedom.

A simple calculation uncovers one of the simplest manifestations of an anomaly. Suppose, first, that m is very large, $m \to M$. In that case we want to integrate out the quarks and obtain a low energy effective theory. To do this, we study the path integral:

$$Z = \int [dA_\mu] \int [dq][d\bar{q}] e^{iS} \tag{13}$$

Again suppose $M = e^{i\theta}|M|$. In order to make m real, we can again make the transformations: $q \to q e^{-i\theta/2}; \bar{q} \to \bar{q} e^{-i\theta/2}$ (in four component language, this is $q \to^{-i\theta/2\gamma_5} q$).) The result of integrating out the quark, i.e. of performing the path integral over q and $\bar{q}$ can be written in the form:

$$Z = \int [dA_\mu] \int e^{iS_{eff}} \tag{14}$$

Here S_{eff} is the effective action which describes the interactions of gluons at scales well below M.

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Figure 1: The triangle diagram associated with the four dimensional anomaly.

Because the field redefinition which eliminates θ is just a change of variables in the path integral, one might expect that there can be no θ-dependence in the effective action. But this is not the case. To see this, suppose that θ is small, and instead of making the transformation, treat the θ term as a small perturbation by expanding the exponential. Now consider a term in the effective action with two external gauge bosons. This is obtained from the Feynman diagram in fig. **??**. The corresponding term in the action is given by

$$\delta\mathcal{L}_{eff} = -i\frac{\theta}{2} g^2 M Tr(T^a T^b) \int \frac{d^p k}{(2\pi)^4} \mathrm{Tr}\gamma_5 \frac{1}{\not{p} + \not{k}_1 - M} \not{\epsilon}_1 \frac{1}{\not{p} - M} \not{\epsilon}_2 \frac{1}{\not{p} - \not{k}_2 - M}. \tag{15}$$

Here, the k_i's are the momenta of the two photons, while the ϵ's are their polarizations and a and b are the color indices of the gluons. Introducing Feynman parameters and shifting the p integral, gives:

$$\delta\mathcal{L}_{eff} = -i\theta g^2 M \mathrm{Tr}(T^a T^b) \int d\alpha_1 d\alpha_2 \int \frac{d^4 p}{(2\pi)^4} \mathrm{Tr}\gamma_5 (\not{p} - \alpha_1 \not{k}_1 + \alpha_2 \not{k}_2 + \not{k}_1 + M) \not{\epsilon}_1 \tag{16}$$

$$\times \frac{(\not p - \alpha_1 \not k_1 + \alpha_2 \not k_2 + M) \not\epsilon_2 (\not p - \alpha_1 \not k_1 + \alpha_2 \not k_2 - \not k_2 + M)}{(p^2 - M^2 + O(k_i^2))^3}.$$

For small k_i, we can neglect the k-dependence of the denominator. The trace in the numerator is easy to evaluate, since we can drop terms linear in p. This gives, after performing the integrals over the α's,

$$\delta\mathcal{L}_{eff} = g^2 M^2 \theta \mathrm{Tr}(T^a T^b) \epsilon_{\mu\nu\rho\sigma} k_1^\mu k_2^\nu \epsilon_1^\rho \epsilon_2^\sigma \int \frac{d^4p}{(2\pi)^4} \frac{1}{(p^2 - M^2)^3}. \tag{17}$$

This corresponds to a term in the effective action, after doing the integral over p and including a combinatoric factor of two from the different ways to contract the gauge bosons:

$$\delta\mathcal{L}_{eff} = \frac{1}{32\pi^2} \theta \mathrm{Tr}(F\tilde{F}). \tag{18}$$

Now why does this happen? At the level of the path integral, the transformation would seem to be a simple change of variables, and it is hard to see why this should have any effect. On the other hand, if one examines the diagram of fig. **??**, one sees that it contains terms which are linearly divergent, and thus it should be regulated. A simple way to regulate the diagram is to introduce a Pauli-Villars regulator, which means that one subtracts off a corresponding amplitude with some very large mass Λ. However, our expression above is independent of Λ. So the θ-depedence from the regulator fields cancels that of eqn. **??**. This sort of behavior is characteristic of an anomaly.

Consider now the case that $m \ll \Lambda_{QCD}$. In this case, we shouldn't integrate out the quarks, but we still need to take into account the regulator diagrams. So if we redefine the fields so that the quark mass is real (γ_5-free, in the four-component description), the low energy theory contains light quarks and the θ term of eqn. [**??**].

We can describe this in a fashion which indicates why this is referred to as an *anomaly.* For small m, the classical theory has an approximate symmetry under which

$$q \to e^{i\alpha} q \qquad \bar{q} \to e^{i\alpha} \bar{q} \tag{19}$$

(in four component language, $q \to e^{i\alpha\gamma_5} q$). In particular, we can define a current:

$$j_5^\mu = \bar{q}\gamma_5\gamma_\mu q, \tag{20}$$

and classically,

$$\partial_\mu j_5^\mu = m\bar{q}\gamma_5 q. \tag{21}$$

Under a transformation by an infinitesimal angle α one would expect

$$\delta L = \alpha \partial_\mu j_5^\mu = m\alpha\bar{q}\gamma_5 q. \tag{22}$$

But the divergence of the current contains another, m-independent, term:

$$\partial_\mu j_5^\mu = m\bar{q}\gamma_5 q + \frac{1}{32\pi^2} F\tilde{F}. \tag{23}$$

The first term just follows from the equations of motion. To see that the second term is present, we can study a three-point function involving the current and two gauge bosons, ignoring the quark mass:

$$\Gamma^{AAj} = T < \partial_\mu j^{5\ \mu} A_\rho A_\sigma > \tag{24}$$

This is essentially the calculation we encountered above. Again, the diagram is linearly divergent and requires regularization. Let's first consider the graph without the regulator mass. The graph is actually two graphs, because we must include the interchange of the two external gluons. The combination is easily seen to vanish, by the sorts of manipulations one usually uses to prove Ward identities:

$$\frac{g^2}{(2\pi)^4}\int d^4p\mathrm{Tr}\ \not{q}\gamma_5\frac{1}{\not{p}+\not{k}_1}\ \not{\epsilon}_1\frac{1}{\not{p}}\ \not{\epsilon}_2\frac{1}{\not{p}-\not{k}_2}+(1\leftrightarrow 2). \tag{25}$$

Writing

$$\not{q}\gamma_5=-\gamma_5(\not{k}_1+\not{p})-(\not{p}-\not{k}_2)\gamma_5 \tag{26}$$

and using the cyclic property of the trace, one can cancel a propagator in each term. This leaves:

$$\int d^4p\mathrm{Tr}(-\gamma_5\ \not{\epsilon}_1\frac{1}{\not{p}}\ \not{\epsilon}_2\frac{1}{\not{p}-\not{k}_2}-\gamma_5\frac{1}{\not{p}+\not{k}_1}\ \not{\epsilon}_1\frac{1}{\not{p}}\ \not{\epsilon}_2+(1\leftrightarrow 2) \tag{27}$$

Now shifting $p\rightarrow p+k_2$ in the first term, and $p\rightarrow p+k_1$ in the second, one finds a pairwise cancellation.

These manipulations, however, are not reliable. In particular, in a highly divergent expression, the shifts do not necessarily leave the result unchanged. With a Pauli-Villars regulator, the integrals are convergent and the shifts are reliable, but the regulator diagram is non-vanishing, and gives the anomaly equation above. One can see this by a direct computation, or relate it to our previous calculation, including the masses for the quarks, and noting that $\not{q}\gamma_5$, in the diagrams with massive quarks, can be replaced by $M\gamma_5$.

This anomaly can be derived in a number of other ways. One can define, for example, the current by "point splitting,"

$$j_5^\mu=\bar{q}(x+i\epsilon)e^{i\int_x^{x+\epsilon}dx^\mu A_\mu}q(x) \tag{28}$$

Because operators in quantum field theory are singular at short distances, the Wilson line makes a finite contribution. Expanding the exponential carefully, one recovers the same expression for the current. We will do this shortly in two dimensions, leaving the four dimensional case for the problems. A beautiful derivation, closely related to that we have performed above, is due to Fujikawa, described in [**?**]. Here one considers the anomaly as arising from a lack of invariance of the path integral measure. One carefully evaluates the Jacobian associated with the change of variables $q\rightarrow q(1+i\gamma_5\alpha)$, and shows that it yields the same result[**?**]. We will do a calculation along these lines in a two dimensional model shortly, leaving the four dimensional case for the problems.