Physics 222. Quantum Field Theory 3. Professor Dine

Spring, 2011. Homework Set 5. **SOLUTIONS**.

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1. Consider a supersymmetric version of the $SU(5)$ grand unified theory. Take Σ to be a chiral superfield in the adjoint representation, and take the superpotential to be

$$W(\Sigma) = m\mathrm{Tr}\Sigma^2 + \frac{\lambda}{3}\mathrm{Tr}\Sigma^3. \tag{1}$$

Verify that there are (up to gauge transformations) three stationary points:

$$\Sigma = 0;\ \Sigma = \frac{m}{\lambda}\mathrm{diag}(1,1,1,1,-4);\ \Sigma = \frac{m}{\lambda}\mathrm{diag}(2,2,2,-3,-3) \tag{2}$$

What is the gauge symmetry in each of these vacua?

Solution: There are a couple of ways to proceed. One is to assume one of the forms above. Write

$$\Sigma = \Sigma_0 + \sigma^a T^a. \tag{3}$$

As in the exercise in the previous problem set, the remaining solution is enough to guarantee that $\frac{\partial W}{\partial \sigma^a}$ vanishes for all but the generator in the putative vev. So it is enough to check for the particular case, as each of you did.

Another approach is to introduce a lagrange multiplier. I.e. one stationarizes

$$\frac{m}{2}\mathrm{Tr}\ \Sigma^2 + \frac{\lambda}{3}\mathrm{Tr}\ \Sigma^3 + \ell\mathrm{Tr}\ \Sigma \tag{4}$$

The ℓ equation enforces the tracelessness condition. Taking the trace of the Σ equation gives:

$$\ell = -\frac{\lambda}{5}\mathrm{Tr}\ \Sigma^2. \tag{5}$$

So the equation for the stationary point is:

$$m\Sigma + \lambda(\Sigma^2 - \frac{1}{5}\mathrm{Tr}\ \Sigma^2) = 0 \tag{6}$$

which is solved by the two types of matrices above (and a vanishing matrix).

2. Consider a $U(1)$ gauge theory, with a neutral field, X, and two charged fields, $\phi^\pm$.

a. Show that the D terms vanish if $\phi^+ = \phi^- = v$ in the vacuum, i.e. that there is a one complex parameter set of vacuum states.

Solution: This one is obvious.

b. For fixed v, compute the spectrum. Basically you should find a massive gauge field, a massive Dirac fermion, arising from the Yukawa couplings between the gaugino and the fermionic components of ϕ^+ and ϕ^- ($g\sqrt{2}\lambda(\phi^{+*}\psi^+ - \phi^{-*}\psi^-)$, and one more massive scalar. This scalar arises from expanding D about the vacuum; you should find

$$D \propto v\Phi$$

where Φ is a (real) scalar field; the square of this is a mass term for Φ.

Solution: Gauge bosons: kinetic terms for gauge fields have a factor of $1/2$. From the kinetic terms for the scalars, one gets $e^2 \times 2v^2$. So the mass-squared is $4e^2v^2$. Gauginos: There is a factor of $\sqrt{2}v$ in the coupling of gauginos to fermions, so the mass term is

$$\sqrt{2}ev\lambda(\psi^+ - \psi^-). \tag{7}$$

Normalizing the fermion, gives a mass term $2ev$. Scalars: Note that $D = v(\phi^+ + \phi^{+*} - \phi^- - \phi^{-*})$. Again one needs to normalize the fields. We have

$$D = v\sqrt{2}\left(\frac{1}{\sqrt{2}}(2\mathrm{Re}\phi^+ - 2\mathrm{Re}\phi^-)\right) \tag{8}$$

Recalling that we started with *complex* scalars, and remembering the $1/2$ in front of the D^2 term, we obtain $m^2 = 4e^2v^2$.