Goldstone Bosons and Chiral Symmetry Breaking in QCD

Physics 222 2011, Advanced Quantum Field Theory

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Before reading this handout, carefully read Peskin and Schroeder's section 7.1

It is easy to prove Goldstone's theorem for theories with fundamental scalar fields. But the theorem is more general, and some of its most interesting applications are in theories without fundamental scalars. We can illustrate this with QCD. In the limit that there are two massless quarks (i.e. in the limit that we neglect the mass of the *u* and *d* quarks), we can write the QCD lagrangian in terms of four-component spinors

$$q = \begin{pmatrix} u \\ d \end{pmatrix}$$

as

$$\mathcal{L} = \bar{q} i \gamma^\mu D_\mu q - \frac{1}{4} F_{\mu\nu}^2 .$$

This lagrangian has symmetries:

$$\Psi \to e^{i\omega^a \frac{\tau^a}{2}} \Psi \qquad \Psi \to e^{i\omega^a \frac{\tau^a}{2} \gamma^5} \Psi$$

(τ^a are the Pauli matrices). In the limit that two quarks are massless, QCD is thus said to have the symmetry $SU(2)_L \times SU(2)_R$.

Exercise: check; make sure you understand what role is played by all of the indices– there are indices for color, flavor, and lorentz transformations.

We can write the theory in terms of two component fermions, so that the symmetry is manifest. Writing a general four-component fermion as

$$q = \begin{pmatrix} q \\ \bar{q}^* \end{pmatrix},$$

the lagrangian has the form:

$$\mathcal{L} = iq\sigma^\mu D_\mu q^* + i\bar{q}\sigma^\mu D_\mu \bar{q}^*$$

Again, it is important to stress that q and $\bar{q}$ are both left-handed fermions. q annihilates quarks and creates (right-handed) andtiquarks; $\bar{q}$ annihilates (left-handed) anti-quarks and creates (right-handed) quarks.

In this form, we have two separate symmetries:

$$q \to e^{i\omega_L^a \frac{\tau^a}{2}} q; \qquad \bar{q} \to e^{i\omega_R^a \frac{\tau^a}{2}} \bar{q}$$

Written in this way, it is clear why the symmetry is called $SU(2)_L \times SU(2)_R$

Now it is believed that in QCD, the operator $\bar{\Psi}\Psi$ has a non-zero vacuum expectation value, i.e.

$$< \bar{\Psi}\Psi > \approx (0.3\mathrm{GeV})^3 \delta_{ff'}.$$

(This is in four component language; in two component language this is:

$$< \bar{\Psi}_f \Psi_{f'} + \bar{\Psi}^*_f \Psi^*_{f'} > \neq 0.)$$

It is easy to see that this leaves unbroken ordinary isospin, the transformation, in four component language, without the γ_5, or in two component language, with $\omega^a_L = -\omega^a_R$.

But there are three broken symmetries. Correspondingly, we expect that there are three Goldstone bosons. How can we prove this statement? Call

$$\mathcal{O} = \bar{\Psi}\Psi; \mathcal{O}^a = \bar{\Psi}\gamma^5 \frac{\tau^a}{2}\Psi.$$

Under an infinitesmal transformation,

$$\delta\mathcal{O} = 2i\omega^a \mathcal{O}^a; \delta\mathcal{O}^a = i\omega^a \mathcal{O}.$$

Exercise: Check this (it's easy; just expand out the transformation matrices to first order in ω^a).

In the quantum theory, these becomes the commutation relations:

$$[Q^a, \mathcal{O}] = 2i\mathcal{O}^a; [Q^a, \mathcal{O}^b] = i\omega\delta^{ab}\mathcal{O}.$$

Now if we take the expectation value of both sides of this relation, we see that there is some sort of statement about the matrix elements of the current. To see precisely what this is, and that there must be a massless particle, we study

$$0 = \int d^4x \partial_\mu [< \Omega | T(j^{\mu a}(x)\mathcal{O}^b(0)) | \Omega > e^{-ip\cdot x}]$$

(this just follows because the integral of a total derivative is zero).

We can evaluate the right hand side, carefully writing out the time-ordered product in terms of θ functions, and noting that ∂_o on the θ functions gives δ-functions:

$$\int d^4x < \Omega|[j^{oa}(x), \mathcal{O}^b(0)]\delta(x^o)|\Omega > e^{-ip\cdot x}$$

$$-ip_\mu \int d^4x < \Omega|T(j^{\mu a}(x)O^b(0))|\Omega > .$$

Exercise: Check this.

Now consider the limit $p^\mu = 0$. The first term on the right hand side becomes the matrix element of $[Q^a, \mathcal{O}^b(0)] = \mathcal{O}(0)$. This is non-zero. The second term must be singular, then, if the equation is to hold. This singularity, as we will now show, requires the presence of a massless particle. As in Peskin and Schroder's discussion of the spectral function in chapter 7, to study this term we introduce a complete set of states, and, say for $x^o > 0$, write it as

$$\sum_\lambda \int \frac{d^3q}{2E_q(\lambda)} < \Omega | j^{\mu a}(x) | \lambda_q >$$

We need a pole at zero momentum, in order to cancel the p_μ. From the structure of the spectral representation, such a pole can only arise if there is a massless particle. We call this particle π^b. On Lorentz-invariance grounds,

$$< \Omega | j^{\mu a} | \pi^b >= f_\pi p^\mu \delta^{ab}.$$

Call

$$< \lambda_q | \mathcal{O}^a(x) | \pi^b >= Z \delta^{ab}.$$

Considering the other time ordering, we obtain for the left hand side a massless scalar propagator, $\frac{i}{p^2}$, multiplied by $Zf_\pi p^\mu$, so the equation is now consistent:

$$< \bar{\Psi}\Psi >= \frac{p^2}{p^2} f_\pi Z.$$

It is easy to see that in this form, Goldstone's theorem generalizes to any theory without fundamental scalars in which a global symmetry is spontaneously broken.

Returning to QCD, what about the fact that the quarks are massive? The quark mass terms break explicitly the symmetries. But if these masses are small, we should be able to think of the potential as “tilted." This gives rise to a small mass for the pions. We can estimate this mass by considering the symmetry-breaking terms in the lagrangian:

$$\mathcal{L}_{sb} = \bar{\Psi} M \Psi$$

where M is the “quark mass matrix",

$$M = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix} .$$

Since the π mesons are, by assumption, light, we can focus on these. If we have a non-zero pion field, we can think of the fermions as being given by:

$$\Psi = e^{i\frac{\pi^a}{f_\pi}\gamma^5\frac{\tau^a}{2}}\Psi.$$

In other words, the pion fields are like symmetry transformations of the vacuum (and everything else).

Now assume that there is an “effective interaction" for the pions containing kinetic terms $\frac{1}{2}(\partial_\mu \pi^a)^2$. Assuming the form above for Ψ, the pions get a potential from the fermion mass terms. To work out this potential, one plugs this form for the fermions into the lagrangian and replaces the fermions by their vacuum expectation value. This gives that

$$V(\pi) = (0.3\ \mathrm{GeV})^3 \mathrm{Tr}(e^{i\omega^2 \gamma_5 \tau^a} M)$$

one can expand to second order in the pion fields, obtaining:

$$m_\pi^2 f_\pi^2 = (m_u + m_d)(0.3)^3.$$

We can describe this by introducing a “non-linear chiral lagrangian". Take the fields to be described by a unitary matrix

$$\Sigma(x) = e^{\frac{i\pi^a\tau^a}{2f_\pi}}$$

Σ transforms as

$$\Sigma \to U_L \Sigma U_R^\dagger$$

Including only terms with two derivatives, the lagrangian is uniquely fixed by the symmetry:

$$\mathcal{L} = f_\pi^2 \mathrm{Tr}(\partial_\mu \Sigma^\dagger \partial^\mu \Sigma).$$

Expanding to second order in the pion fields gives ordinary kinetic terms; at higher orders we obtain derivative interactions.

Now the quark mass terms break the symmetry, but if we pretend that the mass terms are fields, the lagrangian is invariant provided we transform the mass terms as well

$$\mathcal{L}_m = \Lambda^2 \mathrm{Tr}(\Sigma M)$$

Exercise: Derive this pion mass formula, known as the Gell-Mann, Oakes, Renner formula, by considering the non-linear lagrangian.