Physics 222. Group Theory Primer. Professor Dine

In this note, we will focus mainly on special unitary groups, $SU(N)$; later we will discuss orthogonal and possibly other groups.

1 Some Features of $SU(2)$

Physicists are typically interested in *representations* of groups. For $SU(N)$, the fundamental representation is an N component complex object, denoted by N or its conjugate $\bar{N}$. Denoting these by Q and $\bar{Q}$, respectively, we have

$$Q' = U(\omega)Q \qquad \bar{Q}' = U^*(\omega)\bar{Q}. \tag{1}$$

For $SU(2)$,

$$U(\omega) = e^{i\frac{\omega^a \sigma^2}{2}} \equiv e^{i\omega^a T^a}. \tag{2}$$

Particularly important are the infinitesimal transformations:

$$U = 1 + i\omega^a T^a \tag{3}$$

In $SU(2)$, the 2 and $\bar{2}$ representations are equivalent; $SU(2)$ is said to be pseudoreal. This is not true for more general $SU(N)$. For $SU(2)$, this can be seen by exploiting the peculiar properties of σ_2. In particular,

$$\hat{Q} = \sigma_2 Q^* \tag{4}$$

transforms like Q:

$$\begin{aligned}\hat{Q}' &= \sigma_2 U^*(\omega)\sigma_2\sigma_2 Q^* \\ &= \sigma_2 e^{-i\frac{\omega^a \sigma^{a*}}{2}}\sigma_2 \hat{Q}\end{aligned} \tag{5}$$

but

$$\sigma_2 e^{-i\frac{\omega^a \sigma^{a*}}{2}}\sigma_2 = e^{i\frac{\omega^a \sigma^a}{2}} \tag{6}$$

which follows from the fact that $\sigma_2\sigma_a^*\sigma_2 = -\sigma^a$ (**check these statements!**).

Some features of $SU(2)$ generalize to $SU(N)$ (some to all Lie groups):

1. In the fundamental representation, one can take

$$\mathrm{Tr}(T^a T^b) = \frac{1}{2}\delta_{ab} \equiv C(F)\delta_{ab}. \tag{7}$$

2. The commutation relations of the group generators characterize the (local properties) of the group, and are independent of the representation:

$$[T^a, T^b] = i f^{abc} T^c. \tag{8}$$

The structure constants, f^{abc} (ϵ^{abc} in the case of $SU(2)$) are completely antisymmetric. They satisfy an identity, derived from the associative property of the commutator, known as the Jacobi identity (see, e.g., Peskin and Schroder, chapter 16).

These features are familiar from the theory of angular momentum ($T^a \to J^a$).

2 The Fundamental Representation of $SU(3)$

We can write the T^a's as 3×3 matrices and readily obtain a basis with features analogous to those of the Pauli matrices. Writing

$$T^a = \frac{\lambda^a}{2} \tag{9}$$

we can first embed $SU(2)$ simply. E.g. think of

$$Q = \begin{pmatrix} u \\ d \\ s \end{pmatrix} \tag{10}$$

then the up and down quarks form an $SU(2)$ doublet. So there is a natural choice of the matrices $T^a = \frac{\lambda^a}{2}$:

$$\lambda^i = \begin{pmatrix} \sigma^i & 0 \\ 0 & 0 \end{pmatrix}. \tag{11}$$

For the rest, we can take

$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{12}$$

$$\lambda^5 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \tag{13}$$

and two more, similar matrices (for a total of seven). The eighth matrix can be taken to be:

$$\lambda^8 = \lambda^4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{14}$$

These can be thought of as doublets (two) of the $SU(2)$, and a singlet (if this is not clear, try and understand why after you read the section about the adjoint representation, below).

3 The Adjoint Representation

The adjoint representation of a general Lie group can be described quite simply. Start with some representation (e.g. spin 1/2 of $SU(2)$). The generators in the representation are T^a. Take

$$\mathrm{Tr}(T^a T^b) = c_1 \delta_{ab}$$

Now consider an object, A, which satisfies the transformation law (the T^a's are hermitian, and we are specializing to $SU(N)$ groups to write the equations):

$$A \to g A g^\dagger = e^{i\omega^a T^a} A e^{-i\omega^a T^a}$$

For infinitesimal transformations, this becomes:

$$\delta A = i\omega^a [T^a, A]$$

This is transformation law for the adjoint representation.

To see this, write

$$A = \sum A^a T^a$$

. Then

$$A^a = \frac{1}{c_1}\mathrm{Tr}(T^a A)$$

So

$$\begin{aligned}\delta A^a &= \frac{1}{c_1} i\omega^b \mathrm{Tr}([T^b, A]T^a) \\ &= -\omega^b f^{bcd}\mathrm{Tr}(T^d T^a)A^c \\ &= -f^{abc}\omega^b A^c\end{aligned}$$

which is the infinitesimal transformation law for A^a.

This is general, for any group. It is useful for formulating Yang-Mills theory, and explains the homogeneous part of the transformation law, or equivalently the part for constant gauge transformations,

$$A_\mu \to g A_\mu g^\dagger.$$

Exercise: For the rotation group, the vector representation is the adjoint representation. Define the matrix:

$$X = \vec{x} \cdot \vec{\sigma}$$

Check that for a rotation about the z axis, the various components of this matrix transform properly, using the transformation law above, i.e. writing

$$X \to e^{i\vec{\omega}\cdot\vec{\sigma}/2} X e^{-i\vec{\omega}\cdot\vec{\sigma}/2}$$

with $\vec{\omega} = \omega\hat{z}$.

4 Connections to Field Theory

For the quarks, we can define an $SU(3)$ symmetric lagrangian (in the sense of Gell-Mann's eight-fold way) by writing:

$$\mathcal{L} = i\bar{Q}\,\partial\!\!\!/ Q - \bar{Q}mQ \tag{15}$$

If m is proportional to the unit matrix, then this lagrangian respects the full $SU(3)$ symmetry (check if it isn't obvious!). If the mass matrix is more complicated, the mass terms violate the symmetry.

We can introduce the mesons in a simple way. Treat the mesons as *matrix valued fields*,

$$\Pi = \Pi^a T^a \tag{16}$$

i.e. eight fields. These transform in the adjoint of $SU(3)$,

$$\Pi \to U\Pi U^\dagger \tag{17}$$

So a Yukawa coupling:

$$\mathcal{L}_{yuk} = y\bar{Q}\Pi Q \tag{18}$$

is invariant. The eight fields in Π correspond to the eight fields of the meson octet. Based on our discussion of the previous section, you should be able to identify an $SU(2)$ triplet (the pi mesons), two doublets (the K mesons), and a singlet (the η).