An Introduction to QCD

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e^+e^- Annihilation

The total cross section in e^+e^- annihilation is a particularly simple object to study. It is a function of only one dimensional quantity, s. So, provided it is infrared finite, it can only involve $\alpha_s(\mu)$, and $\log(s/\mu^2)$, where μ is a renormalization scale. Simply choosing $s = \mu^2$, gives that

$$R = \frac{\sigma(e^+e^- \to \text{ hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = \sum_f Q_f^2 (1 + \sum_{n=1}^{\infty} c_n \left(\frac{\alpha_s(s)}{4\pi}\right)^n). \quad (1)$$

Establishing the finiteness of the total *inclusive* cross section is easy once one establishes the connection of this quantity to the vacuum polarization tensor. This is a consequence of unitarity, as expressed in Peskin and Schroeder's equation 7.49,

$$2\mathrm{Im}\mathcal{M}(a \to a) = \sum_f \int d\Pi_f \mathcal{M}^*(a \to f)\mathcal{M}(a \to f). \qquad (2)$$

In this case, the process $a \to f$ corresponds to the amplitude to start with the vacuum state and produce quarks, antiquarks, etc. by the action of the electromagnetic current.

$$\sigma(e^+e^- \to \mathrm{hadrons}) = \frac{16\pi^3\alpha^2}{s}\mathrm{Im}\Pi(s). \qquad (3)$$

The first term can be read off from QED calculations; the second was calculated by yours truly and others a long time ago[**?**]. Specifying this term requires precisely defining what is meant by α_s; redefinitions of the form

$$\alpha_s = \alpha_s' + a\alpha_s^2 + \ldots \tag{4}$$

change the expression at higher orders. Such redefinitions correspond to different choices of “renormalization scheme."

Finiteness of $\Pi(q^2)$

The infrared finiteness of $\Pi(q^2)$ follows from general principles of quantum field theory. Indeed, we are familiar from our studies of Feynman graphs, that if all momenta are Euclidean, one can Wick rotate so as to obtain well-behaved Euclidean integrals. By simple power counting arguments one can see that in four dimensions there are no infrared divergences. One can then *analytically continue* in q^2 to timelike momenta (Feynman parameters are analytic functions of the momenta, essentially because they are well-behaved for all Euclidean momenta, including complex; study some simple graphs to understand how this works).

So the discontinuity in Π is also well behaved (it just arrives from the discontinuity of terms like $(\log(-q^2))^n$ upon continuation to timelike q^2.

It is interesting to understand how discontinuities come about in a quantity like $\Pi(q^2)$. Consider, for simplicity, a scalar field theory with ϕ^3 interactions, and study the analog of the vacuum polarization. Take the scalar to have mass m. Then if $q^2 > 4m^2$, the momenta are such that the two internal lines can be on shell; this is precisely when a discontinuity arises. (This is the generalization of the statement that for $q^2 < 0$ one can perform a Wick rotation, avoiding consideration of $i\epsilon$ terms in propagators, and thus any imaginary part in the diagram).

Consider how the propagators behave near their singularities. Writing

$$\frac{1}{k^2 - m^2 + i\epsilon} = \frac{1}{(k^0 - E_k + i\epsilon)(k^2 + E_k - i\epsilon)}, \tag{5}$$

consider the second factor. This is something real plus something imaginary times a δ-function:

$$\int \frac{dk^0}{k^0 - E_k + i\epsilon} = \frac{1}{2}\left(\int \frac{dk^0}{k^0 - E_k + i\epsilon} + \int \frac{dk^0}{k^0 - E_k - i\epsilon}\right) \tag{6}$$

$$`+ \frac{1}{2}\left(\int \frac{dk^0}{k^0 - E_k + i\epsilon} - \int \frac{dk^0}{k^0 - E_k - i\epsilon}\right)$$

The second integral can be done as a contour integral; the first is real, and is called the “Principle Value".

So we have:

$$\frac{1}{k^0 - E_k \pm i\epsilon} = P\frac{1}{k^0 - E_k} \mp i\delta(k^0 - E_k) \tag{7}$$

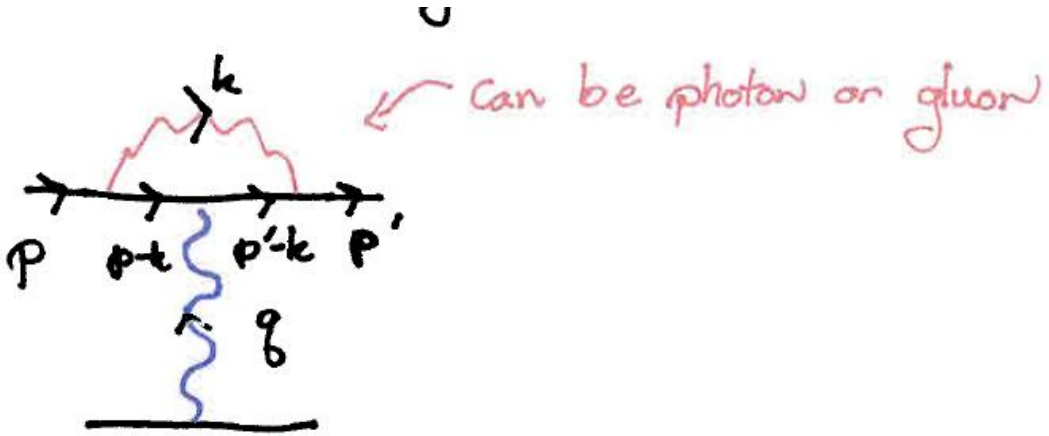
leading to Peskin and Schroeder's equation 7.55:

$$\frac{1}{k^2 - m^2 + i\epsilon} = \mathrm{real} - 2\pi i\delta(k^2 - m^2) \tag{8}$$

This, in turn, leads to the "Cutkosky rules" (p. 236), relating the imaginary parts of individual diagrams to particular contributions to cross sections.

The Infrared Problem in QCD

Let's look at the low momentum behavior of the Feynman diagram for the vertex.



As $k \to 0$, one has

$$\Gamma^\mu \approx -e^3 \int \frac{d^4k}{(2\pi)^4} \frac{\gamma^\rho(\not{p}' + m)\gamma^\mu(\not{p} + m)\gamma_\rho}{k^2(-2p\cdot k + i\epsilon)(-2p'\cdot k + i\epsilon)} \tag{9}$$

$$\approx -4e^3 \int \frac{d^4k}{(2\pi)^4} \frac{p^{\rho\prime} p_\rho \gamma^\mu}{k^2(2p\cdot k)(2p'\cdot k)}$$

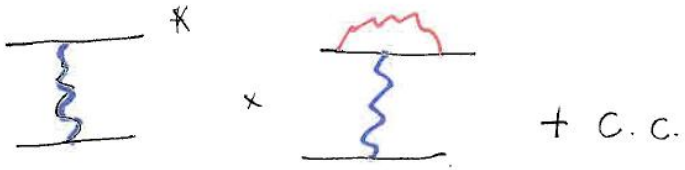
(Dirac eqn. in last step).

Do k^0 integral; close below, so only pick up pole from first propagator.

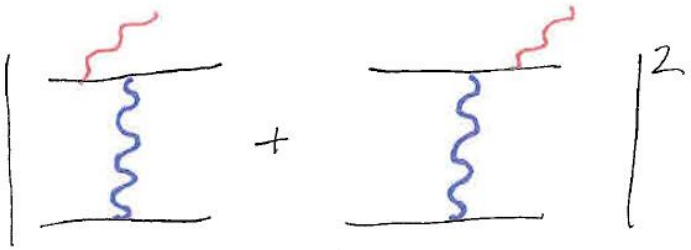
$$\Gamma^\mu = -i\gamma^\mu \left(4e^2 \int \frac{d^3k}{(2\pi)^3} \frac{p \cdot p'}{2|\vec{k}|(2p \cdot k)(2p' \cdot k)} \right). \tag{10}$$

This diverges as $k \to 0$. No excuses! What cancels?

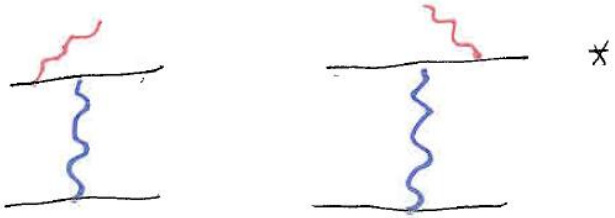
Study cross section:



Compare:



Focus on interference:



$$\mathcal{M}^{(1)} = e^2 \frac{\bar{u}(p')\gamma^\mu(\not{p} - \not{k} + m)\,\not{\epsilon}(k)u(p)}{-2p\cdot k} \times [\text{stuff}] \tag{11}$$

$$= e^2 \frac{2p\cdot\epsilon}{p\cdot k}\bar{u}(p')\gamma^\mu u(p) \times [\text{stuff}]$$

$$= e^2 \frac{p\cdot\epsilon}{p\cdot k} \times \mathcal{M}^{(0)}.$$

Here $\mathcal{M}^{(0)}$ is the zeroth order amplitude for the process without the photon. Similarly

$$\mathcal{M}^{(2)} = e^2 \frac{p'\cdot\epsilon}{p'\cdot k}\bar{u}(p')\gamma^\mu u(p) \times [\text{stuff}] \tag{12}$$

$$= e^2 \frac{p'\cdot\epsilon}{p'\cdot k} \times \mathcal{M}^{(0)}.$$

Squaring (taking interference term) summing over photon polarizations

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_0 e^2 \int \frac{d^3k}{(2\pi)^3} \frac{2p \cdot p'}{(p \cdot k)(p' \cdot d)} \frac{1}{2|\vec{k}|}. \tag{13}$$

Jets in e^+e^- Annihilation

The cancelation of infrared divergences in e^+e^- annihilation is readily understood. Cross sections for particular final states are divergent, but there are cancelations between vertex corrections and soft photon emissions when one considers the inclusive cross section.

Exclusive processes are more subtle. If we consider production of a fixed number of quarks and gluons, this will diverge. But the lesson of the QED infrared cancelation is that we need to be careful what questions we ask. Indeed, this is presumably a good thing, since we don't see final state quarks or gluons, so if such computations were sensible, we would clearly contradict experiment. Instead, we can ask questions like how much energy flows into a cone in a particular solid angle, for example. This leads to notion of *jets*.

In e^+e^- annihilation, want to include soft emissions, and (nearly) colinear emission in the definition of jets. For suitable choice, no small invariants, and still have an expansion in $\alpha_s(s)$.

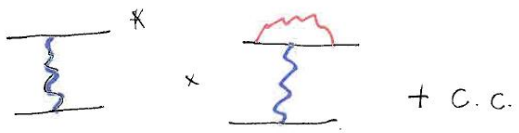
The Parton Model

The parton model originated with Feynman. (The term “parton" seems to have been intended to avoid speaking of quarks and gluons; this may have resulted from his rivalry with Gell-Mann.) Consider two protons colliding at extremely high energy, so that their mass can be neglected (Feynman and others introduced the notion of an “infinite momentum frame"). Idea is that in a hadron, there is a probability of finding a quark or gluon carrying a fraction x of the total longitudinal momentum. At the same time, the transverse momentum, p_t, was assumed to be limited to some characteristic QCD number (consistent with data, which shows rapid fall off of production with p_t. We will discuss these assumptions later. One of the reasons that people were so interested in the possibility of asymptotic freedom is that these phenomena do not occur in non-asymptotically free theories.

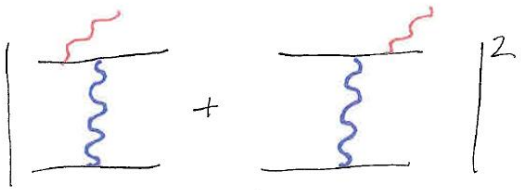
Deep Inelastic Scattering

The prototype for the parton model is deep inelastic scattering. Here one considers scattering of an electron or neutrino off of a nucleon at large momentum transfer. One treats the scattering as scattering off of individual quarks and gluons within the proton, assuming that they carry longitudinal momentum fraction x and essentially zero transverse momentum. The probability for a parton of type i to carry momentum x is given by the “Parton distribution function", $f_i(x)$ (PDF).

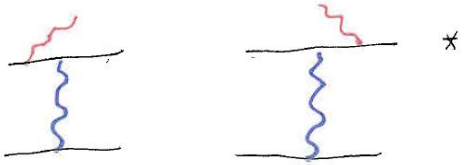
Study cross section:



Compare:



Focus on interference:



We need, first, the cross section for the scattering of a (virtual) photon of momentum $q^2 = -Q^2$ off a quark. Calling the invariants of the “parton level" scattering $\hat{s}$, $\hat{t}$, and $\hat{u}$, this is given by:

$$\frac{d\sigma}{dt}(e^- q \to e^- q) = \frac{2\pi\alpha^2 Q_i^2}{\hat{s}^2}\left[\frac{\hat{s}^2 + \hat{u}^2}{\hat{t}^2}\right]. \tag{14}$$

We need to express the invariants in terms of x and measurable quantities. The actual cross section is then obtained by convoluting the parton cross section with the PDF (sometimes also called “distribution function").

$$\hat{t} = q^2 = -Q^2; \quad \hat{s} = 2p \cdot k = 2\xi P \cdot k = \xi s, \tag{15}$$

and

$$\hat{s} + \hat{t} + \hat{u} = 0. \tag{16}$$

So the actual cross section is given by

$$\sigma(e^-(k) + p(P) \to e^-(k') + X) = \int_0^1 d\xi \sum_i f_i(\xi) \tag{17}$$

$$\times \sigma(e^-(k) q_f(\xi P) \to e^-(k') + q_f(p')).$$

This forms stresses the inclusive nature of the process. Indeed, at the parton level, the cross section would seem to be elastic, but in fact any sort of arrangement of the quarks and gluons into hadrons is considered in the final state.

Finally, the momentum fraction, ξ, is fixed in terms of measurable quantities, given the assumption that the initial and final quarks are on shell (with negligible mass), and noting $p' = \xi P + q$:

$$\xi = x; \quad x = \frac{Q^2}{2P \cdot q}. \tag{18}$$

The cross section can then be expressed in terms of various measurable quantites. A convenient variable is

$$y = \frac{2P \cdot q}{s} \tag{19}$$

In terms of parton variables,

$$y = \frac{\hat{s} + \hat{u}}{\hat{s}} \tag{20}$$

and

$$\frac{d^2\sigma}{dxdy} = (\sum_i x f_i(x) Q_i^2) \frac{2\pi\alpha^2 s}{Q^4} [1 + (1-y)^2]. \tag{21}$$

Similar analysis can be performed for deep inelastic neutrino scattering.

Such processes were studied at SLAC in the late 1960's; one could extract the f's, and they were seen to be roughly independent of Q^2. These provided strong evidence for the quark model.

But in ordinary field theories, this sort of scaling did not seem to hold. If one considered interactions among the partons, one obtained corrections involving $\log(Q^2)$, associated with the fact that the transverse momenta were typically of order Q^2. It is asymptotic freedom of QCD which accounts for the observed approximate scaling.

Other Applications of the Parton Model

Lepton Pair Production (*Drell-Yan*)

$$\sigma(q_i\bar{q}_i \to \ell^+\ell^-) = \frac{1}{3}Q_f^2\frac{4\pi\alpha^2}{3\hat{s}}. \tag{22}$$

Peskin and Schroeder discuss convenient ways to analyze the kinematics. In the end, a convolution of the parton cross section over distribution functions associated with the two incoming hadrons:

$$\frac{d^4\sigma}{dy^3dy^4d^2p_\perp} = x_1f(x_1)x_2f(x_2)\frac{1}{\pi}\frac{d\sigma}{d\hat{t}}(1+2\to 3+4). \tag{23}$$

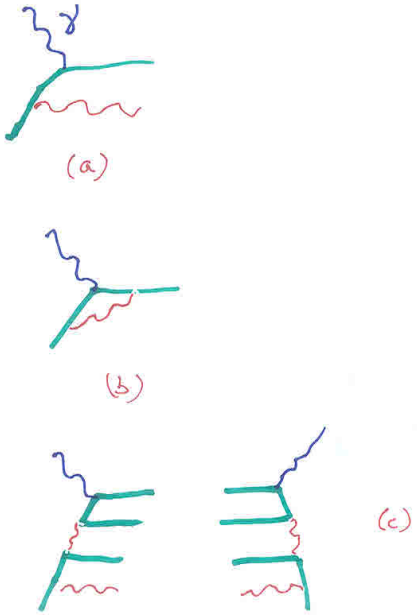
See Peskin and Schroeder for the definitions of the kinematic quantities appearing here.

Jet Pair Production

The formula above, eqn. 23, is applicable to other processes, most notably jet pair production.

Limitations of the Parton Model: Evolution of the Structure Functions with Q^2

Actually field theories don't behave like the parton model. Asymptotically free theories come close. The problem comes from diagrams with emission of quarks and gluons, in addition to the basic parton model process. This leads to two effects: large corrections (behaving as $\alpha(Q^2)\log(Q^2/\lambda^2)$, and also to an apparent breakdown of the notion of probability for finding partons.


γ
(a)
(b)
(c)

In fact, in QCD, amplitudes do appear to factorize*, and a probability interpretation goes through. As in the infrared problem in QED, it is possible to sum the large logarithmic corrections. This is achieved by writing differential equations, with Q^2, for the distribution functions, which are now functions of momentum. These are known as the Altareli-Parisi equations. They relate derivatives with respect to $\log(Q)$ of f to integrals over distribution functions (over x), weighted with functions, known as splitting functions, which describe the amplitude to produce some other parton from the initial one. These are given in PS, eqns. 17.128-17.130.

Everything is in your book or in the PDG!

$$\frac{\sigma(e^+e^- \to \text{hadrons}, Q)}{\sigma(e^+e^- \to \mu^+\mu^-, Q)} \equiv R(Q) = R_{\text{EW}}(Q)(1 + \delta_{\text{QCD}}(Q)) \,, \tag{9.7}$$

where $R_{\text{EW}}(Q)$ is the purely electroweak prediction for the ratio and $\delta_{\text{QCD}}(Q)$ is the correction due to QCD effects. To keep the discussion simple, we can restrict our attention to energies $Q \ll M_Z$, where the process is dominated by photon exchange ($R_{\text{EW}} = 3\sum_q e_q^2$, neglecting finite-quark-mass corrections),

$$\delta_{\text{QCD}}(Q) = \sum_{n=1}^{\infty} c_n \cdot \left(\frac{\alpha_s(Q^2)}{\pi}\right)^n + \mathcal{O}\left(\frac{\Lambda^4}{Q^4}\right) \,. \tag{9.8}$$

The first four terms in the α_s series expansion are then to be found in Refs. 16, 17

$$c_1 = 1 \,, \qquad c_2 = 1.9857 - 0.1152 n_f \,, \tag{9.9a}$$

$$c_3 = -6.63694 - 1.20013 n_f - 0.00518 n_f^2 - 1.240\eta \tag{9.9b}$$

$$c_4 = -156.61 + 18.77 n_f - 0.7974 n_f^2 + 0.0215 n_f^3 + C\eta \,, \tag{9.9c}$$

$$\frac{\partial f_{i/p}(x, \mu_F^2)}{\partial \mu_F^2} = \sum_j \frac{\alpha_s(\mu_F^2)}{2\pi} \int_x^1 \frac{dz}{z} P^{(1)}_{i \leftarrow j}(z) f_{j/p}\left(\frac{x}{z}, \mu_F^2\right) \,, \tag{9.14}$$

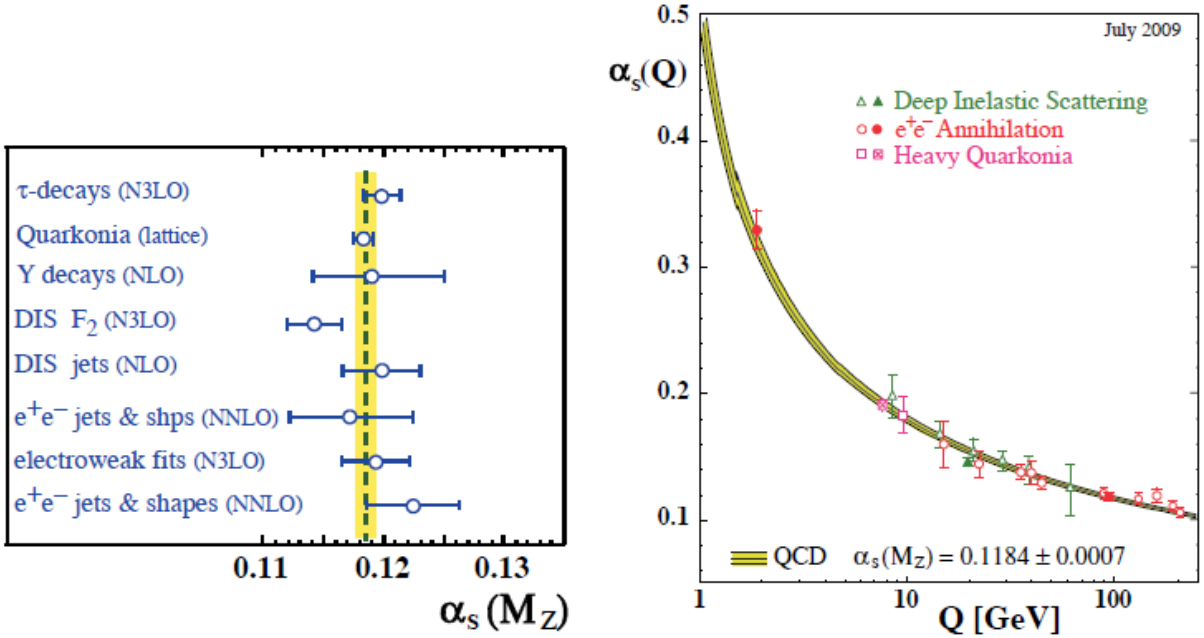


Figure 9.2: Left: Summary of measurements of $\alpha_s(M_Z^2)$, used as input for the world average value; Right: Summary of measurements of α_s as a function of the respective energy scale Q. Both plots are taken from Ref. 172.

Parton Luminosities

A helpful way to think about collider physics is to think about parton luminosities as function of (parton and beam) energy. Here are a couple of examples.

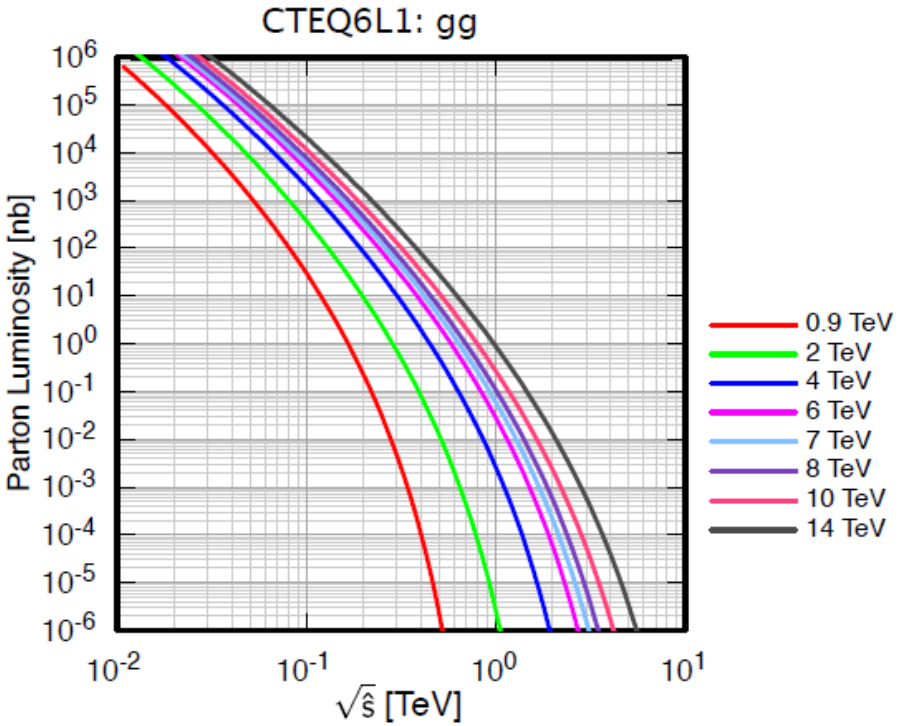


Figure 1: Parton luminosity $(\tau/\hat{s})d\mathcal{L}/d\tau$ for gg interactions.

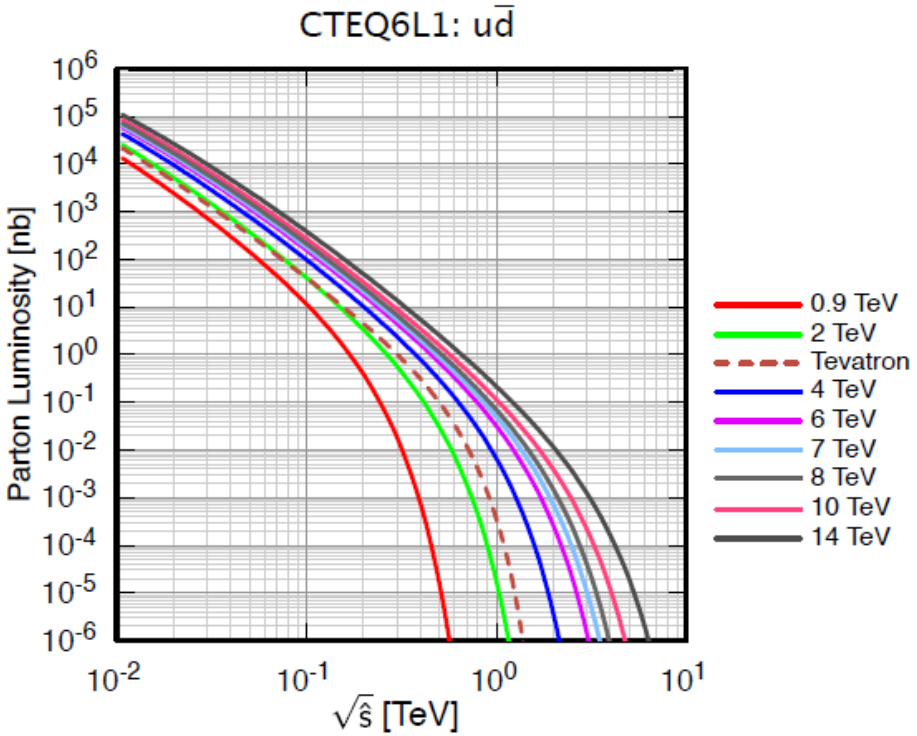


Figure 2: Parton luminosity $(\tau/\hat{s})d\mathcal{L}/d\tau$ for $u\bar{d}$ interactions.