Physics 222. Quantum Field Theory. Professor Dine

Two Component Spinors

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1 Writing a Relativistic Equation for Massless Fermions

If we were living in 1930, and wanted to write a relativistic wave equation for massless fermions, we might proceed as follows. Write:

$$\sigma^\mu \partial_\mu \chi = 0. \tag{1}$$

We want χ to satisfy the Klein-Gordan equation. This will be the case if we can find a set of matrices, $\bar{\sigma}^\mu$, which satisfy

$$\bar{\sigma}^\mu \sigma^\nu + \bar{\sigma}^\nu \sigma^\mu = 2g^{\mu\nu}. \tag{2}$$

Unlike the massive case, we can satisfy this requirement with 2×2 matrices:

$$\sigma^\mu = (1, \vec{\sigma}); \qquad \bar{\sigma}^\mu = (1, -\vec{\sigma}). \tag{3}$$

In momentum space, this equation is remarkably simple:

$$(E - \vec{p} \cdot \vec{\sigma})\chi = 0. \tag{4}$$

For positive energies, this says that the spin is aligned along the momentum. For negative energy spinors, the spin is aligned opposite to the momentum.

Exercise: Write the mode expansion for $\chi(x)$, and identify suitable creation and destruction operators.

To connect to four component spinors, it is convenient to adopt the following basis for the γ matrices:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \tag{5}$$

In this basis,

$$\gamma_5 = i\gamma^o\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{6}$$

so the projectors

$$P_\pm = \frac{1}{2}(1 \pm \gamma_5) \tag{7}$$

are given by:

$$P_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad P_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{8}$$

We will adopt some notation, following the text by Wess and Bagger:

$$\psi = \begin{pmatrix} \chi_\alpha \\ \phi^{*\dot{\alpha}} \end{pmatrix}. \tag{9}$$

Correspondingly, we label the indices on the matrices σ^μ and $\bar{\sigma}^\mu$ as

$$\sigma^\mu = \sigma^\mu_{\alpha\dot{\alpha}} \qquad \bar{\sigma}^\mu = \bar{\sigma}^{\mu\beta\dot{\beta}}. \tag{10}$$

This allows us to match upstairs and downstairs indices, and will prove quite useful. We define complex conjugation to change dotted to undotted indices. So, for example,

$$\phi^{*\dot{\alpha}} = (\phi^{\alpha})^*. \tag{11}$$

Then we define the anti-symmetric matrices $\epsilon_{\alpha\beta}$ and $\epsilon^{\alpha\beta}$ by:

$$\epsilon^{12} = 1 = -\epsilon^{21} \qquad \epsilon_{\alpha\beta} = -\epsilon^{\alpha\beta}. \tag{12}$$

The matrices with dotted indices are defined identically. Note that, with upstairs indices, $\epsilon = i\sigma_2$, $\epsilon_{\alpha\beta}\epsilon^{\beta\gamma} = \delta^{\gamma}_{\alpha}$. We can use these matrices to raise and lower indices on spinors. Define $\phi_\alpha = \epsilon_{\alpha\beta}\phi^\beta$, and similarly for dotted indices. So

$$\phi_\alpha = \epsilon_{\alpha\beta}(\phi^{*\dot{\beta}})^*. \tag{13}$$

Finally, we will define complex conjugation of a product of spinors to invert the order of factors, so, for example, $(\chi_\alpha\phi_\beta)^* = \phi^*_{\dot{\beta}}\chi^*_{\dot{\alpha}}$.

With this in hand, the reader should check that the action for our original four component spinor is:

$$\begin{aligned} S &= \int d^4x\mathcal{L} = \int d^4x \left(i\chi_{\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\partial_\mu\chi_\alpha + i\phi^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu\phi^{*\dot{\alpha}}\right) \\ &= \int d^4x\mathcal{L} = \int d^4x \left(i\chi^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu\chi^{*\dot{\alpha}} + i\phi^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu\phi^{*\dot{\alpha}}\right). \end{aligned} \tag{14}$$

At the level of Lorentz-invariant lagrangians or equations of motion, there is *only one* irreducible representation of the Lorentz algebra for massless fermions.

It is instructive to describe quantum electrodynamics with a massive electron in two-component language. Write

$$\psi = \begin{pmatrix} e \\ \bar{e}^* \end{pmatrix}. \tag{15}$$

In the lagrangian, we need to replace ∂_μ with the covariant derivative, D_μ. e contains annihilation operators for the left-handed electron, and creation operators for the corresponding anti-particle. $\bar{e}$ contains annihilation operators for a particle with the opposite helicity and charge of e, and $\bar{e}^*$, and creation operators for the corresponding antiparticle.

The mass term, $m\bar{\psi}\psi$, becomes:

$$m\bar{\psi}\psi = me^\alpha\bar{e}_\alpha + me^*_{\dot{\alpha}}\bar{e}^{*\dot{\alpha}}. \tag{16}$$

Again, note that both terms preserve electric charge. Note also that the equations of motion now couple e and $\bar{e}$.

It is helpful to introduce one last piece of notation. Call

$$\psi\chi = \psi^\alpha\chi_\alpha = -\psi_\alpha\chi^\alpha = \chi^\alpha\psi_\alpha = \chi\psi. \tag{17}$$

Similarly,

$$\psi^*\chi^* = \psi^*_{\dot{\alpha}}\chi^{*\dot{\alpha}} = -\psi^{*\dot{\alpha}}\chi^*_{\dot{\alpha}}\chi^*_{\dot{\alpha}}\psi^{*\dot{\alpha}} = \chi^*\psi^*. \tag{18}$$

Finally, note that with these definitions,

$$(\chi\psi)^* = \chi^*\psi^*. \tag{19}$$

Exercise: Starting with the action for the four component electron, *with a mass term*, work verify the lagrangian in two component notation for the massive electron. Make sure to work out the covariant derivatives.