Two Component Spinors

1 Writing a Relativistic Equation for Massless Fermions

If we were living in 1930, and wanted to write a relativistic wave equation for massless fermions, we might proceed as follows. Write:

\[ \sigma^\mu \partial_\mu \chi = 0. \] (1)

We want \( \chi \) to satisfy the Klein-Gordon equation. This will be the case if we can find a set of matrices, \( \bar{\sigma}^\mu \), which satisfy

\[ \bar{\sigma}^\mu \sigma^\nu + \sigma^\nu \bar{\sigma}^\mu = 2g^{\mu\nu}. \] (2)

Unlike the massive case, we can satisfy this requirement with \( 2 \times 2 \) matrices:

\[ \sigma^\mu = (1, \vec{\sigma}); \quad \bar{\sigma}^\mu = (1, -\vec{\sigma}). \] (3)

In momentum space, this equation is remarkably simple:

\[ (E - \vec{p} \cdot \vec{\sigma})\chi = 0. \] (4)

For positive energies, this says that the spin is aligned along the momentum. For negative energy spinors, the spin is aligned opposite to the momentum.

**Exercise:** Write the mode expansion for \( \chi(x) \), and identify suitable creation and destruction operators.

To connect to four component spinors, it is convenient to adopt the following basis for the \( \gamma \) matrices:

\[ \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \] (5)

In this basis,

\[ \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \] (6)

so the projectors

\[ P_\pm = \frac{1}{2}(1 \pm \gamma_5) \] (7)

are given by:

\[ P_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \] (8)

We will adopt some notation, following the text by Wess and Bagger:

\[ \psi = \begin{pmatrix} \chi_\alpha \\ \phi^\dot{\alpha} \end{pmatrix}. \] (9)

Correspondingly, we label the indices on the matrices \( \sigma^\mu \) and \( \bar{\sigma}^\mu \) as

\[ \sigma^\mu = \sigma^\mu_\alpha, \quad \bar{\sigma}^\mu = \bar{\sigma}^\mu_\beta. \] (10)
This allows us to match upstairs and downstairs indices, and will prove quite useful. We define complex conjugation to change dotted to undotted indices. So, for example,

$$\phi^* \breve{\alpha} = (\phi^\alpha)^*.$$  \hfill (11)

Then we define the anti-symmetric matrices $\epsilon_{\alpha\beta}$ and $\epsilon^{\alpha\beta}$ by:

$$\epsilon^{12} = 1 = -\epsilon^{21} \quad \epsilon_{\alpha\beta} = -\epsilon^{\alpha\beta}.  \hfill (12)$$

The matrices with dotted indices are defined identically. Note that, with upstairs indices, $\epsilon = i\sigma_2$, $\epsilon^{\alpha\beta} = \delta_3^{\alpha\beta}$. We can use these matrices to raise and lower indices on spinors. Define $\phi_\alpha = \epsilon_{\alpha\beta} \phi^\beta$, and similarly for dotted indices. So

$$\phi_\alpha = \epsilon_{\alpha\beta} (\phi^\beta)^*.  \hfill (13)$$

Finally, we will define complex conjugation of a product of spinors to invert the order of factors, so, for example, $(\chi_\alpha \phi_\beta)^* = \phi^\dagger_\beta \chi^\dagger_\alpha$.

With this in hand, the reader should check that the action for our original four component spinor is:

$$S = \int d^4 x \mathcal{L} = \int d^4 x \left( i \chi_\alpha \sigma^{\mu\alpha} \partial_\mu \chi_\alpha + i \phi^\alpha \sigma^{\mu\alpha} \partial_\mu \phi^* \breve{\alpha} \right)$$

$$= \int d^4 x \mathcal{L} = \int d^4 x \left( i \chi^\alpha \sigma_{\alpha\beta} \partial_\mu \chi^* \breve{\beta} + i \phi^{\alpha} \sigma_{\alpha\beta} \partial_\mu \phi^* \breve{\beta} \right).  \hfill (14)$$

At the level of Lorentz-invariant lagrangians or equations of motion, there is only one irreducible representation of the Lorentz algebra for massless fermions.

It is instructive to describe quantum electrodynamics with a massive electron in two-component language. Write

$$\psi = \begin{pmatrix} e \\ \bar{e}^* \end{pmatrix}.  \hfill (15)$$

In the lagrangian, we need to replace $\partial_\mu$ with the covariant derivative, $D_\mu$. $e$ contains annihilation operators for the left-handed electron, and creation operators for the corresponding anti-particle. $\bar{e}$ contains annihilation operators for a particle with the opposite helicity and charge of $e$, and $\bar{e}^*$, and creation operators for the corresponding antiparticle.

The mass term, $m\bar{\psi}\psi$, becomes:

$$m\bar{\psi}\psi = me^\alpha \bar{e}_\alpha + me^{\dagger}_\alpha \bar{e}^{*\alpha}.  \hfill (16)$$

Again, note that both terms preserve electric charge. Note also that the equations of motion now couple $e$ and $\bar{e}$.

It is helpful to introduce one last piece of notation. Call

$$\psi \chi = \psi^\alpha \chi_\alpha = -\psi_\alpha \chi^\alpha = \chi^\alpha \psi_\alpha = \chi \psi.  \hfill (17)$$

Similarly,

$$\psi^* \chi^* = \psi^\alpha \chi^* \breve{\alpha} = -\psi^* \breve{\alpha} \chi^\alpha = \chi^* \psi^* \chi^*.  \hfill (18)$$

Finally, note that with these definitions,

$$(\chi \psi)^* = \chi^* \psi^*.  \hfill (19)$$

**Exercise:** Starting with the action for the four component electron, with a mass term, work verify the lagrangian in two component notation for the massive electron. Make sure to work out the covariant derivatives.