A Renormalization Group Primer

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Topics in Quantum Field Theory

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Introduction: Some Simple Dimensional Analysis

Consider a theory with only massless fields, or at energy and momentum scales so large that all masses may be neglected. In this case, one might think that one could determine the structure of any amplitude purely by dimensional analysis. For example, at high energies in QED, one might guess that the amplitude for elastic electron-electron scattering would take the form:

$$\mathcal{A}(q) = \frac{f(e^2)}{q^2} \tag{1}$$

where $f(e^2)$ is some dimensionless function of the dimensionless variable e^2, which one might compute order by order in perturbation theory, or in some other way.

But we know that this isn't quite right, once one takes into account the ultraviolet divergences in the theory. In perturbation theory, amplitudes depend also on logarithms of q/μ^2, where μ is a renormalization scale.

Derivation of the renormalization group equations (scalar field theory; generalization to other theories is easy):
Start with the form of the effective action at scale μ:

$$\mathcal{L}_{eff} = Z^{-1}\phi(\partial^2 - m^2)\phi - \frac{\lambda}{4!}\phi^4 + \frac{\delta}{\mu^2}\phi^6 + \ldots .$$

The renormalized field, $\tilde{\phi}$ is defined by a rescaling:

$$\phi = \sqrt{Z}\tilde{\phi}.$$

Now define the renormalized Green's function:

$$\langle\phi(x_1)\ldots\phi(x_n)\rangle = Z^{n/2}\langle\tilde{\phi}(x_1)\ldots\tilde{\phi}(x_n)\rangle$$

$$= Z^{n/2}G(x_1,\ldots,x_n).$$

The left-hand side of this equation is independent of μ; to see this, remember that this is the Green's function, not the effective action. So we can write a differential equation for G by taking the total derivative with respect to $t = \ln(\mu)$ of both sides:

$$Z^{n/2}(\frac{\partial}{\partial t} + \beta(\lambda)\frac{\partial}{\partial \lambda} + \frac{n}{2Z}\frac{\partial Z}{\partial t})G$$

$$(\frac{\partial}{\partial t} + \beta(\lambda)\frac{\partial}{\partial \lambda} + \frac{n}{2}\gamma)G = 0$$

where the "anomalous dimension,"

$$\gamma = \frac{\partial \ln Z}{\partial t}.$$

This equation is known as the Callan-Szymanzik equation. We can obtain what is known as the “renormalization group" equation by dimensional analysis. Suppose we are interested in a Green’s function in momentum space, $G(p_1, \ldots p_n)$. Suppose also that all of the (Euclidean) momentum invariants are comparable, i.e.

$$p_i^2 = x_i^2 M^2; p_i \cdot p_j = x_{ij} M^2; x_i, x_{ij} = \mathcal{O}(1).$$

We can determine how G depends on M. Define

$$t = \ln(\mu^2/M^2).$$

Here we are using the fact that by dimensional analysis, all M dependence is related to μ-dependence. It is convenient, if G has naive dimension $-d$, to take

$$G = M^{-d} f(t, x_i, x_{ij}).$$

Then f satisfies (why?)

$$(\frac{\partial}{\partial t} + \beta(\lambda)\frac{\partial}{\partial \lambda} + \frac{n}{2}\gamma)f = 0$$

The solution can be found by the fluid mechanics analogy, or simply by making a good guess. Define the running coupling constant

$$\bar{\lambda}(t) : \frac{\partial \bar{\lambda}(t)}{\partial t} = \beta(\bar{\lambda}(t)).$$

Then

$$f(t, \lambda, M) = f(\bar{\lambda}(t))e^{-\int_{\lambda_o}^{\bar{\lambda}} \frac{n}{2}\frac{\gamma(\lambda')}{\beta(\lambda')}d\lambda'}.$$

Plugging in, one can see that this satisfies the original equation.

One can consider, instead, the renormalization group equation for terms in the effective action. Returning to the effective action for ϕ, after the rescaling,

$$\mathcal{L}_{eff} = \frac{1}{2}\phi(\partial^2 - m^2)\phi - Z^{4/2}\frac{\lambda}{4!}\phi^4 + Z^{6/2}\frac{\delta}{\mu^2}\phi^6 + \ldots.$$

The Z factors are just what are required to renormalize the couplings, i.e. the terms in the effective action can be written in terms of the renormalized couplings, plus explicit cutoff and μ dependence in the original one-particle irreducible diagram (the μ-dependence can be thought of as arising from the counterterm).

Let's apply this to the renormalization of a fermion mass in a non-abelian gauge theory. This is a term in the effective action. At one loop, the mass renormalization is:

$$\delta m = \frac{6 C_F g^2}{16\pi^2} m \ln(\Lambda/\mu).$$

So the mass satisfies a renormalization group equation:

$$(\frac{\partial}{\partial t} + \beta(g)\frac{\partial}{\partial g} + \gamma_m) m = 0,$$

with $\gamma_m = \frac{6g^2 C_F}{16\pi^2}$. We know how to solve this equation:

$$m = m_o e^{-\int_{g_o}^{\bar{g}} \frac{n}{2}\frac{\gamma(g')}{\beta(g')} dg'}$$

$$= m_o(\frac{g(M)}{g_o})^{3\frac{N^2-1}{Nb_o}}.$$

Wilsonian Description: Integrating Out

For simplicity we write the equations for ϕ^4 theory. We consider here a sharp momentum cutoff, integrating out physics between the scales μ and $b\mu$, $b < 1$. We break up the field, ϕ, into a low momentum ("background", in the sense of the background field method) and high momentum part:

$$\phi = \Phi + \phi$$

Then, for ϕ^4 theory, the action, up to terms quadratic in ϕ, becomes:

$$\mathcal{L} = \frac{1}{2}\Phi(-\partial^2)\Phi + \frac{\lambda}{4!}\Phi^4 + \frac{1}{2}\phi(-\partial^2 + \frac{\lambda}{2}\Phi^2)\phi.$$

So at one loop, the result of the ϕ integral is:

$$\int [d\phi] exp(-\int d^4x \phi(-\partial^2 + \frac{\lambda}{4}\Phi^2)\phi)$$

$$= \det(-\partial^2 + \frac{\lambda}{4}\Phi^2)^{-1/2}$$

where it is understood that the determinant is over $b\mu < |k| < \mu$

To obtain the Φ^4 term, expand the determinant:

$$\Gamma = \frac{1}{2}\mathrm{Tr}\ln(-\partial^2) + \ln(1 - \partial^{-2}\frac{\lambda}{2}\Phi^2).$$

The quadratic term gives the mass renormalization. The quartic term is:

$$\frac{\lambda^2}{32}\Phi^4 \int_{b\mu}^{\mu} \frac{d^4k}{(2\pi)^4}\frac{1}{k^4}$$

$$= 3\frac{1}{4!}\frac{\lambda^2}{16\pi^2}\ln(b).$$

(Note, by the way, how easy this calculation is in the background field method). One can also compute higher order terms in the effective action, by just expanding out the logarithm. Because of the integration limits, these will come with powers of $\frac{1}{b\mu}$.

At this order, the effective action has the structure:

$$\Gamma = \int d^4x(\frac{1}{2}\partial\Phi^2 - \frac{1}{2}m^2\Phi^2 - \frac{1}{4!}(\lambda(1-\frac{3}{32\pi^2})\ln(b))\Phi^4 + \frac{1}{\mu^2}\Phi^6 + \dots)$$

It comes with cutoff $b\mu$. We can change this to a theory with a fixed cutoff by introducing $k = k'b$, $x = \frac{x'}{b}$. In terms of these, the cutoff is μ, and the lagrangian has extra factors of b:

$$\Gamma = \int d^4x'(\frac{b^{-2}}{2}\partial\Phi^2 - \frac{b^{-4}}{2}m^2\Phi^2$$

$$-\frac{b^{-4}}{4!}(\lambda(1-\frac{3}{32\pi^2})\ln(b))\Phi^4 + b^{-4}\frac{1}{\mu^2}\Phi^6 + \dots)$$

Rescaling $\phi \rightarrow b\phi$, gives:

$$\Gamma = \int d^4x'(\frac{1}{2}\partial\Phi^2 - \frac{b^{-2}}{2}m^2\Phi^2 - (\lambda(1 - \frac{3}{32\pi^2})\ln(b))\Phi^4 + b^2\frac{1}{\mu^2}\Phi^6 + \dots)$$

Note that as $b \rightarrow 0$, the mass term becomes more important ("relevant"), while the non-renormalizable terms become less so. The Φ^4 term flows more slowly.

Fixed Points

Suppose the β function has a zero, as indicated in the figure:

$$\beta = \pm C(g - g_o).$$

Then the running coupling is given by:

$$g - g_o = e^{\pm C(t)}.$$

So the running coupling goes to the fixed point either as $t = \ln(\mu/p) \to \mp\infty$. These are referred to, accordingly, as ultraviolet or infrared fixed points. Operators of a given anomalous dimension then tend to zero at the fixed points with powers of p different than expected from classical dimensional analysis. This is the origin of the term “anomalous dimension."

The “Banks-Zaks" Fixed Point

In condensed matter physics, such conformal fixed points are important in critical phenomena in range of systems, with varying dimensionality. This is discussed in chapter 13 of your text. There are now also a wide range of four dimensional field theories known with non-trivial conformal fixed points. This has emerged from the work of Seiberg; the theories of this sort which are understood are supersymmetric (at some point, if we do a supersymmetry course, we can discuss this).

But there is a simple class of theories where a fixed point can be found in perturbation theory. For these, we consider theories with a large number of colors and flavors. It is simplest to consider here also the supersymmetric case. Then, to two loop order, the beta function is given by:

$$b_o = -(3N - N_f)\frac{g^3}{16\pi^2} \quad b_1 = -[6N^2 - 2NN_f - 4N_f\frac{N^2-1}{2N}](\frac{g^2}{16\pi^2})^2 g. \tag{2}$$

Vanishing of the beta function gives, to this order,

$$N_f = 3N - \epsilon \qquad \frac{g^{*2}}{16\pi^2} = \frac{\epsilon}{6N^2}. \tag{3}$$

Here we will think of N as extremely large, and ϵ as an integer of order 1. There will be corrections to this result at higher orders in g. In particular, at higher orders, g^* *is scheme dependent*.

Note that near the fixed point,

$$g = g^* + \delta \qquad \beta = \beta'(g)\delta = b\delta = \frac{\epsilon^2}{3N^2}\delta \tag{4}$$

So

$$\delta(t) = \delta(0)e^{-bt}. \tag{5}$$

One can consider the behavior of various quantities. For example, one can add a small mass term for the scalar fields and think of this as a perturbation. One finds that this runs slowly to zero in the infrared. It is a “relevant" operator. At some point, $m > \mu$, and the theory is no longer conformal.

Applications

Here I just mention a few:

1. The behavior of masses such as the proton mass in asymptotically free theories:

$$m_p = c \, \mu \, e^{-\int \frac{dg'}{\beta(g')}} \tag{6}$$

2. Behavior of high momentum/short distance quantities in asymptotically free theories like QCD:

$$\mathcal{A} \approx f(g^2(s)) \tag{7}$$

   At high energies, because the coupling is small, the behavior of the amplitude can be computed (deep inelastic scattering, e^+e^- annihilation, high transverse momentum scattering...)

3. Unification of masses in grand unified theories
4. Critical phenomena in statistical mechanics and condensed matter physics
5. Two dimensional conformal field theories crucial in string theory

Unification of fermion masses in grand unified theories:

In $SU(5)$, we have two types of allowed Yukawa couplings:

$$H1010 \tag{8}$$

(where the indices are contracted with an ϵ tensor) and

$$H^*\bar{5}10. \tag{9}$$

The first coupling gives masses to the up type quarks; the second to the down type quarks and to the leptons. In fact, for each generation, ignoring mixing, this would predict:

$$m_d = m_\ell. \tag{10}$$

This is clearly not a terribly successful prediction. There are various types of phenomena which might correct them. For example, consider non-renormalizable couplings involving the adjoint field, Φ, and the various fields here. One such coupling is:

$$\frac{y}{M_p} H_i^* \bar{5}_j \Phi^i_k 10^{kj} \tag{11}$$

This is a non-renormalizable coupling and we might be tempted to ignore it. But assuming its coefficient is $\mathcal{O}(1/M_p)$, the effects are not negligible for the light quarks; the ration $\langle \Phi \rangle / M_P$ is of order $10^{-2} - 10^{-3}$. So this yields a contribution to the first two generation Yukawa couplings easily of the same order as the renormalizable term.

The third generation is more interesting. Here the tree level relation is not wildly off (a factor of three) and much work has gone intro trying to understand this. As for the gauge couplings, we need to consider renormalization corrections. At very high energies, much greater than the quark masses, we can treat the mass terms as perturbative terms in the lagrangian, and study their renormalization and their evolution with scale. Indeed, we have already calculated the mass renormalization in QED. In the case of a non-abelian group, the result (for fermions in the fundamental representation, i.e. for $SU(3)$ or $SU(2)$, in the case of the b and the τ) one just multiplies by $1/2$, for the trace of the fermion generators. So, in the effective action, integrating out physics between scale Λ and scale μ, we have

$$m_0 \left(1 + 3/2 \frac{g^2}{16\pi^2} \ln(\frac{\Lambda^2}{\mu^2})\right) \bar{\psi}\psi \tag{12}$$

where the g is that appropriate to either b or τ.

From this, we can write a renormalization group equation for m:

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} + 3\frac{g^2}{16\pi^2}\right] = 0. \tag{13}$$

The solution, to lowest non-trivial order in g^2 is, from our formulas above:

$$m(\mu) = m(\mu_0)\left(\frac{\alpha(\mu)}{\alpha(\mu_0)}\right)^{3/b_0}. \tag{14}$$

(note that expanding in powers of g^2 reproduces our formula above). Since, for most of the range, the strong coupling is largest, and the weak the second, we can estimate the size of this effect by just including it; much more detailed and careful analyses have been performed. Just using the observed values for α_s and m_Z (about 0.1) and taking for the unified coupling (about 0.04), and then evolving further to 5 GeV, I get $m_b/m_\tau \approx 2.34$, vs. the observed valued of about 2.4. While very crude, this is encouraging.