Wilson's Approach to Renormalization

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Topics in Quantum Field Theory

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Introduction

Wilson gave a description of renormalization in which the problem of infinities is not a nuisance or an embarrassment, but inevitable. This approach makes the distinction between renormalizable and non-renormalizable theories, which at first brush seems artificial, again a consequence of simple physical ideas. In particular, the sense in which non-renormalizable operators are irrelevant is clear. This note is a brief overview of material in chapter 12 of Peskin and Schroder.

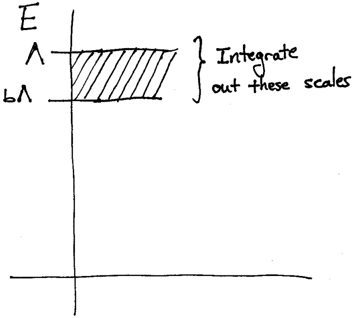
The basic idea is to imagine that at some very high energy one has some sort of fundamental theory, in which all questions are answered (i.e. there are no infinities). String theory, whether a correct description of nature or not, is a model for this, in which the parameters of low energy physics are determined by a finite, high energy theory. One might imagine that there is some other structure, e.g. some discreteness of space and time (though string theory has candidates for this as well).

An Aside

Rather than think about the effective action by first introducing sources, calculating the path integral as a functional of the sources, and performing a Legendre transformation, in many (most interesting) cases, one can simply calculate the path integral, setting fields equal to particular values. We will discuss this in the context of the background field method shortly. But at a conceptual level, this is the content of the more formal development; one *chooses* sources, J, which give rise to values of the fields, ϕ.

The output of this high energy theory is a lagrangian, $\mathcal{L}(\phi)$, and some cutoff Λ (above which one needs to keep all of the degrees of freedom of the underlying fundamental theory). Now suppose one wants to study physics at scales $q \ll \Lambda$. Here Λ might be of order the Planck mass, 10^{10} GeV, and q might be of order 100 GeV. Choose a number $b < 1$, and integrate out all modes with

$$q << b\Lambda < k < \Lambda. \tag{1}$$


E
Λ
bΛ
Integrate
out these scales

(For this discussion, we will think of all momenta as Euclidean, so the notion of “less than" is sharp.) In other words, write (for simplicity, we will write formulas as if we are dealing with a single scalar field):

$$\phi = \sum_{|k|<b\Lambda} \phi(k) e^{ik\cdot x} + \sum_{b\Lambda<|k|<\Lambda} \phi(k) e^{ik\cdot x} \tag{2}$$
$$= \phi_b(x) + \hat{\phi}(x)$$

where the subscript b refers to “background." The effective action appropriate to a theory with cutoff $b\Lambda$ is:

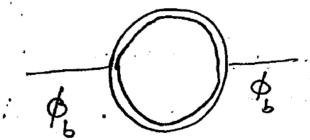
$$e^{iS_{eff}(b)} = \int \prod_{b\Lambda<k<\Lambda} \phi(k) e^{iS} \tag{3}$$
$$= \int [d\hat{\phi}] e^{iS}.$$

Here, in line with our remarks above, we calculate $S(\phi)$ by by performing the path integral with $\phi(x)$ set equal to some particular value.

You know from your study of effective actions that S_{eff} is obtained from diagrams with external ϕ_b lines, and internal $\hat{\phi}$ lines. For example, in ϕ^4 theory, this means that the ϕ^2 term in the effective action is, at one loop:

$$\Gamma^2 = \phi_b(k)[k^2 + m^2]\phi_b(-k) + \delta m^2 \phi_b(k)\phi_b(-k) \tag{4}$$

where δm^2 arises from



The $\hat{\phi}$ propagator is

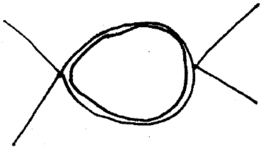
$$\frac{i}{k^2 + m^2} \quad b\Lambda < k\Lambda. \tag{5}$$

(zero otherwise).
So this diagram is given by:

$$\delta m^2 = \frac{\lambda^2}{16\pi^4} \int_{b\Lambda}^{\Lambda} \frac{d^4 k}{k^2 + m^2} \approx \frac{\lambda^2}{16\pi^2} \int_{b\Lambda}^{\Lambda} dkk \tag{6}$$

$$= \frac{\lambda^2}{32\pi^2} \Lambda^2 (1 - b^2).$$

Similarly, the ϕ^4 correction has the form:



$$\delta\Gamma^4 = \phi^4\lambda^2 \int_{b\Lambda}^{\Lambda} \frac{d^4k}{(2\pi)^4(k^2+m^2)^2} \approx \frac{\lambda^2}{16\pi^2}\ln(b^{-1}). \quad (7)$$

Exercise: Work out Γ^6. Show that it is proportional (for small b) to $\frac{1}{(b\Lambda)^2}$.

Note the connection to the background field method. At one loop, we can compute the effective action for the background field ϕ_b as a functional determinant.

$$\int [d\phi_b] e^{iS_{eff}}(\phi_b) = \int [d\phi_b] \int [d\hat{\phi}] e^{iS} \tag{8}$$

Exercise: Expand S to second order in $\hat{\phi}$. Show that the functional integral is a determinant of the form:

$$e^{\mathrm{Tr}\,\log(p^2+m^2+\frac{\lambda}{2}\phi_b^2)}. \tag{9}$$

Show that to second order in λ, the result is:

$$\mathcal{L}_{eff}(\phi_b) = \mathcal{L}_o + \frac{1}{2}\delta m^2 \phi_b^2 + \frac{\delta\lambda}{4!}\phi_b^4. \tag{10}$$

Here

$$\mathcal{L}_o = \frac{1}{2}\phi_b(-\partial^2 + m^2)\phi_b + \frac{\lambda}{4!}\phi_b^4$$