# QCD Instantons: SUSY and NOT

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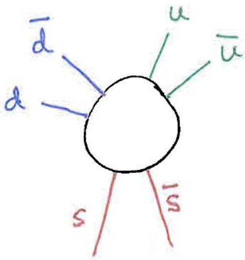
# Reliable Semiclassical Computations in QCD

## Motivation: The Strong CP Problem and $m_u = 0$

There are three plausible solutions to the strong CP problem.

1. Axions
2. Spontaneous CP Violation (Nelson-Barr mechanism)
3. $m_u = 0$.

The last seems, at first, inconsistent with current algebra estimates of meson masses. But, as pointed out by Kaplan and Manohar, this may be naive (earlier work: MacArthur and Georgi, others).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This can be understood naively by thinking about instanton effects in QCD. If $m_u = 0$ at, say, 100 GeV, then instanton corrections generate a $u$ quark mass of order

$$m_u = \frac{m_d m_s}{\lambda}. \tag{1}$$

where $\lambda$ is an infrared cutoff. If $\lambda = \Lambda_{qcd}$, this is not much different than the standard $m_u$.

Suppose, for a moment, that this is consistent with observed facts of strong interactions. One can ask: how plausible is it that $m_u = 0$. After all, $m_u = 0$ is not protected by any (non-anomalous) symmetry.

But in response:

1. The same objection can be raised for the axion.
2. As for the axion, one can find explanations for a symmetry violated only (or more precisely dominantly) by QCD.
    - In string theory, anomalous *discrete* symmetries common. Could forbid $m_u$. Broken only by stringy instantons, low energy effects like QCD.
    - Models of quark masses (Leurer, Nir, Seiberg) often make such a prediction.

To determine these masses, the only tool we have is lattice QCD; one fits to the spectrum as a function of quark masses (in practice, one studies a lattice version of the chiral lagrangian).

With improvements in computing power and algorithms for handling fermions, results for quark masses have been obtained, especially by the MILC collaboration. For our purposes the most important is $m_u$, which is *at least seven standard deviations from zero*.

Using the two-loop perturbative calculation of the mass renormalization constant $Z_m$ (Mason *et al.*, 2006),[11] absolute quark masses can be found,

$$\begin{aligned} m_s &= 88(0)(3)(4)(0)\ \text{MeV}, \\ \hat{m} &= 3.2(0)(1)(2)(0)\ \text{MeV}, \\ m_u &= 1.9(0)(1)(1)(1)\ \text{MeV}, \\ m_d &= 4.6(0)(2)(2)(1)\ \text{MeV}. \end{aligned} \tag{139}$$

The errors are statistical, lattice systematic, perturbative, and electromagnetic (from continuum estimates). Non-perturbative computations of $Z_m$ are in progress.

## This subject is not without controversy. E.g. on the arXiv:

**Quark mass dependence of two-flavor QCD**

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Abstract

I explore the rich phase diagram of two-flavor QCD as a function of the quark masses. The theory involves three parameters, including one that is CP violating. As the masses vary, regions of both first and second order transitions are expected. For non-degenerate quarks, non-perturbative effects cease to be universal, leaving individual quark mass ratios with a renormalization scheme dependence. This raises complications in matching lattice results with perturbative schemes and demonstrates the tautology of attacking the strong CP problem via a vanishing up quark mass.

This is a sufficiently important and dramatic claim that it is worthy of scrutiny. There is, growing corroboration from other groups I should say that if true, it will make me, many others very happy as it points towards an axion; but precisely because of this, I would like to be sure.

As always with experimental science, one would like to have a *calibration*, i.e. an estimate of errors independent of the experimental facts one is trying to explain. One would like some effect which one could calculate analytically and compare with simulations. The beta function at weak coupling is one such quantity. But we would like something non-perturbative, specific to the chiral limit. Such an objection would be theoretically interesting in its own right.

# Ancient history

Instanton calculations in QCD typically ir divergent. Not surprising; a strongly coupled theory.
E.g. calculation of some quantity with dimensions of mass:

$$m = \Lambda^{b_0} \int d\rho \rho^{b_0 - 2}. \quad (2)$$

Similarly, our effective $m_u$:

$$m_u = \Lambda^{b_0} m_d m_s \int d\rho \rho^{b_0}. \quad (3)$$

In the bad old days, this lead to an elaborate program (Callan, Dashen and Gross) to try and use instantons as the basis of a solution of QCD. It was doomed to failure; there was no small parameter which could be the basis of any systematic approximation.

# Can one calculate short distance Green's functions?

Still in these bad old days, people asked: can one calculate instanton contributions to short distance Green's functions? E.g. could one calculate such a contribution to $R_{e^+e^-}$, by computing an instanton correction to $\Pi_{\mu\nu}$ for Euclidean separations, and continue to Minkowski space. But even for short distances, one finds $\Pi$ is infrared divergent.

Typical expression in coordinate space:(schematic):

$$\langle J_\mu(x_1) J_\nu(x_2)\rangle \sim (g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu) \tag{4}$$

$$\times \int d\rho \Lambda^{b_0} \rho^{b_0-1} \frac{d^4 x_0}{[(x_1-x_0)^2+\rho^2]^2[(x_2-x_0)^2+\rho^2]^2}$$

The qualitative features of this result follow from dimensional analysis. For large $\rho$, this behaves as $d\rho\rho^{b_0-5}$ (diverges unless $n_f > 9$).

Appelquist and Shankar, Gross and Andrei, and Ellis and collaborators noticed that the *Fourier transform* of $\Pi$ is infrared finite. Traces to fact that Fourier transform of instanton solution (and zero modes, etc.) behaves as

$$\int d^4x \frac{e^{ip\cdot x}}{(x^2+\rho^2)^2} \sim e^{-p\rho}.$$

Results fall as high powers of momentum.

Suggests a lattice calibration. Calculate momentum space Green's functions which have *no perturbative contribution*, and take Fourier transform. E.g., in three flavor, massless QCD:

$$\Delta(x) = \langle \bar{u}(x)u(x)\bar{d}(0)d(0)\bar{s}(0)s(0)\rangle. \tag{5}$$

Fourier transform will behave as:

$$\Delta(p) = \frac{\Lambda^9}{p^5}. \tag{6}$$

Very rapid fall off with momentum, so probably impossible in practice. But proof of principle?

# What can we actually calculate?

Finiteness is fine, but are these actually the dominant contributions to anything?

Why can't we calculate quantities at short distances?

Organize using the language of the *operator product expansion*.

Start with QCD with two flavors. Consider the operator:

$$\bar{u}(x)u(x)\bar{d}(0)d(0) \tag{7}$$

In perturbation theory, the leading term on the right hand side is non-singular,

$$\bar{u}(x)u(x)\bar{d}(0)d(0) \sim (1 + \mathcal{O}(\alpha_s(x)))\,\bar{u}(0)u(0)\bar{d}(0)d(0) \tag{8}$$

The matrix elements of the operator $\bar{u}(0)u(0)\bar{d}(0)d(0)$ are infrared divergent in an instanton background. This why there is an ir divergence in this correlation function, even at short distances.

In momentum space, this translates to $\delta(p)$, $1/p^4$.

If claiming to isolate an instanton effect, would require $1/p^a$, $a < 4$.

# The unit operator

Beyond perturbation theory, the unit operator can appear in the expansion. Consider $N$ colors, $N_f$ flavors. Study maximally chirality violating Green's functions:

$$\Delta(x) = \langle \prod_{f=1}^{N_f - A} \bar{q}_f(x) q_f(x) \prod_{g=N_f-A+1}^{N_f} \bar{q}_g(0) q_g(0) \rangle \tag{9}$$

$$\prod_{f=1}^{N_f-A} \bar{q}_f(x) q_f(x) \prod_{g=N_f-A+1}^{N_f} \bar{q}_g(0) q_g(0) + \dots \tag{10}$$

$$= C(x) + D(x) \prod_{f=1}^{N_f} \bar{q}_f(0) q_f(0)$$

$D(x)$ has the form $1 + a\alpha_s(x)$; it is logarithmically singular in perturbation theory. It's Fourier transform behaves roughly as $1/p^4$.

If generated by instantons $C(x)$ has the form

$$C(x) \approx \Lambda^{11/3N - 2/3N_f} |x|^{11/3N - 11/3N_f}. \tag{11}$$

So power law singularity if $N_f > N$, non-singular (i.r. divergent) if $N_f < N$, borderline (logarithm) if $N_f = N$.

$SU(2)$ with three flavors first interesting (singular at short distances) case.

Momentum space:

$$C(p) \sim \Lambda^{11/3N-2/3N_f} p^{-4} p^{(11/3N_f-11/3N)} \tag{12}$$

Lesson: if instanton contributions are to dominate, the coefficient of the unit operator in the OPE must be singular.

# The Instanton Computation

Verification that infrared finite.

Fermion zero modes:

$$q^i_\alpha = \rho \frac{\sqrt{2}}{\pi} \frac{\delta^i_\alpha}{[(x-x_0)^2+\rho^2]^{3/2}}, \tag{13}$$

So

$$\Delta(x) = C \int d^4x_0 d\rho \frac{(\Lambda\rho)^{\frac{11}{3}N-\frac{2}{3}N_f} \rho^{3N_f-5}}{[(x-x_0)^2+\rho^2]^{3A}[x_0^2+\rho^2]^{3(N_f-A)}} \tag{14}$$

where $C$ is a constant obtained from the non-zero modes, $x_0$ and $\rho$ are the translational and rotational collective coordinates.

Perform the integral over $x_0$ using Feynman parameters:

$$\Delta(x) = C' \int d\alpha[\alpha^{3A-1}(1-\alpha)^{3(N_f-A)-1}]d\rho \frac{(\Lambda\rho)^{\frac{11}{3}N-\frac{2}{3}N_f}\rho^{3N_f-5}}{[x^2\alpha(1-\alpha)+\rho^2]^{3N_f-2}}. \quad (15$$

For large $\rho$:

$$\Delta \sim \int_{|x|}^{\infty} \frac{d\rho}{\rho}\rho^{\frac{11}{3}(N-N_f)}. \quad (16)$$

The integral converges for large $\rho$ if $N_f > N$, exhibits a power law divergence for $N_f < N$, and diverges logarithmically for $N_f = N$.

# Significance of the Infrared Divergences:

In the infrared divergent cases, the divergent part is identical to the (similarly ill-defined) instanton contribution to $\langle \mathcal{O}_1(0)\mathcal{O}_2(0)\rangle$. For $N_f < N$, the (cutoff) integral is non-singular for small $x$, corresponding to non-singular corrections to the coefficients of operators appearing in the OPE.

For the case $N_f = N$, the expression also has a logarithmic singularity for small $x$, indicating the appearance of the unit operator in the OPE, with a coefficient function behaving as $\log(x)$. It is necessary to define the operators appearing in these expressions at a scale $M$, and this introduces a mass scale both into the matrix element and into the coefficient of the unit operator.

We see that the coefficient of the unit operator is proportional to a single power of a logarithm. We will see that the unit operator can dominate, but only by a fractional power of a logarithm.

# Systematic approximation for small $x$

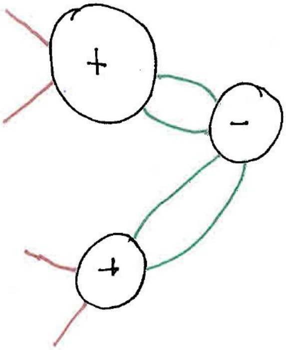
Two types of contributions to $\Delta$:

1. Perturbative corrections to the instanton: For $N_f > N$, the contributions to the unit operator are infrared finite. Controlled by $\alpha_s(x)$.

$$e^{-\frac{8\pi^2}{g^2(x)}} \left(1 + \sum c_n \alpha_s(x)^n\right) . \qquad (17)$$

2. Dilute gas contributions to the leading instanton result: These are infrared divergent, but the divergence, again, is a contribution to the matrix elements of higher dimension operators.

So calculation is systematic.




Dilute gas corrections to the single instanton result

For $N_f = N$ at best the instanton wins (or loses) by a power of a logarithm. To analyze one must compute the anomalous dimensions of the various operators appearing in the OPE. Indeed one finds that for suitable operators, the instant on does win. Will interesting in principle, in practice, e.g. in a lattice computation, this would be numerically hard, if possible at all.

# Applications to Lattice Gauge Theory

Real lattice computations: finite quark masses ($m_q \geq 10$ MeV). So correlation functions like $\Delta$ receive contributions already in perturbation theory. It is necessary that quark masses be very small if the instanton computation is to dominate.
E.g. $SU(2)$ with three flavors:

$$\langle \mathcal{O}_1(x)\mathcal{O}_2(0)\rangle = C\frac{\Lambda^{16/3}}{x^{11/3}}(1+\mathcal{O}(\alpha_s(x))). \tag{18}$$

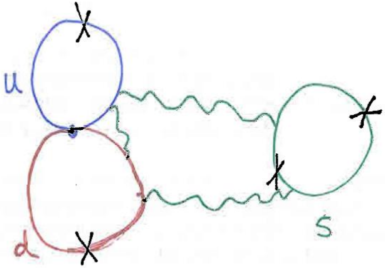
where

$$C = 9\times 10^3 \qquad \Lambda^{16/3} = \mu^{16/3}e^{\frac{-8\pi^2}{g^2(\mu)}} \tag{19}$$

in the $\overline{MS}$ scheme.

With finite quark mass, helpful to consider operators with parity properties such that they have vanishing expectation values (acknowledgment to Steve Sharpe). Then leading diagrams are high loop, so suppressed both by quark masses and $\alpha_s$, and seem small enough, given masses used in practice.

So tests of lattice computations seem feasible, and potentially interesting.




Feynman diagram including mass insertions giving a perturbative contribution to the correlator.
Behaves as $\alpha_s(a)^3 m_u m_d m_s a^{-3}$.

# Applications to Supersymmetric Gauge Theories

Novikov, Shifman, Vainshtein, Zakharov: (1982-    ) – in a program with far reaching implications, studied instanton contributions in supersymmetric gauge theories.
Prototype: $SU(2)$ gauge theory, with chiral fermions in the adjoint representation (gluinos), $\lambda$.

$$\Delta(x) = \langle \lambda(x)\lambda(x)\lambda(0)\lambda(0) \rangle \tag{20}$$

Supersymmetry implies that this correlator is independent of $x$. Argued

1. One can compute at short distances. Found a finite result (i.e. no ir divergence) for the leading contribution:

$$\Delta(x) = C\Lambda^6 \tag{21}$$

2. There is a non-renormalization theorem, and no corrections to the result.
3. Taking $x$ large, by cluster, gives value of gluino condensate.

Inconsistent, however, with result obtained using arguments of Seiberg.

Calculate in theory with a single quark flavor of mass *m*. For small *m*, compute $< \lambda\lambda >$ (or $\Delta$). Find discrepancy by a constant factor $(3/5)$.

What went wrong?

Does analysis in terms of OPE provide insight?

Some observations:

1. We have seen that short distance, by itself, does not guarantee calculability.
2. If there are infrared divergences, these can lead to order one corrections. E.g.

$$\delta\Delta = \Lambda^6 \int \frac{d\rho}{\rho} g^2 = g^2 \log(M/\lambda) \tag{22}$$

with $\lambda$ an ir cutoff. $\lambda \propto (Me^{\frac{-8\pi^2}{g^2}})$ gives an order one correction (consistent with holomorphy).
3. Non-perturbatively (dilute gas corrections)

$$\delta\Delta = \frac{\Lambda^{b_0}(\bar{\Lambda}\Lambda)^{nb_0}}{\lambda^{2nb_0}} \tag{23}$$

behaves like the lowest order result, up to a numerical constant.

We will demonstrate that this is the issue.

# Exact $\beta$ function

Novikov, Shifman, Vainshtein and Zakharov: exact instanton result

$$\frac{\langle\lambda\lambda^{2N}\approx 1}{g^{2N}e^{-\frac{8\pi^2}{g^2}}M^{3N}}$$

leads to an "exact beta function":

$$\beta(g) = -\frac{3N\frac{g^3}{16\pi^2}}{(1-2N\frac{g^2}{16\pi^2})}. \tag{24}$$

Agrees with two loop beta function (universal). But beyond two loops, scheme dependent. What is the scheme? Here a simple explanation (building in part on work of Arkani-Hamed and Murayama) of the result, and a clear identification of the scheme – and why it is not singled out by any compelling physical consideration.

Observation of AHM: $N = 4$ theory, suitably deformed, can serve as a (holomorphic) regulator for the $N = 1$ theory.

In a popular presentation of the theory, in $N = 1$ language, there are three adjoint chiral fields, $\Phi_i$, $i = 1, 2, 3$, and an $SU(4)$ $R$ symmetry.

$$\mathcal{L} = \int d^4\theta \frac{1}{g^2} \Phi_i^\dagger \Phi_i - \frac{1}{32\pi^2} \int d^2\theta \left( \frac{8\pi^2}{g^2} + i\theta \right) W_\alpha^2 \qquad (25)$$

$$+ \int d^2\theta \frac{1}{g^2} \Phi_1 \Phi_2 \Phi_3 + \mathrm{c.c.}$$

*Action is not manifestly holomorphic in $\tau$.* To exploit the power of holomorphy, necessary to rescale the adjoints so that there are no factors of $g$ in the superpotential:

$$\Phi_i \to g^{2/3} \Phi_i. \qquad (26)$$

We can add mass terms for the $\Phi_i$'s (for simplicity, we will take all masses the same, but this is not necessary, and allowing them to differ allows one to consider other questions):

$$\mathcal{L} = \int d^4\theta \frac{1}{g^{2/3}} \Phi_i^\dagger \Phi_i - \frac{1}{32\pi^2} \int d^2\theta \left( \frac{8\pi^2}{g^2} + i\theta \right) W_\alpha^2 \tag{27}$$

$$+ \int d^2\theta (\Phi_1 \Phi_2 \Phi_3 + m_{hol} \Phi_i \Phi_i + \text{c.c.}).$$

*Holomorphic presentation* of the $N = 4$ theory.

Under a renormalization group transformation (a change from cutoff $m_{hol}^{(1)}$ to $m_{hol}^{(2)}$,

$$\frac{8\pi^2}{g^2(m_2)} = \frac{8\pi^2}{g^2(m_1)} + 3N\log(m_{hol}^{(2)}/m_{hol}^{(1)}) \tag{28}$$

But the holomorphic masses don't correspond to the masses of physical particles; these are, at tree level:

$$m_{phys} = g^{2/3}m_{hol}. \tag{29}$$

Substituting in the eqn. for $g$, yields the NSVZ beta function.

So it is tempting to say that the NSVZ scheme is that associated with the physical masses of the cutoff fields, i.e. some “physical" cutoff scale. However, at higher orders, the actual physical masses of the adjoints receive perturbative corrections (indeed already at one loop). So the NSVZ scheme, beyond two loop order, while easy to specify, is just one of an infinite class of schemes:

$$m_{cut} = g^{2/3}(m_{cut})(1 + \frac{g^2}{16\pi^2} f(g^2)) m_{hol}. \tag{30}$$

An alternative regulator is provided by the compactification of the theory on $R^3 \times S^1$ (Hollowood, Mattis, Dorey...). This is essentially a three dimensional gauge theory with scalars ($A_4$) in the adjoint representation. The theory has a classical flat direction ($SU(2)$)

$$A_4 = v\frac{\sigma_3}{2}. \tag{31}$$

In this theory, the simplest instantons are (from a four dimensional perspective) static magnetic monopoles. There are actually two types of monopoles, the usual BPS monopole, and a transformation of the monopole (the “KK monopole) by

$$U = e^{-i\frac{\pi x_4}{\beta}\sigma_3}. \tag{32}$$

One can compute a superpotential in this theory; the potential has two minima, corresponding to breaking the $Z_2$ symmetry. In each vacuum, one can calculate $\langle\lambda\lambda\rangle$ by summing the contributions from these two monopoles (each of which has two zero modes),

$$\langle\lambda\lambda\rangle = \langle\lambda\lambda\rangle_{BPS} + \langle\lambda\lambda\rangle_{KK}, \tag{33}$$

remarkably obtaining the weak coupling result for the condensate.

To understand the failure of the strong coupling calculation, study:

$$\Delta(x) = \langle (\lambda(x)\lambda(x))\,(\lambda(0)\lambda(0))\rangle. \quad (34)$$

This is generated by instantons which can be described as monopole pairs. We can label the three types of solution as $BPS - BPS$, $BPS - KK$, and $KK - KK$. Consider, first, the limit $|x| \gg \beta$. Then $\Delta$ will be generated by configurations where one monopole is near $x$ and one near 0. For large separations we have

$$\Delta = \langle (\lambda\lambda)(\lambda\lambda)\rangle_{BPS-BPS} + \langle (\lambda\lambda)(\lambda\lambda)\rangle_{BPS-KK} + \langle (\lambda\lambda)(\lambda\lambda)\rangle_{KK-KK} \quad (35)$$

$$= \langle \lambda\lambda\rangle\langle\lambda\lambda\rangle.$$

To compare with the strong coupling calculation, we are interested in the opposite limit, $|x| \ll \beta$. Because the correlation function is a constant in $x$, the result obtained by the simple, factorized computation still holds in this other limit.

The BPS-KK instanton is known explicitly from finite temperature studies (Lee). If one takes the limit $\beta \to \infty$, this solution reduces to the infinite volume solution.

Much is known about the BPS-BPS solution, though an analytic form is not available. It is independent of $x^4$. So it cannot lead, in general, to $O(4)$ invariant expressions for Green's functions. On the other hand, its contribution to $\langle \lambda\lambda \rangle$ is a constant. So this is a contribution to $\langle \lambda\lambda \rangle$ which survives in the large $\beta$ limit. The $KK - KK$ instanton contributions are identical to the $BPS - BPS$ contributions.

The BPS-KK instanton, while formally identical to the infinite volume theory at large $\beta$, also makes a different contribution than that found by NSVZ. This is because the limit $\beta \to \infty$ does not commute with the collective coordinate integrals.

We can summarize as follows. If one works in the formal infinite volume limit, there is no systematic computation of $\Delta$; the infrared is not under control, and there is no approximation scheme. In a situation where systematic computations are possible, an infrared cutoff is present. There are contributions to $\Delta$ which survive in the limit the cutoff is taken to infinity, So the formal infinite volume result has, as expected, infrared sensitivity.

**By definition, contributions in the winding number one sector from configurations which are not identical to the usual instant are, at infinite volume, instanton-anti-instanton effects.**

So it is the dilute gas which is the problem, as expected.

**THE END**