The complex logarithm, exponential and power functions

In this note, we examine the logarithm, exponential and power functions, where the arguments\(^*\) of these functions can be complex numbers. In particular, we are interested in how their properties differ from the properties of the corresponding real-valued functions.

1. Properties of the real-valued logarithm, exponential and power functions

Consider the logarithm of a positive real number. This function satisfies a number of properties:

\[ e^{\ln x} = x, \quad \text{(1)} \]
\[ \ln(e^a) = a, \quad \text{(2)} \]
\[ \ln(xy) = \ln(x) + \ln(y), \quad \text{(3)} \]
\[ \ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y), \quad \text{(4)} \]
\[ \ln\left(\frac{1}{x}\right) = -\ln(x), \quad \text{(5)} \]
\[ \ln x^p = p\ln x, \quad \text{(6)} \]

for positive real numbers \(x\) and \(y\) and arbitrary real number \(a\). Likewise, the power function defined over the real numbers satisfies:

\[ x^a = e^{a\ln x}, \quad \text{(7)} \]
\[ x^a x^b = x^{a+b}, \quad \text{(8)} \]
\[ \frac{x^a}{x^b} = x^{a-b}, \quad \text{(9)} \]
\[ \frac{1}{x^a} = x^{-a}, \quad \text{(10)} \]
\[ (x^a)^b = x^{ab}, \quad \text{(11)} \]

\(^*\)Note that the word argument has two distinct meanings. In this context, given a function \(w = f(z)\), we say that \(z\) is the argument of the function \(f\). This should not be confused with the argument of a complex number \(z = r^i\theta\), which is defined by \(\arg z = \theta + 2\pi n\) for \(n = 0, \pm 1, \pm 2, \pm 3, \ldots\).
\[(xy)^a = x^a y^a, \quad (12)\]
\[(\frac{x}{y})^a = x^a y^{-a}, \quad (13)\]

for positive real numbers \(x\) and \(y\) and arbitrary real numbers \(a\) and \(b\). Closely related to the power function is the generalized exponential function defined over the real numbers. This function satisfies:

\[a^x = e^{x \ln a}, \quad (14)\]

\[a^x a^y = a^{x+y}, \quad (15)\]

\[\frac{a^x}{a^y} = a^{x-y}, \quad (16)\]

\[\frac{1}{a^x} = a^{-x}, \quad (17)\]

\[(a^x)^y = a^{xy}, \quad (18)\]

\[(ab)^x = a^x b^x, \quad (19)\]

\[(\frac{a}{b})^x = a^x b^{-x}. \quad (20)\]

for positive real numbers \(a\) and \(b\) and arbitrary real numbers \(x\) and \(y\).

We would like to know which of these relations are satisfied when these functions are extended to the complex plane. It is dangerous to assume that all of the above relations are valid in the complex plane without modification, as this assumption can lead to seemingly paradoxical conclusions. Here are three examples:

1. Since \(1/(-1) = (-1)/1 = -1\),

\[\sqrt{\frac{1}{-1}} = \frac{1}{i} = \sqrt{\frac{-1}{1}} = \frac{i}{1}.\]

Hence, \(1/i = i\) or \(i^2 = 1\). But \(i^2 = -1\), so we have proven that \(1 = -1\).

2. Since \(1 = (-1)(-1)\),

\[1 = \sqrt{1} = \sqrt{(-1)(-1)} = (\sqrt{-1})(\sqrt{-1}) = i \cdot i = -1.\]

3. To prove that \(\ln(-z) = \ln(z)\) for all \(z \neq 0\), we proceed as follows:

\[\ln(z^2) = \ln[(-z)^2],\]
\[\ln(z) + \ln(z) = \ln(-z) + \ln(-z),\]
\[2 \ln(z) = 2 \ln(-z),\]
\[\ln(z) = \ln(-z).\]
Of course, all these “proofs” are faulty. The fallacy in the first two proofs can be traced back to eqs. (12) and (13), which are true for real-valued functions but not true in general for complex-valued functions. The fallacy in the third proof is more subtle, and will be addressed later in these notes. A careful study of the complex logarithm, power and exponential functions will reveal how to correctly modify eqs. (1)–(20) and avoid pitfalls that can lead to false results.

2. Definition of the complex exponential function

We begin with the complex exponential function, which is defined via its power series:

\[ e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \]

where \( z \) is any complex number. Using this power series definition, one can verify that:

\[ e^{z_1+z_2} = e^{z_1}e^{z_2}, \quad \text{for all complex } z_1 \text{ and } z_2. \tag{21} \]

In particular, if \( z = x + iy \) where \( x \) and \( y \) are real, then it follows that

\[ e^z = e^{x+iy} = e^x e^{iy} = e^x (\cos y + i \sin y). \]

One can quickly verify that eqs. (14)–(17) are satisfied by the complex exponential function. In addition, eq. (18) clearly holds when the outer exponent is an integer:

\[ (e^z)^n = e^{nz}, \quad n = 0, \pm 1, \pm 2, \ldots. \tag{22} \]

If the outer exponent is a non-integer, then the resulting expression is a multi-valued power function. We will discuss this case in more detail in section 10.

Before moving on, we record one key property of the complex exponential:

\[ e^{2\pi in} = 1, \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots. \tag{23} \]

3. Definition of the argument function

The argument of a complex number is a multi-valued function which plays a key role in understanding the properties of the complex logarithm and power functions. Any complex number \( z \) can be written in polar form

\[ z = |z|e^{i\arg z}, \]

where \( \arg z \) is a multi-valued function given by:

\[ \arg z = \theta + 2\pi n, \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots. \]
Here, $\theta \equiv \text{Arg} \, z$ is the so-called principal value of the argument, which by convention is taken to lie in the range $-\pi < \theta \leq \pi$. That is,\(^\dagger\)

$$\text{arg} \, z = \text{Arg} \, z + 2\pi n, \quad n = 0, \pm 1, \pm 2, \pm 3, \cdots, \quad -\pi < \text{Arg} \, z \leq \pi. \quad (24)$$

It will be useful to have an explicit formula for $\text{Arg} \, z$ in terms of $\text{arg} \, z$. First, we introduce some notation: $[x]$ means the largest integer less than or equal to the real number $x$. That is, $[x]$ is the unique integer that satisfies the inequality

$$x - 1 < [x] \leq x, \quad \text{for real } x \text{ and integer } [x]. \quad (25)$$

For example, $[1.5] = [1] = 1$ and $[-0.5] = -1$. With this notation, one can write $\text{Arg} \, z$ in terms of $\text{arg} \, z$ as follows:

$$\text{Arg} \, z = \text{arg} \, z + 2\pi \left[1 - \frac{[\text{arg} \, z]}{2\pi}\right], \quad (26)$$

where $[ ]$ denotes the bracket (or greatest integer) function introduced above. It is straightforward to check that $\text{Arg} \, z$ as defined by eq. (26) does indeed fall inside the principal interval [eq. (24)].

4. Properties of the multi-valued argument function

We can view a multi-valued function $f(z)$ evaluated at $z$ as a set of values, where each element of the set corresponds to a different choice of some integer $n$. For example, given the multi-valued function $\text{arg} \, z$ whose principal value is $\text{Arg} \, z \equiv \theta$, then $\text{arg} \, z$ consists of the set of values:

$$\text{arg} \, z = \{\theta, \theta + 2\pi, \theta - 2\pi, \theta + 4\pi, \theta - 4\pi, \cdots\}. \quad (27)$$

Given two multi-valued functions, e.g., $f(z) = F(z) + 2\pi n$ and $g(z) = G(z) + 2\pi n$, where $F(z)$ and $G(z)$ are the principal values of $f(z)$ and $g(z)$ respectively, then $f(z) = g(z)$ if and only if for each point $z$, the corresponding set of values of $f(z)$ and $g(z)$ precisely coincide:

$$\{F(z), F(z) + 2\pi, F(z) - 2\pi, \cdots\} = \{G(z), G(z) + 2\pi, G(z) - 2\pi, \cdots\}. \quad (28)$$

Sometimes, one refers to the equation $f(z) = g(z)$ as a set equality since all the elements of the two sets in eq. (28) must coincide. We add two additional rules to our concept of set equality. First, the ordering of terms within the set is

\(^\dagger\)The choice of the principal range $-\pi < \theta \leq \pi$ is conventional and the one most often used in textbooks. However, other choices are sometimes made—McQuarrie chooses $0 \leq \theta < 2\pi$. Some of the results of this note involving principal values would change slightly if another convention were chosen. We leave it up to the reader to modify the relevant results if a different convention is employed.
unimportant. Second, we only care about the distinct elements of each set. That is, if our list of set elements has repeated entries, we omit all duplicate elements.

To see how the set equality of two multi-valued functions works, let us consider the multi-valued function \( \text{arg} \). One can prove that:

\[
\text{arg}(z_1 z_2) = \text{arg} z_1 + \text{arg} z_2, \tag{29}
\]

\[
\text{arg} \left( \frac{z_1}{z_2} \right) = \text{arg} z_1 - \text{arg} z_2. \tag{30}
\]

\[
\text{arg} \left( 
\frac{1}{z} \right) = - \text{arg} z. \tag{31}
\]

To prove eq. (29), consider \( z_1 = |z_1|e^{i\text{Arg} z_1} \) and \( z_2 = |z_2|e^{i\text{Arg} z_2} \). The arguments of these two complex numbers are: \( \text{arg} z_1 = \text{Arg} z_1 + 2\pi n_1 \) and \( \text{arg} z_2 = \text{Arg} z_2 + 2\pi n_2 \), where \( n_1 \) and \( n_2 \) are arbitrary integers. [One can also write \( \text{arg} z_1 \) and \( \text{arg} z_2 \) in set notation as in eq. (27).] Then it follows that

\[
z_1 z_2 = |z_1 z_2|e^{i(\text{Arg} z_1 + \text{Arg} z_2)},
\]

where we have used \( |z_1||z_2| = |z_1 z_2| \) and made use of eq. (21). Thus, \( \text{arg}(z_1 z_2) = \text{Arg} z_1 + \text{Arg} z_2 + 2\pi n_{12} \), where \( n_{12} \) is also an arbitrary integer. Therefore, we have established that:

\[
\text{arg} z_1 + \text{arg} z_2 = \text{Arg} z_1 + \text{Arg} z_2 + 2\pi(n_1 + n_2),
\]

\[
\text{arg}(z_1 z_2) = \text{Arg} z_1 + \text{Arg} z_2 + 2\pi n_{12},
\]

where \( n_1, n_2 \) and \( n_{12} \) are arbitrary integers. Thus, \( \text{arg} z_1 + \text{arg} z_2 \) and \( \text{arg}(z_1 z_2) \) coincide as sets, and so eq. (29) is confirmed. You can easily prove eqs. (30) and (31) by a similar method.

Now, for a little surprise:

\[
\text{arg} z^2 \neq 2 \text{arg} z. \tag{32}
\]

To see why this statement is surprising, consider the following false proof. Use eq. (29) with \( z_1 = z_2 = z \) to derive:

\[
\text{arg} z^2 = \text{arg} z + \text{arg} z \overset{?}{=} 2 \text{arg} z, \quad \text{[FALSE!!]}.
\]

The false step is the one indicated by \( \overset{?}{=} \). Given, \( z = |z|e^{i\text{Arg} z} \), one finds that \( z^2 = |z|^2 e^{2i\text{Arg} z} \), and so the possible values of \( \text{arg}(z^2) \) are:

\[
\text{arg}(z^2) = \{2\text{Arg} z, 2\text{Arg} z + 2\pi, 2\text{Arg} z - 2\pi, 2\text{Arg} z + 4\pi, 2\text{Arg} z - 4\pi, \cdots \},
\]

whereas the possible values of \( 2 \text{arg} z \) are:

\[
2 \text{arg}(z) = \{2\text{Arg} z, 2(\text{Arg} z + 2\pi), 2(\text{Arg} z - 2\pi), 2(\text{Arg} z + 4\pi), \cdots \}
\]

\[
= \{2\text{Arg} z, 2\text{Arg} z + 4\pi, 2\text{Arg} z - 4\pi, 2\text{Arg} z + 8\pi, 2\text{Arg} z - 8\pi, \cdots \}.
\]
Thus, $2 \arg z$ is a *subset* of $\arg(z^2)$, but half the elements of $\arg(z^2)$ are missing from $2 \arg z$. These are therefore unequal sets, as indicated by eq. (32). Now, you should be able to see what is wrong with the statement:

$$\arg z + \arg z = 2 \arg z.$$ \hspace{1cm} (34)

When you add $\arg z$ as a set to itself, the element you choose from the first $\arg z$ need not be the same as the element you choose from the second $\arg z$. In contrast, $2 \arg z$ means take the set $\arg z$ and multiply each element by two. The end result is that $2 \arg z$ contains only half the elements of $\arg z + \arg z$ as shown above.

Here is one more example of an incorrect proof. Consider eq. (30) with $z_1 = z_2 \equiv z$. Then, you might be tempted to write:

$$\arg \left( \frac{z}{z} \right) = \arg(1) = \arg z - \arg z = 0.$$  

This is clearly wrong since $\arg(1) = 2\pi n$, where $n$ is the set of integers. Again, the error occurs with the step:

$$\arg z - \arg z = 0.$$ \hspace{1cm} (35)

The fallacy of this statement is the same as above. When you subtract $\arg z$ as a set from itself, the element you choose from the first $\arg z$ need not be the same as the element you choose from the second $\arg z$.

5. Properties of the principal value of the argument

The properties of the principal value $\text{Arg} z$ are not as simple as those given in eqs. (29)–(31), since the range of $\text{Arg} z$ is restricted to lie within the principal range $-\pi < \text{Arg} z \leq \pi$. Instead, the following relations are satisfied:

$$\text{Arg} (z_1 z_2) = \text{Arg} z_1 + \text{Arg} z_2 + 2\pi N_+,$$ \hspace{1cm} (36)

$$\text{Arg} (z_1 / z_2) = \text{Arg} z_1 - \text{Arg} z_2 + 2\pi N_-,$$ \hspace{1cm} (37)

where the integers $N_\pm$ are determined as follows:

$$N_\pm = \begin{cases} 
-1, & \text{if } \text{Arg} z_1 \pm \text{Arg} z_2 > \pi, \\
0, & \text{if } -\pi < \text{Arg} z_1 \pm \text{Arg} z_2 \leq \pi, \\
1, & \text{if } \text{Arg} z_1 \pm \text{Arg} z_2 \leq -\pi. 
\end{cases} \hspace{1cm} (38)$$

If we set $z_1 = 1$ in eq. (37), we find that

$$\text{Arg}(1/z) = \begin{cases} 
\pi, & \text{if } z \text{ is real and negative}, \\
-\text{Arg}(z), & \text{otherwise}. 
\end{cases} \hspace{1cm} (39)$$
Note that for \( z \) real and negative, \( 1/z \) is also real and negative so that in this case, \( \text{Arg}(1/z) = \text{Arg}(z) = \pi \).

If \( n \) is an integer, then

\[
\text{arg } z^n = \text{arg } z + \text{arg } z + \cdots + \text{arg } z \neq n \text{ arg } z ,
\]

where the final inequality above was noted in the case of \( n = 2 \) in section 4. The corresponding property of \( \text{Arg } z \) is much simpler:

\[
\text{Arg}(z^n) = n\text{Arg } z + 2\pi N_n ,
\]

where the integer \( N_n \) is given by:

\[
N_n = \left\lfloor \frac{1}{2} - \frac{n}{2\pi} \text{Arg } z \right\rfloor ,
\]

and \( \lfloor \cdot \rfloor \) is the greatest integer bracket function introduced in eq. (25). It is straightforward to verify eqs. (36)–(39) and eq. (41). These formulae follow from the corresponding properties of \( \text{arg } z \), taking into account the requirement that \( \text{Arg } z \) must lie within the principal interval.

The properties of \( \text{arg } z \) and \( \text{Arg } z \) will be crucial for determining the properties of the complex logarithm and power functions.

**6. Definition of the complex logarithm**

In order to define the complex logarithm, one must solve the complex equation:

\[
z = e^w,
\]

for \( w \), where \( z \) is any non-zero complex number. If we write \( w = u + iv \), then eq. (43) can be written as

\[
e^u e^{iv} = |z| e^{i\text{arg } z} .
\]

Eq. (44) implies that:

\[
|z| = e^u , \quad v = \text{arg } z .
\]

The equation \( |z| = e^u \) is a real equation, so we can write \( u = \ln |z| \), where \( \ln |z| \) is the ordinary logarithm acting on positive real numbers. Thus,

\[
w = u + iv = \ln |z| + i \text{arg } z = \ln |z| + i(\text{Arg } z + 2\pi n) , \quad n = 0, \pm 1, \pm 2, \pm 3, \cdots
\]

We call \( w \) the complex logarithm and write \( w = \ln z \). This is a somewhat awkward notation since in eq. (45) we have already used the symbol \( \ln \) for the real logarithm. We shall finesse this notational quandary by denoting the real logarithm henceforth by the symbol \( \text{Ln} \). That is, \( \text{Ln } z \) means the ordinary real logarithm.
of \(z\) for real positive values of \(z\). (In eq. (47) below, we shall extend the definition of \(\text{Ln} z\) to complex \(z\).) With this notational convention, we rewrite eq. (45) as:

\[
\ln z = \text{Ln}|z| + i \arg z = \text{Ln}|z| + i(\text{Arg} z + 2\pi n), \quad n = 0, \pm 1, \pm 2, \pm 3, \cdots
\]

(46)

for any non-zero complex number \(z\).

Clearly, \(\ln z\) is a multi-valued function (as its value depends on the integer \(n\)). It is useful to define a single-valued function \(\text{Ln} z\) called the principal value of \(\ln z\) as follows:

\[
\text{Ln} z = \text{Ln}|z| + i \text{Arg} z, \quad -\pi < \text{Arg} z \leq \pi.
\]

(47)

Note that for positive \(z\), we have \(\text{Arg} z = 0\), so that eq. (47) simply reduces to the usual real logarithmic function. Thus, we have extended the definition of \(\text{Ln} z\) to the entire complex plane (excluding the origin, \(z = 0\), where the logarithmic function is singular). The relation between \(\ln z\) and its principal value is simple:

\[
\ln z = \text{Ln} z + 2\pi in, \quad n = 0, \pm 1, \pm 2, \pm 3, \cdots.
\]

7. Properties of the complex logarithm

We now consider which of the properties given in eqs. (1)–(6) apply to the complex logarithm. Since we have defined the multi-value function \(\ln z\) and the single-valued function \(\text{Ln} z\), we should examine the properties of both these functions. We begin with the multi-valued function \(\ln z\). First, we examine eq. (1). Using eq. (46), it follows that:

\[
e^{\ln z} = e^{\text{Ln}|z|}e^{i\text{Arg} z}e^{2\pi in} = |z|e^{i\text{Arg} z} = z.
\]

(48)

Thus, eq. (1) is satisfied. Next, we examine eq. (2) for \(z = x + iy\):

\[
\ln(e^z) = \text{Ln}|e^z| + i(\arg e^z) = \text{Ln}(e^x + iy) = x + iy + 2\pi ik = z + 2\pi ik,
\]

where \(k\) is an arbitrary integer. In deriving this result, we used the fact that \(e^z = e^x e^{iy}\), which implies that \(\arg(e^z) = y + 2\pi k\).\(^\dagger\) Thus,

\[
\ln(e^z) = z + 2\pi ik \neq z, \quad \text{unless} \quad k = 0.
\]

(49)

This is not surprising, since \(\ln(e^z)\) is a multi-valued function, which cannot be equal to the single-valued function \(z\). Indeed eq. (2) is false for the multi-valued complex logarithm.

As a check, let us compute \(\ln(e^{\ln z})\) in two different ways First, using eq. (48), it follows that \(\ln(e^{\ln z}) = \ln z\). Second, using eq. (49), \(\ln(e^{\ln z}) = \ln z + 2\pi ik\). This

\(^\dagger\)Note that \(\arg e^z = y + 2\pi N\), where \(N\) is chosen such that \(-\pi < y + 2\pi N \leq \pi\). Moreover, eq. (24) implies that \(\arg e^z = \text{Arg} e^z + 2\pi n\) where \(n\) is an element of the set of all integers. Hence, \(\arg(e^z) = y + 2\pi k\), where \(k = n + N\) is still an element of the set of all integers.
seems to imply that \( \ln z = \ln z + 2\pi i k \). In fact, the latter is completely valid as a set equality in light of eq. (46).

We now consider the properties exhibited in eqs. (3)–(6). Using the definition of the multi-valued complex logarithms and the properties of \( \arg z \) given in eqs. (29)–(31), it follows that eqs. (3)–(5) are satisfied as set equalities:

\[
\ln(z_1 z_2) = \ln z_1 + \ln z_2 ,
\]

\[
\ln \left( \frac{z_1}{z_2} \right) = \ln z_1 - \ln z_2 .
\]

\[
\ln \left( \frac{1}{z} \right) = - \ln z .
\]

However, one must be careful in employing these results. One should not make the mistake of writing, for example, \( \ln z + \ln z \equiv 2 \ln z \) or \( \ln z - \ln z \equiv 0 \). Both these latter statements are false for the same reasons that eqs. (34) and (35) were false. In particular, the multi-valued complex logarithm does not satisfy eq. (6) when \( p \) is an integer \( n \):

\[
\ln z^n = \ln z + \ln z + \cdots + \ln z \not\equiv n \ln z ,
\]

which follows from a similar property of \( \arg z \) [eq. (40)]. If \( p \) is not an integer, then \( z^p \) is a complex multi-valued function, and one needs further analysis to determine whether eq. (6) is valid. In section 8, we will prove [see eq. (63)] that eq. (3) is satisfied by the complex logarithm only if \( p = 1/n \) where \( n \) is an integer. In this case,

\[
\ln(z^{1/n}) = \frac{1}{n} \ln z , \quad n = 1, 2, 3, \cdots .
\]

We next examine the properties of the single-valued function \( \text{Ln} z \). Again, we examine the six properties given by eqs. (1)–(6). First, eq. (1) is trivially satisfied since

\[
e^{\text{Ln} z} = e^{\text{Ln}|z|e^{i\arg z}} = |z|e^{i\arg z} = z .
\]

However, eq. (2) is generally false. In particular, for \( z = x + iy \)

\[
\text{Ln}(e^z) = \text{Ln} |e^z| + i(\arg e^z) = \text{Ln}(e^x) + i(\arg e^{iy}) = x + i\arg(e^{iy})
\]

\[
= x + i \arg(e^{iy}) + 2\pi i \left[ \frac{1}{2} - \frac{\arg(e^{iy})}{2\pi} \right] = x + iy + 2\pi i \left[ \frac{1}{2} - \frac{y}{2\pi} \right]
\]

\[
= z + 2\pi i \left[ \frac{1}{2} - \frac{\text{Im} z}{2\pi} \right],
\]

after using eq. (26), where \([ \ ]\) is the greatest integer bracket function [eq. (25)]. Thus, eq. (2) is satisfied only when \(-\pi < y \leq \pi\). For values of \( y \) outside the principal interval, eq. (2) contains an additive correction term as shown in eq. (56).
As a check, let us compute \( \log(e^{\log z}) \) in two different ways. First, using eq. (55), it follows that \( \log(e^{\log z}) = \log z \). Second, using eq. (56),

\[
\log(e^{\log z}) = \log z + 2\pi i \left[ \frac{1}{2} - \frac{\text{Im} \log z}{2\pi} \right] = \log z + 2\pi i \left[ \frac{1}{2} - \frac{\text{Arg} z}{2\pi} \right] = \log z,
\]

where we have used \( \text{Im} \log z = \text{Arg} z \) [see eq. (47)], and noted that for \( \text{Arg} z \) in the principal range, \( 0 \leq \frac{1}{2} - \text{Arg}(z)/(2\pi) < 1 \). Thus, the two computations agree.

We now consider the properties exhibited in eqs. (3)–(6). \( \log z \) may not satisfy any of these properties due to the fact that the principal value of the complex logarithm must satisfy \(-\pi < \text{Im} \log z \leq \pi\). Using the results of eqs. (36)–(42), it follows that

\[
\log(z_1 z_2) = \log z_1 + \log z_2 + 2\pi i N_+ , \quad (57)
\]

\[
\log(z_1/z_2) = \log z_1 - \log z_2 + 2\pi i N_- , \quad (58)
\]

\[
\log(z^n) = n \log z + 2\pi i N_n \quad \text{ (integer } n), \quad (59)
\]

where the integers \( N_+ = -1, 0 \) or \(+1\) and \( N_n \) are determined by eqs. (38) and (42), respectively, and

\[
\log(1/z) = \begin{cases} 
-\log(z) + 2\pi i , & \text{if } z \text{ is real and negative} , \\
-\log(z) , & \text{otherwise} .
\end{cases} \quad (60)
\]

Note that eq. (3) is satisfied if \( \text{Re} z_1 > 0 \) and \( \text{Re} z_2 > 0 \) (in which case \( N_+ = 0 \)).\(^5\) In other cases, \( N_+ \neq 0 \) and eq. (3) fails. Similar considerations apply to eqs. (4)–(6). For example, eq. (5) is satisfied by \( \log z \) unless \( \text{Arg} z = \pi \) (equivalently for negative real values of \( z \)), as indicated by eq. (60). In particular, one may use eq. (59) to verify that:

\[
\log([-1]) = -\log(1) + 2\pi i = -\pi i + 2\pi i = \pi i = \log(-1),
\]

as expected.

We cannot yet check whether eq. (6) is satisfied if \( p \) is a non-integer, since in this case \( z^p \) is a multi-valued function. Thus, we now turn our attention to the complex power functions (and the related generalized exponential functions).

### 8. Definition of the generalized power and exponential functions

The generalized complex power function is defined via the following equation:

\[
w = z^c = e^{c \log z} , \quad z \neq 0 .
\]

\(^5\)This last statement [and eq. (38)] would be modified if a different convention for the principal interval were employed. For example, if \( 0 \leq \text{Arg} z < 2\pi \), then eq. (3) would be satisfied if \( \text{Im} z_1 > 0 \) and \( \text{Im} z_2 > 0 \).
To motivate this definition, we first note that if \( c = k \) is an integer, then for \( z = |z| e^{i \arg z} \),
\[
    z^k = |z|^k e^{k i \arg z} = e^{k \ln |z| + 2\pi i n} e^{k i \arg z} = e^{k \ln z}.
\]
In this case, \( w = z^k \) is a single-valued function, since
\[
    z^k = e^{k \ln z} = e^{k (\ln |z| + 2\pi i n)} = e^{k \ln z}.
\]
If \( c = 1/k \) (where \( k \) is an integer), then we have:
\[
    z^{1/k} = |z|^{1/k} e^{i \arg(z)/k} = e^{\ln |z|/k + 2\pi i n/k} = e^{(\ln z)/k},
\]
where \(|z|^{1/k}\) refers to the positive real \( k \)th root of \(|z|\). Combining the two results just obtained, we can easily prove that eq. (61) holds for any rational real number \( c \). Since any irrational real number can be approximated (to any desired accuracy) by a rational number, it follows by continuity that eq. (61) must hold for any real number \( c \). These arguments provide the motivation for defining the generalized complex power function as in eq. (61) for an arbitrary complex power \( c \).

Note that due to the multi-valued nature of \( \ln z \), it follows that \( w = z^c = e^{c \ln z} \) is also multi-valued for any non-integer value of \( c \), with a branch point at \( z = 0 \):
\[
    w = z^c = e^{c \ln z} = e^{c \ln z + 2\pi i m}, \quad m = 0, \pm 1, \pm 2, \pm 3, \cdots. \quad (62)
\]
If \( c \) is a rational number of the form \( c = m/k \), where \( m \) and \( k \) are integers with no common divisor, then we may take \( n = 0, 1, 2, \ldots, k-1 \) in eq. (62), since other values of \( n \) will not produce any new values of \( z^{m/k} \). It follows that the multi-valued function \( w = z^{m/k} \) has precisely \( k \) distinct branches. If \( c \) is irrational or complex, then the number of branches is infinite (with one branch for each possible choice of integer \( n \)).

Having defined the multi-valued complex power function, we are now able to compute \( \ln(z^c) \):
\[
    \ln(z^c) = \ln(e^{c \ln z}) = \ln(e^{c (\ln z + 2\pi i m)}) = \ln(e^{c \ln z} e^{2\pi i mc})
    = \ln(e^{c \ln z}) + \ln(e^{2\pi i mc}) = c (\ln z + 2\pi i m) + 2\pi i k
    = c \ln z + 2\pi i k = c \left( \ln z + \frac{2\pi i k}{c} \right), \quad (63)
\]
where \( k \) and \( m \) are arbitrary integers. Thus, \( \ln(z^c) = c \ln z \) if and only if \( k/c \) is an integer for all values of \( k \). The only possible way to satisfy this latter requirement is to take \( c = 1/n \), where \( n \) is an integer. Thus, eq. (54) is now verified.

We can define a single-valued power function by selecting the principal value of \( \ln z \) in eq. (61). Consequently, the \textit{principal value} of \( z^c \) is defined by
\[
    Z^c = e^{c \ln z}, \quad z \neq 0.
\]
For a lack of a better notation, I will indicate the principal value by capitalizing the variable $Z$ as above. The principal value definition of $z^c$ can lead to some unexpected results. For example, consider the principal value of the cube root function $w = Z^{1/3} = e^{\text{Ln}(z)/3}$. Then, for $z = -1$, the principal value of

$$\sqrt[3]{-1} = e^{\text{Ln}(-1)/3} = e^{\pi i/3} = \frac{1}{2} \left( 1 + i\sqrt{3} \right).$$

This may have surprised you, if you were expecting that $\sqrt[3]{-1} = -1$. To obtain the latter result would require a different choice of the principal interval in the definition of the principal value of $z^{1/3}$.

We are now in the position to check eq. (6) in the case that both the complex logarithm and complex power function are defined by their principal values. That is, we compute:

$$\text{Ln}(Z^c) = \text{Ln}(e^{c \text{Ln} z}) = c \text{Ln} z + 2\pi i N_c,$$

after using eq. (56), where $N_c$ is an integer determined by

$$N_c \equiv \left[ \frac{1}{2} - \frac{\text{Im} (c \text{Ln} z)}{2\pi} \right],$$

and $[ \ ]$ is the greatest integer bracket function [eq. (25)]. $N_c$ can be evaluated by noting that:

$$\text{Im} (c \text{Ln} z) = \text{Im} \{c (\text{Ln}|z| + i \text{Arg} z)\} = \text{Arg} z \text{Re} c + \text{Ln}|z| \text{Im} c.$$

Note that if $c = n$ where $n$ is an integer, then eq. (64) simply reduces to eq. (59), as expected. We conclude that eq. (6) is generally false both for the multi-valued complex logarithm and its principal value.

A function that in some respects is similar to the complex power function is the generalized exponential function. A possible definition of the generalized exponential function for $c \neq 0$ is:

$$w = c^z = e^{z \text{Ln} c} = e^{z(\text{Ln} c + 2\pi in)}, \quad n = 0, \pm 1, \pm 2, \pm 3, \cdots.$$

However, the multi-valued nature of this function differs somewhat from the multi-valued power function. In contrast to the latter, the generalized exponential function possesses no branch point in the finite complex $z$-plane. Thus, one can regard eq. (66) as defining a set of independent single-valued functions for each value of $n$. Typically, the $n = 0$ case is the most useful, in which case, we would simply define:

$$w = c^z = e^{z \text{Ln} c}, \quad c \neq 0.$$
This conforms with our definition of the exponential function in section 2 (where \( c = e \)). Henceforth, we shall employ eq. (67) as the definition of the single-valued generalized exponential function.

Some results for the principal value of the complex power function can be immediately adapted to the generalized exponential function. For example, by an almost identical computation as in eqs. (64) and (65), we find that:

\[
\ln(c^z) = \ln(e^{z \ln c}) = z \ln c + 2\pi i N'_c, \tag{68}
\]

where \( N'_c \) is an integer determined by:

\[
N'_c \equiv \left[ \frac{1}{2} - \frac{\text{Im} (z \ln c)}{2\pi} \right]. \tag{69}
\]

9. Properties of the generalized power function

Let us examine the properties listed in eqs. (7)–(13). Eq. (7) defines the complex power function. It is tempting to write:

\[
z^a z^b = e^{a \ln z} e^{b \ln z} = e^{a \ln z + b \ln z} = e^{(a+b) \ln z} = z^{a+b}. \tag{70}
\]

However, consider the case of non-integer \( a \) and \( b \) where \( a + b \) is an integer. In this case, eq. (70) cannot be correct since it would equate a multi-valued function \( z^a z^b \) with a single-valued function \( z^{a+b} \). In fact, the questionable step in eq. (70) is false:

\[
a \ln z + b \ln z \neq (a + b) \ln z \quad \text{[FALSE!!].} \tag{71}
\]

We previously noted that eq. (71) is false in the case of \( a = b = 1 \) [c.f. eq. (33)]. A more careful computation yields:

\[
z^a z^b = e^{a \ln z} e^{b \ln z} = e^{a \ln z + 2\pi i n} e^{b (\ln z + 2\pi i k)} = e^{(a+b) \ln z} e^{2\pi i (na+kb)},
\]

\[
z^{a+b} = e^{(a+b) \ln z} = e^{(a+b) \ln z + 2\pi i k} = e^{(a+b) \ln z} e^{2\pi i (a+b)}, \tag{72}
\]

where \( k \) and \( n \) are arbitrary integers. Hence, \( z^{a+b} \) is a subset of \( z^a z^b \). Whether the set of values for \( z^a z^b \) and \( z^{a+b} \) does or does not coincide depends on \( a \) and \( b \). However, in general, eq. (8) does not hold.

Similarly,

\[
z^a \over z^b = \frac{e^{a \ln z}}{e^{b \ln z}} = \frac{e^{a (\ln z + 2\pi i n)}}{e^{b (\ln z + 2\pi i k)}} = e^{(a-b) \ln z} e^{2\pi i (na-kb)},
\]

\[
z^{a-b} = e^{(a-b) \ln z} = e^{(a-b) \ln z + 2\pi i k} = e^{(a-b) \ln z} e^{2\pi i (a-b)}, \tag{73}
\]

Again, we emphasize the difference between the principal value of the power function \( Z^c \), which possesses a branch point at \( z = 0 \), and the single-valued exponential function of eq. (67) which does not possess a branch point (or any other type of singularity) in the finite complex \( z \)-plane.
where \( k \) and \( n \) are arbitrary integers. Hence, \( z^{a-b} \) is a subset of \( z^{a/b} \). Whether the set of values \( z^{a/b} \) and \( z^{a-b} \) does or does not coincide depends on \( a \) and \( b \). However, in general, eq. (9) does not hold. Setting \( a = b \) in eq. (73) yields the expected result:

\[
z^0 = 1, \quad z \neq 0
\]

for any non-zero complex number \( z \). Setting \( a = 0 \) in eq. (73) yields the set equality:

\[
z^{-b} = \frac{1}{z^b}, \quad (74)
\]

i.e., the set of values for \( z^{-b} \) and \( 1/z^b \) coincide. Thus, eq. (10) is satisfied. Note, however, that

\[
z^a z^{-a} = e^{a \ln z} e^{-a \ln z} = e^{a(\ln z - \ln z)} = e^{a \ln 1} = e^{2\pi i k a},
\]

where \( k \) is an arbitrary integer. Hence, if \( a \) is a non-integer, then \( z^a z^{-a} \neq 1 \) for \( k \neq 0 \). This is not in conflict with the set equality given in eq. (74) since there always exists at least one value of \( k \) (namely \( k = 0 \)) for which \( z^a z^{-a} = 1 \).

To show that eq. (11) can fail, we use eq. (49) in concluding that

\[
(z^a)^b = e^{a \ln z} e^{b \ln(\ln z)} = e^{b(a \ln z + 2\pi i k)} = e^{b \ln z} e^{2\pi i b k} = z^{ab} e^{2\pi i b k},
\]

where \( k \) is an arbitrary integer. Thus, \( z^{ab} \) is a subset of \( (z^a)^b \). The elements of \( z^{ab} \) and \( (z^a)^b \) coincide if and only if \( b \) is an integer. For example, if \( z = a = b = i \), we find that:

\[
(i)^i = i^i e^{-2\pi k} = i^{-1} e^{-2\pi k} = -i e^{-2\pi k}, \quad k = 0, \pm 1, \pm 2, \ldots. \quad (75)
\]

On the other hand, eqs. (12) and (13) are satisfied by the multi-valued power function, since

\[
(z_1 z_2)^a = e^{a \ln(z_1 z_2)} = e^{a(\ln z_1 + \ln z_2)} = e^{a \ln z_1} e^{a \ln z_2} = z_1^a z_2^a,
\]

\[
\left(z_1 \over z_2\right)^a = e^{a \ln(z_1/z_2)} = e^{a(\ln z_1 - \ln z_2)} = e^{a \ln z_1} e^{-a \ln z_2} = z_1^a z_2^{-a}.
\]

We now repeat the above analysis for the principal value of the power function, \( Z^c = e^{c \ln z} \). In this case, the results are somewhat reversed from the case of the multi-valued power function. In particular, eqs. (8)–(10) are satisfied, whereas eqs. (11)–(13) may be violated. For example, for the single-valued power function,

\[
Z^a Z^b = e^{a \ln z} e^{b \ln z} = e^{(a+b) \ln z} = Z^{a+b}, \quad (76)
\]

\[
\frac{Z^a}{Z^b} = \frac{e^{a \ln z}}{e^{b \ln z}} = e^{(a-b) \ln z} = Z^{a-b}, \quad (77)
\]

\[
Z^a Z^{-a} = e^{a \ln z} e^{-a \ln z} = 1. \quad (78)
\]
Setting $a = b$ in eq. (77) yields $Z^0 = 1$ (for $z \neq 0$) as expected.

Eq. (11) may be violated since eq. (56) implies that

$$(Z^c)^b = (e^{c\ln z})^b = e^{b\ln(e^{c\ln z})} = e^{bc\ln z} e^{2\pi ibN_e} = Z^{cb} e^{2\pi ibN_e},$$

where $N_e$ is an integer determined by eq. (65). As an example, if $z = b = c = i$, eq. (65) gives $N_e = 0$, which yields the principal value of $(i^i)^i = i^{i^i} = i^{-1} = -i$.

However, in general $N_e \neq 0$ is possible in which case eq. (11) does not hold.

Eqs. (12) and (13) may also be violated since eqs. (57) and (58) imply that

Eq. (11) may be violated since eq. (56) implies that

$$(Z_1Z_2)^a = e^{a \ln(z_1z_2)} = e^{a \ln(z_1 + \ln z_2 + 2\pi i N_+) = Z_1^a Z_2^a e^{2\pi i a N_+},} \quad (79)$$

$$(Z_1/Z_2)^a = e^{a \ln(z_1/z_2)} = e^{a \ln(z_1 -\ln z_2 + 2\pi i N_-) = Z_1^a Z_2^a e^{2\pi i a N_-},} \quad (80)$$

where the integers $N_\pm$ are determined from eq. (38).

10. Properties of the generalized exponential function

The generalized exponential function, $u = e^z$ ($c \neq 0$), is a single-valued function defined by eq. (67). Using this definition and the properties of the complex exponential function $e^z$, one can quickly check whether eqs. (15)–(20) hold in the complex plane. The proof of eqs. (15)–(17) is nearly identical to the one given in eqs. (76)–(78):

$$c^{z_1}e^{z_2} = e^{z_1 \ln c} e^{z_2 \ln c} = e^{(z_1+z_2) \ln c} = e^{z_1+z_2}, \quad (81)$$

$$\frac{c^{z_1}}{c^{z_2}} = \frac{e^{z_1 \ln c}}{e^{z_2 \ln c}} = e^{(z_1-z_2) \ln c} = e^{z_1-z_2}, \quad (82)$$

$$c^z c^{-z} = e^{z \ln c} e^{-z \ln c} = 1. \quad (83)$$

However, eq. (18) does not generally hold. Using eq. (68),

$$(e^{z_1})^{z_2} = e^{z_2 \ln(e^{z_1})} = e^{z_2(z_1 \ln c+2\pi i N'_e)} = e^{z_2 z_1 \ln c} e^{2\pi i z z_2 N'_e} = e^{z_1 z_2} e^{2\pi i z z_2 N'_e}, \quad (84)$$

where $N'_e$ is determined by eq. (69) [with $z$ replaced by $z_1$].

The case of $c = e$ is noteworthy. Eq. (84) reduces to:

$$(e^{z_1})^{z_2} = e^{z_1 z_2} e^{2\pi i z z_2 N'_e}, \quad N'_e = \left[\frac{1}{2} - \frac{\text{Im } z_1}{2\pi}\right]. \quad (85)$$

If $z_2 = n$ where $n$ is any integer, then $e^{2\pi in N'_e} = 1$ (since $N'_e$ is an integer by definition of the bracket notation). Thus, we recover eq. (22).

Let us test eq. (85) by substituting $z_1 = -i\pi$. Then, $N'_e = 1$ and hence

$$(e^{-i\pi})^{z} = e^{i\pi z}.$$
This result may seem strange, but it is a consequence of our definition of the generalized exponential function, \( c^z = e^{z \ln c} \), which employs the principal value of the logarithm. Indeed

\[
(e^{-i\pi})^z = (-1)^z = e^{z \ln(-1)} = e^{i\pi z},
\]
since \( \ln(-1) = i\pi \). We conclude that eq. (18) can be violated, even for the ordinary exponential function.

Ultimately, the real difficulty with \((c^{z_1})^{z_2}\) is that it is simultaneously a generalized exponential function and a generalized power function. Thus, if \(z_2\) is a non-integer, it may be more convenient to treat \((c^{z_1})^{z_2}\) as a multi-valued function. That is, in this latter convention, we treat the generalized exponential function \(c^{z_1} = e^{z_1 \ln c}\) as a single-valued function (using the principal value definition of the logarithm in the exponent), whereas we treat the generalized power function \((c^{z_1})^{z_2} = e^{z_2 \ln(c^{z_1})}\) as a potential multi-valued function:

\[
(c^{z_1})^{z_2} = e^{z_2 \ln(c^{z_1})} = e^{z_2 \ln(c^{z_1 \ln c})} = e^{z_2(z_1 \ln c + 2\pi i k)} = e^{z_2 z_1 \ln c} e^{2\pi i z_2 k} = c^{z_1 z_2} e^{2\pi i z_2 k},
\]

where \(k\) is an arbitrary integer [see eq. (49)]. In particular, for \(z_2\) a non-integer, \((c^{z_1})^{z_2}\) is a multi-valued function, with branches corresponding to different choices of \(k\). For example, for \(c = z_1 = z_2 = i\), we recover eq. (75). One might be tempted to call the \(k = 0\) branch the principal value of \((c^{z_1})^{z_2}\), in which case eq. (18) would be valid. Clearly, we must define our conventions carefully if we wish to manipulate expressions involving exponentials of exponentials.

Finally, eqs. (19) and (20) may be violated. The calculation is nearly identical to the one given in eqs. (79) and (80):

\[
(ab)^z = e^{z \ln(ab)} = e^{z(\ln a + \ln b + 2\pi i N_+)} = a^z b^z e^{2\pi i z N_+},
\]

\[
\left(\frac{a}{b}\right)^z = e^{z \ln(a/b)} = e^{z(\ln a - \ln b + 2\pi i N_-)} = \frac{a^z}{b^z} e^{2\pi i z N_-},
\]

where the integers \(N_{\pm}\) are determined from eq. (38) [with \(z_1\) and \(z_2\) replaced by \(a\) and \(b\), respectively]. If \(\text{Re } a > 0\) and \(\text{Re } b > 0\), then \(N_{\pm} = 0\), and eqs. (19) and (20) are satisfied.

\[\text{In practice, many textbooks treat the generalized exponential function as a single-valued function, } \quad c^z = e^{z \ln c} \text{, only when } c \text{ is a positive real number. For any other value of } c, \text{ the multi-valued function } c^z = e^{z \ln c} \text{ is preferred. In this convention, } e^z \text{ is single-valued but } (e^{z_1})^{z_2} \text{ is multi-valued when } z_1 \text{ is not a real number (and } z_2 \text{ is a non-integer). We shall not pursue this approach further.}\]
References

The following three books were particularly useful in the preparation of these notes:

