ELEMENTARY REAL ANALYSIS:

DRIPPED VERSION

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D.R.I.P. = Dump the Riemann Integral Project. This version of the text includes an account of the natural integral on the real line and removes the development of the Riemann integral and the improper Riemann integral. It should be considered experimental.

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PREFACE

Dripped Preface, January 2008

> *Does anyone believe that the difference between the Lebesgue and Riemann integrals can have any physical significance, and that, whether, say, an airplane would or would not fly could depend on this difference? If such were claimed, I should not clare to fly in that plane.*[1]

The purpose of the dripped edition of our text is to replace the Riemann integral in the original with the natural integral on the real line that includes the Lebesgue integral. The presentation is much more elementary than usual, starting with Riemann sums and not starting with measure theory, although measure theory is added later. The "will it fly" question that introduces the preface is left to the instructor to answer; the serious limitations of the Riemann integral are well known, even if not appreciated by all users of mathematical analysis.

[1] . . . from *Yes, but will it fly?*, Nexus Network Journal–Vol. 4, no. 2, 2002, p. 9.

This *dripped*[2] version of the text was prepared for January 1, 2008 and made available then on our web set www.classicalrealanalysis.com. We are grateful to Alan Smithee for providing us with access to his notes and allowing us to select and copy freely from them. His originals are also posted on the web site www.classicalrealanalysis.com.

The intention is to provide instructors with an alternative to teaching the Riemann integral in undergraduate analysis courses. Chapters 8, 9, 10, 11, 12, 13, 15 and 18 now replace the original chapter where the Riemann integral was introduced and its properties developed. Use as much of this (or as little) as you like and the course should still be a fairly unified treatment of the subject.

There is a danger of putting in way too much and we have (barely) suppressed our natural inclination to do this. A full account of Lebesgue's integration theory along with all the methods of measure theory can easily be developed by adding material to what we have so far included. The exercises may hint at directions that might be pursued, but the incurious student will not likely be drawn in any deeper than he should.

Suggestions for introducing integration to analysis students

- [*Short drip*] Replace the Riemann integral with the calculus integral and the Newton integral only [Chapter 8].

- [*Medium drip*] Add sufficient elements of the theory to give a rudimentary picture of the integral and its properties [Chapters 9 and 10]. This is easier than the usual introductions to the Riemann integral, but it happens to include the calculus integral, the Newton integral, and the Lebesgue integral.

- [*Strong drip*] Throw in the basics of zero measure and zero variation [Chapters 11, 12, and 13]. While still at an elementary level, this allows a full treatment of the Lebesgue differentiation theorem, absolute continuity, and the most general version of the fundamental theorem of the calculus.

[2] *D.R.I.P.=Dump.the.Riemann.Integral.* In this we are following the suggestions of a number of professional mathematicians who believe (or did believe at some point) that a reasonable analysis course could be constructed with only historical reference to the Riemann integral, by using some aspects of the integration methods of R. Henstock and J. Kurzweil.

- [*Full drip*] Add parts of the advanced material as desired [Chapters 15, 18, and 17]. This brings the student up to a serious level in integration theory but not in the usual direction. The culmination, in Chapter 17, is the Lebesgue program for the measure-theoretic development of his integral. The standard approach starts with measure theory and slowly (some say painfully) develops the integral and its theory; this easier version *starts* with the integral and the measure theory develops naturally from it.

Summaries of the added *drip* chapters

Chapter 8 just recounts the calculus version of the integral and suggests that a study of Riemann sums should lead to an adequate theory of integration. This is at an entirely elementary level and, given a minimal ambition for the course, could be used alone (with no further drip material) for a simple course.

Chapter 9 gives the basics of the covering argument approach to elementary analysis, the use of partitions, Riemann sums, full and fine covers, and the Cousin covering lemma.

Chapter 10 contains the integration theory. There is enough theory on integrability criteria so that one can demonstrate just how large the class of integrable functions are. This chapter includes proofs that continuous functions and derivatives are integrable, in fact that the integral includes a fairly general version of the Newton integral. The usual (and some unusual) simple properties of integrals are proved.

Chapter 11 gives the theory of sets of measure zero and the usual applications for the integral. Sets of measure zero play a peculiar (but important) role in the theory of the Riemann integral. Since we have “dumped” that integral the measure zero sets now play a natural and compelling role. We include an elementary proof of the Lebesgue differentiation theorem, that functions of bounded variation are a.e. differentiable. The chapter contains a narrow version of the Vitali covering theorem, showing that null sets can be characterized by fine covers. The proof is elementary and is a convenient way to introduce Vitali covering arguments at an elementary level.

Chapter 13 gives a complete account of the fundamental theorem of the calculus for this integral. This goes far beyond what would be done in a traditional undergraduate class, and even exceeds somewhat what is done in many graduate classes, albeit using here fairly elementary methods. Indeed, the elementary version of the fundamental theorem of the calculus in Chapter 10 is already beyond what most courses would attempt.

Chapter 15 completes the previous chapter on integration of sequences and series by proving the monotone convergence theorem. This is done with no measure theory which many consider one of the strongest reasons for dumping the Riemann integral in favor of this integral.

Chapter 18 develops basic material on functions of bounded variation and Stieltjes integral. The Jordan decomposition theorem is proved. This material is more frequently reserved for advanced courses but there is little trouble in presenting this at an undergraduate level if you have sufficient reason to do so. It can be entirely skipped without interfering with any later chapters.

Finally, Chapter 17 gives Lesbesgue's measure-theoretic program for the integral. It starts with a proof of the Vitali covering theorem that should be accessible, especially since it is well anticipated by the mini-Vitali version of Chapter 11. One main difference with the standard treatment is that the integral is not *defined* by the measure methods, but is *characterized* by them. This would be a suitable elementary introduction to measure theory, preparatory to the student taking an abstract course in the subject. This chapter, too, can be entirely skipped without interfering with any later chapters.

Original Preface (2001)

University mathematics departments have for many years offered courses with titles such as *Advanced Calculus* or *Introductory Real Analysis*. These courses are taken by a variety of students, serve a number of purposes, and are written at various levels of sophistication. The students range from ones who have just completed a course in elementary calculus to beginning graduate students in mathematics. The purposes are multifold:

1. To present familiar concepts from calculus at a more rigorous level.

2. To introduce concepts that are not studied in elementary calculus but that are needed in more advanced undergraduate courses. This would include such topics as point set theory, uniform continuity of functions, and uniform convergence of sequences of functions.

3. To provide students with a level of mathematical sophistication that will prepare them for graduate work in mathematical analysis, or for graduate work in several applied fields such as engineering or economics.

4. To develop many of the topics that the authors feel all students of mathematics should know.

There are now many texts that address some or all of these objectives. These books range from ones that do little more than address objective (1) to ones that try to address all four objectives. The books of the first extreme are generally aimed at one-term courses for students with minimal background. Books at the other extreme often contain substantially more material than can be covered in a one-year course.

The level of rigor varies considerably from one book to another, as does the style of presentation. Some books endeavor to give a very efficient streamlined development; others try to be more user friendly. We have opted for the user-friendly approach. We feel this approach makes the concepts more meaningful to the student.

Our experience with students at various levels has shown that most students have difficulties when topics that are entirely new to them first appear. For some students that might occur almost immediately when rigorous proofs are required, for example, ones needing ε-δ arguments. For others, the difficulties begin with elementary point set theory, compactness arguments, and the like.

To help students with the transition from elementary calculus to a more rigorous course, we have included motivation for concepts most students have not seen before and provided more details in proofs when we introduce new methods. In addition, we have tried to give students ample opportunity to see the new tools in action.

For example, students often feel uneasy when they first encounter the various compactness arguments (Heine-Borel theorem, Bolzano-Weierstrass theorem, Cousin's lemma, introduced in Section 4.5). To help the student see why such theorems are useful, we pose the problem of determining circumstances under which local boundedness of a function f on a set E implies global boundedness of f on E. We show by example that some conditions on E are needed, namely that E be closed and bounded, and then show how each of several theorems could be used to show that closed and boundedness of the set E suffices. Thus we introduce students to the theorems by showing how the theorems can be used in natural ways to solve a problem.

We have also included some optional material, marked as "Advanced" or "Enrichment" and flagged with the symbol ✂.

Enrichment

We have indicated as "Enrichment"' some relatively elementary material that could be added to a longer course to provide enrichment and additional examples. For example, in Chapter 3 we have added to the study of series a section on infinite products. While such a topic plays an important role in the representation of analytic functions, it is presented here to allow the instructor to explore ideas that are closely related to the study of series and that help illustrate and review many of the fundamental ideas that have played a role in the study of series.

Advanced

We have indicated as "Advanced" material of a more mathematically sophisticated nature that can be omitted without loss of continuity. These topics might be needed in more advanced courses in real analysis or in certain of the marked sections or exercises that appear later in this book. For example, in Chapter 2 we have added to the study of sequence limits a section on lim sups and lim infs. For an elementary first course this can be considered somewhat advanced and skipped. Later problems and text material that require these concepts are carefully indicated. Thus, even though the text carries on to relatively advanced undergraduate analysis, a first course can be presented by avoiding these advanced sections.

We apply these markings to some entire chapters as well as to some sections within chapters and even to certain exercises. We do not view these markings as absolute. They can simply be interpreted in the following ways. Any unmarked material will not depend, in any substantial way, on earlier marked sections. In addition, if a section has been flagged and will be used in a much later section of this book, we indicate where it will be required.

The material marked "Advanced" is in line with goals (2) and (3). We resist the temptation to address objective (4). There are simply too many additional topics that one might feel every student should know (e.g., functions of bounded variation, Riemann-Stieltjes and Lebesgue integrals). To cover these topics in the manner we cover other material would render the book more like a reference book than a text that could reasonably be covered in a year. Students who have completed this book will be in a good position to study such topics at rigorous levels.

We include, however, a chapter on metric spaces. We do this for two reasons: to offer a more general framework for viewing concepts treated in earlier chapters, and to illustrate how the abstract viewpoint can be applied to solving concrete problems. The metric space presentation in Chapter 13 can be considered more advanced as the reader would require a reasonable level of preparation. Even so, it is more readable and accessible than many other presentations of metric space theory, as we have prepared it with the assumption that the student has just the minimal background. For example, it is easier than the corresponding chapter in our graduate level text (*Real Analysis*, Prentice Hall, 1997) in which the student is expected to have studied the Lebesgue integral and to be at an appropriately sophisticated level.

The Exercises

The exercises form an integral part of the book. Many of these exercises are routine in nature. Others are more demanding. A few provide examples that are not usually presented in books of this type but that students have found challenging, interesting, and instructive.

Some exercises have been flagged with the ✂ symbol to indicate that they require material from a flagged section. For example, a first course is likely to skip over the section on lim sups and lim infs of sequences. Exercises that require those concepts are flagged so that the instructor can decide whether they can be used or not. Generally, that symbol on an exercise warns that it might not be suitable for routine assignments.

The exercises at the end of some of the chapters can be considered more challenging. They include some Putnam problems and some problems from the journal *American Mathematical Monthly*. They do not require more knowledge than is in the text material but often need a bit more persistence and some clever ideas. Students should be made aware that solutions to Putnam problems can be found on various Web sites and that solutions to *Monthly* problems are published; even so, the fun in such problems is in the attempt rather than in seeing someone else's solution.

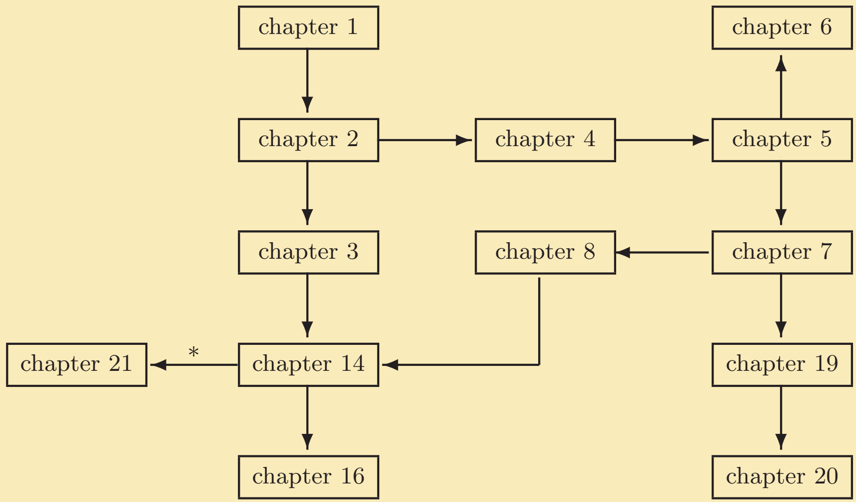

Figure 0.1. Chapter Dependencies (Unmarked Sections). Chapter numbers refer to this dripped edition, not the original.

Designing a Course

We have attempted to write this book in a manner sufficiently flexible to make it possible to use the book for courses of various lengths and a variety of levels of mathematical sophistication.

Much of the material in the book involves rigorous development of topics of a relatively elementary nature, topics that most students have studied at a nonrigorous level in a calculus course. A short course of moderate mathematical sophistication intended for students of minimal background can be based entirely on this material. Such a course might meet objective (1).

We have written this book in a leisurely style. This allows us to provide motivational discussions and historical perspective in a number of places. Even though the book is relatively large (in terms of number of pages), we can comfortably cover most of the main sections in a full-year course, including many of the interesting exercises.

Instructors teaching a short course have several options. They can base a course entirely on the unmarked material of Chapters 1, 2, 4, 5, and 7. As time permits, they can add the early parts of Chapters 3 and 8 or parts of Chapters 11 and 12 and some of the enrichment material.

Background

We should make one more point about this book. We do assume that students are familiar with nonrigorous calculus. In particular, we assume familiarity with the elementary functions and their elementary properties. We also assume some familiarity with computing derivatives and integrals. This allows us to illustrate various concepts using examples familiar to the students. For example, we begin Chapter 2, on sequences, with a discussion of approximating $\sqrt{2}$ using Newton's method. This is merely a motivational discussion, so we are not bothered by the fact that we don't treat the derivative formally until Chapter 7 and haven't yet proved that $\frac{d}{dx}(x^2 - 2) = 2x$. For students with minimal background we provide an appendix that informally covers such topics as notation, elementary set theory, functions, and proofs.

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Chapter 1

PROPERTIES OF THE REAL NUMBERS

Often I have considered the fact that most difficulties which block the progress of students trying to learn analysis stem from this: that although they understand little of ordinary algebra, still they attempt this more subtle art. From this it follows not only that they remain on the fringes, but in addition they entertain strange ideas about the concept of the infinite, which they must try to use. — Leonhard Euler (1748).

1.1 Introduction

The goal of any analysis course is to do some analysis. There are some wonderfully important and interesting facts that can be established in a first analysis course.

Unfortunately, all of the material we wish to cover rests on some foundations, foundations that may not have been properly set down in your earlier courses. Calculus courses traditionally avoid any foundational problems by simply not proving the statements that would need them. Here we cannot do this. We must start with the real number system.

Historically much of real analysis was undertaken without any clear understanding of the real numbers. To be sure the mathematicians of the time had a firm intuitive grasp of the nature of the real numbers and often found precisely the right tool to use in their proofs, but in many cases the tools could not be justified by any line of reasoning.

By the 1870s mathematicians such as Georg Cantor (1845–1918) and Richard Dedekind (1831–1916) had found ways to describe the real numbers that seemed rigorous. We could follow their example and find a presentation of the real numbers that starts at the very beginning and leads up slowly (very slowly) to the exact tools that we need to study analysis. This subject is, perhaps, best left to courses in logic, where other important foundation issues can be discussed.

The approach we shall take (and most textbooks take) is simply to list all the tools that are needed in such a study and take them for granted. You may consider that the real number system is exactly as you have always imagined it. You can sketch pictures of the real line and measure distances and consider the order just as before. Nothing is changed from high school algebra or calculus. But when we come to prove assertions about real numbers or real functions or real sets, we must use exactly the tools here and not rely on our intuition.

1.2 The Real Number System

To do real analysis we should know exactly what the real numbers are. Here is a loose exposition, suitable for calculus students but (as we will see) not suitable for us.

The Natural Numbers We start with the natural numbers. These are the counting numbers

$$1, 2, 3, 4, \dots .$$

The symbol $\mathbb{N}$ is used to indicate this collection. Thus $n \in \mathbb{N}$ means that n is a natural number, one of these numbers $1, 2, 3, 4, \dots$.

There are two operations on the natural numbers, addition and multiplication:

$$m + n \text{ and } m \cdot n.$$

There is also an order relation

$$m < n.$$

Large amounts of time in elementary school are devoted to an understanding of these operations and the order relation.

(Subtraction and division can also be defined, but not for all pairs in $\mathbb{N}$. While $7-5$ and $10/5$ are assigned a meaning [we say $x = 7-5$ if $x+5 = 7$ and we say $x = 10/5$ if $5 \cdot x = 10$] there is no meaning that can be attached to $5-7$ and $5/10$ in this number system.)

The Integers For various reasons, usually well motivated in the lower grades, the natural numbers prove to be rather limited in representing problems that arise in applications of mathematics to the real world. Thus they are enlarged by adjoining the negative integers and zero. Thus the collection

$$\dots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \dots$$

is denoted $\mathbb{Z}$ and called the integers. (The symbol $\mathbb{N}$ seems obvious enough [N for "natural"] but the symbol $\mathbb{Z}$ for the integers originates in the German word for whole number.)

Once again, there are two operations on $\mathbb{Z}$, addition and multiplication:

$$m + n \ \text{ and } m \cdot n.$$

Again there is an order relation

$$m < n.$$

Fortunately, the rules of arithmetic and order learned for the simpler system $\mathbb{N}$ continue to hold for $\mathbb{Z}$, and young students extend their abilities perhaps painlessly.

(Subtraction can now be defined in this larger number system, but division still may not be defined. For example, $-9/3$ is defined but $3/(-9)$ is not.)

The Rational Numbers At some point the problem of the failure of division in the sets $\mathbb{N}$ and $\mathbb{Z}$ becomes acute and the student must progress to an understanding of fractions. This larger number system is denoted $\mathbb{Q}$, where the symbol chosen is meant to suggest quotients, which is after all what fractions are.

The collection of all "numbers" of the form

$$\frac{m}{n},$$

where $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ is called the set of *rational numbers* and is denoted $\mathbb{Q}$.

A higher level of sophistication is demanded at this stage. Equality has a new meaning. In $\mathbb{N}$ or $\mathbb{Z}$ a statement $m = n$ meant merely that m and n were the same object. Now

$$\frac{m}{n} = \frac{a}{b}$$

for m, $a \in \mathbb{Z}$ and n, $b \in \mathbb{N}$ means that

$$m \cdot b = a \cdot n.$$

Addition and multiplication present major challenges too. Ultimately the students must learn that

$$\frac{m}{n} + \frac{a}{b} = \frac{mb + na}{nb}$$

and

$$\frac{m}{n} \cdot \frac{a}{b} = \frac{ma}{nb}.$$

Subtraction and division are similarly defined. Fortunately, once again the rules of arithmetic are unchanged. The associative rule, distributive rule, etc. all remain true even in this number system.

Again, too, an order relation

$$\frac{m}{n} < \frac{a}{b}$$

is available. It can be defined by requiring, for m, $a \in \mathbb{Z}$ and n, $b \in \mathbb{N}$,

$$mb < na.$$

The same rules for inequalities learned for integers and natural numbers are valid for rationals.

The Real Numbers Up to this point in developing the real numbers we have encountered only arithmetic operations. The progression from $\mathbb{N}$ to $\mathbb{Z}$ to $\mathbb{Q}$ is simply algebraic. All this algebra might have been a burden to the weaker students in the lower grades, but conceptually the steps are easy to grasp with a bit of familiarity.

The next step, needed for all calculus students, is to develop the still larger system of real numbers, denoted as $\mathbb{R}$. We often refer to the real number system as *the real line* and think about it as a geometrical object, even though nothing in our definitions would seem at first sight to allow this.

Most calculus students would be hard pressed to say exactly what these numbers are. They recognize that $\mathbb{R}$ includes all of $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ and also many new numbers, such as $\sqrt{2}$, e, and π. But asked what a real number is, many would return a blank stare. Even just asked what $\sqrt{2}$, e, or π are often produces puzzlement. Well, $\sqrt{2}$ is a number whose square is 2. But is there a number whose square is 2? A calculator might oblige with 1.4142136, but

$$(1.4142136)^2 \neq 2.$$

So what exactly "is" this number $\sqrt{2}$? If we are unable to write down a number whose square is 2, why can we claim that there is a number whose square is 2? And π and e are worse.

Some calculus texts handle this by proclaiming that real numbers are obtained by infinite decimal expansions. Thus while rational numbers have infinite decimal expansions that terminate (e.g., $1/4 = 0.25$) or repeat (e.g., $1/3 = 0.333333\dots$), the collection of real numbers would include *all* infinite decimal expansions whether repeating, terminating, or not. In that case the claim would be that there is some infinite decimal expansion $1.414213\dots$ whose square really is 2 and that infinite decimal expansion is the number we mean by the symbol $\sqrt{2}$.

This approach is adequate for applications of calculus and is a useful way to avoid doing any hard mathematics in introductory calculus courses. But you should recall that, at certain stages in the calculus textbook that you used, appeared a phrase such as "the proof of this next theorem is beyond the level of this text." It was beyond the level of the text only because the real numbers had not been properly treated and so there was no way that a proof could have been attempted.

We need to construct such proofs and so we need to abandon this loose, descriptive way of thinking about the real numbers. Instead we will define the real numbers to be a complete, ordered field. In the next sections each of these terms is defined.

1.3 Algebraic Structure

We describe the real numbers by assuming that they have a collection of properties. We do not construct the real numbers, we just announce what properties they are to have. Since the properties that we develop are familiar and acceptable and do in fact describe the real numbers that we are accustomed to using, this approach should not cause any distress. We are just stating rather clearly what it is about the real numbers that we need to use.

We begin with the algebraic structure.

In elementary algebra courses one learns many formulas that are valid for real numbers. For example, the formula

$$(x+y)+z = x+(y+z)$$

called the associative rule is learned. So also is the useful factoring rule

$$x^2 - y^2 = (x-y)(x+y).$$

It is possible to reduce the many rules to one small set of rules that can be used to prove all the other rules.

These rules can be used for other kinds of algebra, algebras where the objects are not real numbers but some other kind of mathematical constructions. This particular structure occurs so frequently, in fact, and in so many different applications that it has its own name. Any set of objects that has these same features is called a *field*. Thus we can say that the first important structure of the real number system is the field structure.

The following nine properties are called the *field axioms*. When we are performing algebraic manipulations in the real number system it is the field axioms that we are really using.

Assume that the set of real numbers $\mathbb{R}$ has two operations, called addition "+" and multiplication "·" and that these operations satisfy the field axioms. The operation $a \cdot b$ (multiplication) is most often written without the dot as ab.

A1 For any a, $b \in \mathbb{R}$ there is a number $a+b \in \mathbb{R}$ and $a+b = b+a$.

A2 For any a, b, $c \in \mathbb{R}$ the identity

$$(a+b)+c = a+(b+c)$$

is true.

A3 There is a unique number $0 \in \mathbb{R}$ so that, for all $a \in \mathbb{R}$,

$$a+0 = 0+a = a.$$

A4 For any number $a \in \mathbb{R}$ there is a corresponding number denoted by $-a$ with the property that

$$a+(-a) = 0.$$

M1 For any a, $b \in \mathbb{R}$ there is a number $ab \in \mathbb{R}$ and $ab = ba$.

M2 For any a, b, $c \in \mathbb{R}$ the identity

$$(ab)c = a(bc)$$

is true.

M3 There is a unique number $1 \in \mathbb{R}$ so that

$$a1 = 1a = a$$

for all $a \in \mathbb{R}$.

M4 For any number $a \in \mathbb{R}$, $a \neq 0$, there is a corresponding number denoted a^{-1} with the property that

$$aa^{-1} = 1.$$

AM1 For any a, b, $c \in \mathbb{R}$ the identity

$$(a+b)c = ac+bc$$

is true.

Note that we have labeled the axioms with letters indicating which operations are affected, thus A for addition and M for multiplication. The distributive rule AM1 connects addition and multiplication.

How are we to use these axioms? The answer likely is that, in an analysis course, you would not. You might try some of the exercises to understand what a field is and why the real numbers form a field. In an algebra course it would be most interesting to consider many other examples of fields and some of their applications. For an analysis course, understand that we are trying to specify exactly what we mean by the real number system, and these axioms are just the beginning of that process. The first step in that process is to declare that the real numbers form a field under the two operations of addition and multiplication.

Exercises

1.3.1 The field axioms include rules known often as associative rules, commutative rules and distributive rules. Which are which and why do they have these names?

1.3.2 To be precise we would have to say what is meant by the operations of addition and multiplication. Let S be a set and let $S \times S$ be the set of all ordered pairs (s_1, s_2) for $s_1,\ s_2 \in S$. A *binary operation* on S is a function $B : S \times S \to S$. Thus the operation takes the pair (s_1, s_2) and outputs the element $B(s_1, s_2)$. For example, addition is a binary operation. We could write $(s_1, s_2) \to A(s_1, s_2)$ rather than the more familiar $(s_1, s_2) \to s_1 + s_2$.

(a) Rewrite axioms A1–A4 using this notation $A(s_1, s_2)$ instead of the sum notation.

(b) Define a binary operation on $\mathbb{R}$ different from addition, subtraction, multiplication, or division and determine some of its properties.

(c) For a binary operation B define what you might mean by the commutative, associative, and distributive rules.

(d) Does the binary operation of subtraction satisfy any one of the commutative, associative, or distributive rules?

1.3.3 If in the field axioms for $\mathbb{R}$ we replace $\mathbb{R}$ by any other set with two operations $+$ and $\cdot$ that satisfy these nine properties, then we say that that structure is a field. For example, $\mathbb{Q}$ is a field. The rules are valid since $\mathbb{Q} \subset \mathbb{R}$. The only thing that needs to be checked is that $a + b$ and $a \cdot b$ are in $\mathbb{Q}$ if both a and b are. For this reason $\mathbb{Q}$ is called a *subfield* of $\mathbb{R}$. Find another subfield.

See Note 1

1.3.4 Let S be a set consisting of two elements labeled as A and B. Define $A+A=A$, $B+B=A$, $A+B=B+A=B$, $A\cdot A=A$, $A\cdot B=B\cdot A=A$, and $B\cdot B=B$. Show that all nine of the axioms of a field hold for this structure.

1.3.5 Using just the field axioms, show that

$$(x+1)^2 = x^2+2x+1$$

for all $x\in\mathbb{R}$. Would this identity be true in any field?

See Note 2

1.3.6 Define operations of addition and multiplication on $\mathbb{Z}_5=\{0,1,2,3,4\}$ as follows:

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

×	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

Show that $\mathbb{Z}_5$ satisfies all the field axioms.

1.3.7 Define operations of addition and multiplication on $\mathbb{Z}_6=\{0,1,2,3,4,5\}$ as follows:

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

×	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

Which of the field axioms does $\mathbb{Z}_6$ fail to satisfy?

1.4 Order Structure

The real number system also enjoys an order structure. Part of our usual picture of the reals is the sense that some numbers are “bigger” than others or more to the “right” than others. We express this by using inequalities $x < y$ or $x \leq y$. The order structure is closely related to the field structure. For example, when we use inequalities in elementary courses we frequently use the fact that if $x < y$ and $0 < z$, then $xz < yz$ (i.e., that inequalities can be multiplied through by positive numbers).

This structure, too, can be axiomatized and reduced to a small set of rules. Once again, these same rules can be found in other applications of mathematics. When these rules are added to the field axioms the result is called an *ordered field*.

The real number system is an ordered field, satisfying the four additional axioms. Here $a < b$ is now a statement that is either true or false. (Before $a + b$ and $a \cdot b$ were not statements, but elements of $\mathbb{R}$.)

O1 For any a, $b \in \mathbb{R}$ exactly one of the statements $a = b$, $a < b$ or $b < a$ is true.

O2 For any a, b, $c \in \mathbb{R}$ if $a < b$ is true and $b < c$ is true, then $a < c$ is true.

O3 For any a, $b \in \mathbb{R}$ if $a < b$ is true, then $a + c < b + c$ is also true for any $c \in \mathbb{R}$.

O4 For any a, $b \in \mathbb{R}$ if $a < b$ is true, then $a \cdot c < b \cdot c$ is also true for any $c \in \mathbb{R}$ for which $c > 0$.

Exercises

1.4.1 Using just the axioms, prove that $ad + bc < ac + bd$ if $a < b$ and $c < d$.

1.4.2 Show for every $n \in \mathbb{N}$ that $n^2 \geq n$.

1.4.3 Using just the axioms, prove the *arithmetic-geometric mean inequality*:

$$\sqrt{ab} \leq \frac{a+b}{2}$$

for any a, $b \in \mathbb{R}$ with $a > 0$ and $b > 0$. (Assume, for the moment, the existence of square roots.)

See Note 3

1.5 Bounds

Let E be some set of real numbers. There may or may not be a number M that is bigger than every number in the set E. If there is, we say that M is an upper bound for the set. If there is no upper bound, then the set is said to be *unbounded above* or to have no upper bound. This is a simple enough idea, but it is critical to an understanding of the real numbers and so we shall look more closely at it and give some precise definitions.

Definition 1.1: (Upper Bounds) Let E be a set of real numbers. A number M is said to be an *upper bound* for E if $x \leq M$ for all $x \in E$.

Definition 1.2: (Lower Bounds) Let E be a set of real numbers. A number m is said to be a *lower bound* for E if $m \leq x$ for all $x \in E$.

It is often important to note whether a set has bounds or not. A set that has an upper bound and a lower bound is called *bounded*.

A set can have many upper bounds. Indeed every number is an upper bound for the empty set $\emptyset$. A set may have no upper bounds. We can use the phrase "E is unbounded above" if there are no upper bounds. For some sets the most natural upper bound (from among the infinitely many to choose) is just the largest member of the set. This is called the maximum. Similarly, the most natural lower bound for some sets is the smallest member of the set, the minimum.

Definition 1.3: (Maximum) Let E be a set of real numbers. If there is a number M that belongs to E and is larger than every other member of E, then M is called the maximum of the set E and we write $M = \max E$.

Definition 1.4: (Minimum) Let E be a set of real numbers. If there is a number m that belongs to E and is smaller than every other member of E, then m is called the minimum of the set E and we write $m = \min E$.

Example 1.5: The interval

$$[0, 1] = \{x : 0 \leq x \leq 1\}$$

has a maximum and a minimum. The maximum is 1 and 1 is also an upper bound for the set. (If a set has a maximum, then that number must certainly be an upper bound for the set.) Any number larger than 1 is also an upper bound. The number 0 is the minimum and also a lower bound. ◀

Example 1.6: The interval

$$(0, 1) = \{x : 0 < x < 1\}$$

has no maximum and no minimum. At first glance some novices insist that the maximum should be 1 and the minimum 0 as before. But look at the definition. The maximum must be both an upper bound and also a member of the set. Here 1 and 0 are upper and lower bounds, respectively, but do not belong to the set. ◀

Example 1.7: The set $\mathbb{N}$ of natural numbers has a minimum but no maximum and no upper bounds at all. We would say that it is bounded below but not bounded above. ◀

1.6 Sups and Infs

Let us return to the subject of maxima and minima again. If E has a maximum, say M, then that maximum could be described by the statement

M is the least of all the upper bounds of E,

that is to say, M is the minimum of all the upper bounds. The most frequent language used here is "M is the least upper bound." It is possible for a set to have no maximum and yet be bounded above. In any example that comes to mind you will see that the set appears to have a least upper bound.

Example 1.8: The open interval $(0, 1)$ has no maximum, but many upper bounds. Certainly 2 is an upper bound and so is 1. The least of all the upper bounds is the number 1. Note that 1 cannot be described as a maximum because it fails to be in the set. ◀

Definition 1.9: (Least Upper Bound/Supremum) Let E be a set of real numbers that is bounded above and nonempty. If M is the least of all the upper bounds, then M is said to be *the least upper bound of* E or *the supremum* of E and we write $M = \sup E$.

Definition 1.10: (Greatest Lower Bound/Infimum) Let E be a set of real numbers that is bounded below and nonempty. If m is the greatest of all the lower bounds of E, then m is said to be *the greatest lower bound of* E or *the infimum* of E and we write $M = \inf E$.

To complete the definition of $\inf E$ and $\sup E$ it is most convenient to be able write this expression even for $E = \emptyset$ or for unbounded sets. Thus we write

1. $\inf \emptyset = \infty$ and $\sup \emptyset = -\infty$.
2. If E is unbounded above, then $\sup E = \infty$.
3. If E is unbounded below, then $\inf E = -\infty$.

The Axiom of Completeness Any example of a nonempty set that you are able to visualize that has an upper bound will also have a least upper bound. Pages of examples might convince you that all nonempty sets bounded above must have a least upper bound. Indeed your intuition will forbid you to accept the idea that this could not always be the case. To prove such an assertion is not possible using only the axioms for an ordered field. Thus we shall assume one further axiom, known as the axiom of completeness.

> **Completeness Axiom** A nonempty set of real numbers that is bounded above has a least upper bound (i.e., if E is nonempty and bounded above, then $\sup E$ exists and is a real number).

This now is the totality of all the axioms we need to assume. We have assumed that $\mathbb{R}$ is a field with two operations of addition and multiplication, that $\mathbb{R}$ is an ordered field with an inequality relation "$<$", and finally that $\mathbb{R}$ is a complete ordered field. This is enough to characterize the real numbers and the phrase "complete ordered field" refers to the system of real numbers and to no other system. (We shall not prove this statement; see Exercise 1.11.3 for a discussion.)

Exercises

1.6.1 Show that a set of real numbers E is bounded if and only if there is a positive number r so that $|x| < r$ for all $x \in E$.

1.6.2 Find $\sup E$ and $\inf E$ and (where possible) $\max E$ and $\min E$ for the following examples of sets:

(a) $E = \mathbb{N}$

(b) $E = \mathbb{Z}$

(c) $E = \mathbb{Q}$

(d) $E = \mathbb{R}$

(e) $E = \{-3, 2, 5, 7\}$

(f) $E = \{x : x^2 < 2\}$

(g) $E = \{x : x^2 - x - 1 < 0\}$

(h) $E = \{1/n : n \in \mathbb{N}\}$

(i) $E = \{\sqrt[n]{n} : n \in \mathbb{N}\}$

1.6.3 Under what conditions does $\sup E = \max E$?

1.6.4 Show for every nonempty, finite set E that $\sup E = \max E$.

See Note 4

1.6.5 For every $x \in \mathbb{R}$ define

$$[x] = \max\{n \in \mathbb{Z} : n \leq x\}$$

called the *greatest integer function.* Show that this is well defined and sketch the graph of the function.

1.6.6 Let A be a set of real numbers and let $B = \{-x : x \in A\}$. Find a relation between $\max A$ and $\min B$ and between $\min A$ and $\max B$.

1.6.7 Let A be a set of real numbers and let $B = \{-x : x \in A\}$. Find a relation between $\sup A$ and $\inf B$ and between $\inf A$ and $\sup B$.

1.6.8 Let A be a set of real numbers and let $B = \{x + r : x \in A\}$ for some number r. Find a relation between $\sup A$ and $\sup B$.

1.6.9 Let A be a set of real numbers and let $B = \{xr : x \in A\}$ for some positive number r. Find a relation between $\sup A$ and $\sup B$. (What happens if r is negative?)

1.6.10 Let A and B be sets of real numbers such that $A \subset B$. Find a relation among $\inf A$, $\inf B$, $\sup A$, and $\sup B$.

1.6.11 Let A and B be sets of real numbers and write $C = A \cup B$. Find a relation among $\sup A$, $\sup B$, and $\sup C$.

1.6.12 Let A and B be sets of real numbers and write $C = A \cap B$. Find a relation among $\sup A$, $\sup B$, and $\sup C$.

1.6.13 Let A and B be sets of real numbers and write

$$C = \{x + y : x \in A,\ y \in B\}.$$

Find a relation among $\sup A$, $\sup B$, and $\sup C$.

1.6.14 Let A and B be sets of real numbers and write

$$C = \{x + y : x \in A,\ y \in B\}.$$

Find a relation among $\inf A$, $\inf B$, and $\inf C$.

1.6.15 Let A be a set of real numbers and write $A^2 = \{x^2 : x \in A\}$. Are there any relations you can find between the infs and sups of the two sets?

1.6.16 Let E be a set of real numbers. Show that x is not an upper bound of E if and only if there exists a number $e \in E$ such that $e > x$.

1.6.17 Let A be a set of real numbers. Show that a real number x is the supremum of A if and only if $a \leq x$ for all $a \in A$ and for every positive number ε there is an element $a' \in A$ such that $x - \varepsilon < a'$.

1.6.18 Formulate a condition analogous to the preceding exercise for an infimum.

1.6.19 Using the completeness axiom, show that every nonempty set E of real numbers that is bounded below has a greatest lower bound (i.e., $\inf E$ exists and is a real number).

1.6.20 A function is said to be *bounded* if its range is a bounded set. Give examples of functions $f : \mathbb{R} \to \mathbb{R}$ that are bounded and examples of such functions that are unbounded. Give an example of one that has the property that

$$\sup\{f(x) : x \in \mathbb{R}\}$$

is finite but $\max\{f(x) : x \in \mathbb{R}\}$ does not exist.

1.6.21 The rational numbers $\mathbb{Q}$ satisfy the axioms for an ordered field. Show that the completeness axiom would not be satisfied. That is show that this statement is false: Every nonempty set E of rational numbers that is bounded above has a least upper bound (i.e., $\sup E$ exists and is a rational number).

1.6.22 Let F be the set of all numbers of the form $x + \sqrt{2}y$, where x and y are rational numbers. Show that F has all the properties of an ordered field but does not have the completeness property.

1.6.23 Let A and B be nonempty sets of real numbers and let

$$\delta(A,B) = \inf\{|a-b| : a \in A,\ b \in B\}.$$

$\delta(A,B)$ is often called the "distance" between the sets A and B.

(a) Let $A = \mathbb{N}$ and $B = \mathbb{R} \setminus \mathbb{N}$. Compute $\delta(A,B)$

(b) If A and B are finite sets, what does $\delta(A,B)$ represent?

(c) Let $B = [0,1]$. What does the statement $\delta(\{x\},B) = 0$ mean for the point x?

(d) Let $B = (0,1)$. What does the statement $\delta(\{x\},B) = 0$ mean for the point x?

1.7 The Archimedean Property

There is an important relationship holding between the set of natural numbers $\mathbb{N}$ and the larger set of real numbers $\mathbb{R}$. Because we have a well-formed mental image of what the set of reals "looks like," this property is entirely intuitive and natural. It hardly seems that it would require a proof. It says that the set of natural numbers $\mathbb{N}$ has no upper bound (i.e., that there is no real number x so that $n \le x$ for all $n = 1, 2, 3, \dots$).

At first sight this seems to be a purely algebraic and order property of the reals. In fact it cannot be proved without invoking the completeness property of Section 1.6.

The property is named after the famous Greek mathematician known as Archimedes of Syracuse (287 B.C.–212 B.C.).[1]

[1] Archimedes seems to be the archetypical absent-minded mathematician. The historian Plutarch tells of his death at the hand of an invading army: "As fate would have it, Archimedes was intent on working out some problem by a diagram, and

Theorem 1.11 (Archimedean Property of $\mathbb{R}$) *The set of natural numbers $\mathbb{N}$ has no upper bound.*

Proof. The proof is obtained by contradiction. If the set $\mathbb{N}$ does have an upper bound, then it must have a least upper bound. Let $x = \sup \mathbb{N}$, supposing that such does exist as a finite real number. Then $n \leq x$ for all natural numbers n but $n \leq x - 1$ cannot be true for all natural numbers n. Choose some natural number m with $m > x - 1$. Then $m + 1$ is also an natural number and $m + 1 > x$. But that cannot be so since we defined x as the supremum. From this contradiction the theorem follows. ■

The archimedean theorem has some consequences that have a great impact on how we must think of the real numbers.

1. No matter how large a real number x is given, there is always a natural number n larger.

2. Given any positive number y, no matter how large, and any positive number x, no matter how small, one can add x to itself sufficiently many times so that the result exceeds y (i.e., $nx > y$ for some $n \in \mathbb{N}$).

3. Given any positive number x, no matter how small, one can always find a fraction $1/n$ with n a natural number that is smaller (i.e., so that $1/n < x$).

Each of these is a consequence of the archimedean theorem, and the archimedean theorem in turn can be derived from any one of these.

Exercises

1.7.1 Using the archimedean theorem, prove each of the three statements that follow the proof of the archimedean theorem.

having fixed both his mind and eyes upon the subject of his speculation, he did not notice the entry of the Romans nor that the city was taken. In this transport of study a soldier unexpectedly came up to him and commanded that he accompany him. When he declined to do this before he had finished his problem, the enraged soldier drew his sword and ran him through." For this biographical detail and many others on all the mathematicians in this book consult *http://www-history.mcs.st-and.ac.uk/history.*

1.7.2 Suppose that it is true that for each $x > 0$ there is an $n \in \mathbb{N}$ so that $1/n < x$. Prove the archimedean theorem using this assumption.

1.7.3 Without using the archimedean theorem, show that for each $x > 0$ there is an $n \in \mathbb{N}$ so that $1/n < x$.

See Note 5

1.7.4 Let x be any real number. Show that there is a $m \in \mathbb{Z}$ so that

$$m \leq x < m + 1.$$

Show that m is unique.

1.7.5 The mathematician Leibniz based his calculus on the assumption that there were "infinitesimals," positive real numbers that are extremely small—smaller than all positive rational numbers certainly. Some calculus students also believe, apparently, in the existence of such numbers since they can imagine a number that is "just next to zero." Is there a positive real number smaller than all positive rational numbers?

1.7.6 The archimedean property asserts that if $x > 0$, then there is a natural number N so that $1/N < x$. The proof requires the completeness axiom. Give a proof that does not use the completeness axiom that works for x rational. Find a proof that is valid for $x = \sqrt{y}$, where y is rational.

1.7.7 In Section 1.2 we made much of the fact that there is a number whose square is 2 and so $\sqrt{2}$ does exist as a real number. Show that

$$\alpha = \sup\{x \in \mathbb{R} : x^2 < 2\}$$

exists as a real number and that $\alpha^2 = 2$.

See Note 6

1.8 Inductive Property of $\mathbb{N}$

Since the natural numbers are included in the set of real numbers there are further important properties of $\mathbb{N}$ that can be deduced from the axioms. The most important of these is the principle of induction. This is the basis for the technique of proof known as induction, which is often used in this text. For an elementary account and some practice, see Section A.8 in the appendix.

We first prove a statement that is equivalent.

Theorem 1.12 (Well-Ordering Property) *Every nonempty subset of* $\mathbb{N}$ *has a smallest element.*

Proof. Let $S \subset \mathbb{N}$ and $S \neq \emptyset$. Then $\alpha = \inf S$ must exist and be a real number since S is bounded below. If $\alpha \in S$, then we are done since we have found a minimal element.

Suppose not. Then, while α is the greatest lower bound of S, α is not a minimum. There must be an element of S that is smaller than $\alpha + 1$ since α is the greatest lower bound of S. That element cannot be α since we have assumed that $\alpha \notin S$. Thus we have found $x \in S$ with

$$\alpha < x < \alpha + 1.$$

Now x is not a lower bound of S, since it is greater than the greatest lower bound of S, so there must be yet another element y of S such that

$$\alpha < y < x < \alpha + 1.$$

But now we have reached an impossibility, for x and y are in S and both natural numbers, but $0 < x - y < 1$, which cannot happen. From this contradiction the proof now follows. ■

Now we can state and prove the principle of induction.

Theorem 1.13 (Principle of Induction) *Let* $S \subset \mathbb{N}$ *so that* $1 \in S$ *and, for every natural number* n*, if* $n \in S$ *then so also is* $n + 1$*. Then* $S = \mathbb{N}$*.*

Proof. Let $E = \mathbb{N} \setminus S$. We claim that $E = \emptyset$ and then it follows that $S = \mathbb{N}$ proving the theorem. Suppose not (i.e., suppose $E \neq \emptyset$). By Theorem 1.12 there is a first element α of E. Can $\alpha = 1$? No, because $1 \in S$ by hypothesis. Thus $\alpha - 1$ is also a natural number and, since it cannot be in E it must be in S. By hypothesis it follows that $\alpha = (\alpha - 1) + 1$ must be in S. But it is in E. This is impossible and so we have obtained a contradiction, proving our theorem. ■

Exercises

1.8.1 Show that any bounded, nonempty set of natural numbers has a maximal element.

1.8.2 Show that any bounded, nonempty subset of $\mathbb{Z}$ has a maximum and a minimum.

1.8.3 For further exercises on proving statements using induction as a method, see Section A.8.

1.9 The Rational Numbers Are Dense

There is an important relationship holding between the set of rational numbers $\mathbb{Q}$ and the larger set of real numbers $\mathbb{R}$. The rational numbers are dense. They make an appearance in every interval; there are no gaps, no intervals that miss having rational numbers.

For practical purposes this has great consequences. We need never actually compute with arbitrary real numbers, since close by are rational numbers that can be used. Thus, while π is irrational, in routine computations with a practical view any nearby fraction might do. At various times people have used 3, 22/7, and 3.14159, for example.

For theoretical reasons this fact is of great importance too. It allows many arguments to replace a consideration of the set of real numbers with the smaller set of rationals. Since every real is as close as we please to a rational and since the rationals can be carefully described and easily worked with, many simplifications are allowed.

Definition 1.14: (Dense Sets) A set E of real numbers is said to be *dense* (or *dense in* $\mathbb{R}$) if every interval (a, b) contains a point of E.

Theorem 1.15: *The set $\mathbb{Q}$ of rational numbers is dense.*

Proof. Let $x < y$ and consider the interval (x, y). We must find a rational number inside this interval.

By the archimedean theorem, Theorem 1.11, there is a natural number

$$n > \frac{1}{y - x}.$$

This means that $ny > nx + 1$.

Let m be chosen as the integer just less than $nx + 1$; more precisely (using Exercise 1.7.4), find $m \in \mathbb{Z}$ so that

$$m \leq nx + 1 < m + 1.$$

Now some arithmetic on these inequalities shows that

$$m - 1 \leq nx < ny$$

and then

$$x < \frac{m}{n} \leq x + \frac{1}{n} < y$$

thus exhibiting a rational number m/n in the interval (x, y). ■

Exercises

1.9.1 Show that the definition of "dense" could be given as

> A set E of real numbers is said to be *dense* if every interval (a, b) contains infinitely many points of E.

1.9.2 Find a rational number between $\sqrt{10}$ and π.

1.9.3 If a set E is dense, what can you conclude about a set $A \supset E$?

1.9.4 If a set E is dense, what can you conclude about the set $\mathbb{R} \setminus E$?

1.9.5 If two sets E_1 and E_2 are dense, what can you conclude about the set $E_1 \cap E_2$?

1.9.6 Show that the *dyadic rationals* (i.e., rational numbers of the form $m/2^n$ for $m \in \mathbb{Z}$, $n \in \mathbb{N}$) are dense.

1.9.7 Are the numbers of the form

$$\pm m/2^{100}$$

for $m \in \mathbb{N}$ dense? What is the length of the largest interval that contains no such number?

1.9.8 Show that the numbers of the form

$$\pm m\sqrt{2}/n$$

for m, $n \in \mathbb{N}$ are dense.

SEE NOTE 7

1.10 The Metric Structure of $\mathbb{R}$

In addition to the algebraic and order structure of the real numbers, we need to make measurements. We need to describe distances between points. These are the metric properties of the reals, to borrow a term from the Greek for measure (*metron*).

As usual, the distance between a point x and another point y is either $x-y$ or $y-x$ depending on which is positive. Thus the distance between 3 and -4 is 7. The distance between π and $\sqrt{10}$ is $\sqrt{10}-\pi$. To describe this in general requires the absolute value function which simply makes a choice between positive and negative.

Definition 1.16: (Absolute Value) For any real number x write

$$|x| = x \quad \text{if } x \geq 0$$

and

$$|x| = -x \quad \text{if } x < 0\ .$$

(Beginners tend to think of the absolute value function as "stripping off the negative sign," but the example

$$|\pi - \sqrt{10}| = \sqrt{10} - \pi$$

shows that this is a limited viewpoint.)

Properties of the Absolute Value Since the absolute value is defined directly in terms of inequalities (i.e., the choice $x \geq 0$ or $x < 0$), there are a number of properties that can be proved directly from properties of inequalities. These properties are used routinely and the student will need to have a complete mastery of them.

Theorem 1.17: *The absolute value function has the following properties:*

1. *For any* $x \in \mathbb{R}$, $-|x| \leq x \leq |x|$.
2. *For any* x, $y \in \mathbb{R}$, $|xy| = |x|\,|y|$.
3. *For any* x, $y \in \mathbb{R}$, $|x + y| \leq |x| + |y|$.
4. *For any* x, $y \in \mathbb{R}$, $|x| - |y| \leq |x - y|$ *and* $|y| - |x| \leq |x - y|$.

Distances on the Real Line Using the absolute value function we can define the distance function or metric.

Definition 1.18: (Distance) The distance between two real numbers x and y is

$$d(x, y) = |x - y|.$$

We hardly ever use the notation $d(x, y)$ in elementary analysis, preferring to write $|x - y|$ even while we are thinking of this as the distance between the two points. Thus if a sequence of points x_1, x_2, x_3, ... is growing ever closer to a point c, we should perhaps describe $d(x_n, c)$ as getting smaller and smaller, thus emphasizing that the distances are shrinking; more often we would simply write $|x_n - c|$ and expect you to interpret this as a distance.

Properties of the Distance Function The main properties of the distance function are just interpretations of the absolute value function. Expressed in the language of a distance function, they are geometrically very intuitive:

1. $d(x, y) \geq 0$

   (all distances are positive or zero).

2. $d(x, y) = 0$ if and only if $x = y$

   (different points are at positive distance apart).

3. $d(x, y) = d(y, x)$

   (distance is symmetric, that is the distance from x to y is the same as from y to x)).

4. $d(x, y) \leq d(x, z) + d(z, y)$

   (the triangle inequality, that is it is no longer to go directly from x to y than to go from x to z and then to y).

In Chapter **??** we will study general structures called metric spaces, where exactly such a notion of distance satisfying these four properties is used. For now we prefer to rewrite these properties in the language of the absolute value, where they lose some of their intuitive appeal. But it is in this form that we are likely to use them.

1. $|a| \geq 0$.

2. $|a| = 0$ if and only if $a = 0$.

3. $|a| = |-a|$.

4. $|a + b| \leq |a| + |b|$ (the triangle inequality).

Exercises

1.10.1 Show that $|x| = \max\{x, -x\}$.

1.10.2 Show that $\max\{x, y\} = |x - y|/2 + (x + y)/2$. What expression would give $\min\{x, y\}$?

1.10.3 Show that the inequalities $|x - a| < \varepsilon$ and

$$a - \varepsilon < x < a + \varepsilon$$

are equivalent.

1.10.4 Show that if $\alpha < x < \beta$ and $\alpha < y < \beta$, then $|x - y| < \beta - \alpha$ and interpret this geometrically as a statement about the interval (α, β).

1.10.5 Show that $||x| - |y|| \leq |x - y|$ assuming the triangle inequality (i.e., that $|a + b| \leq |a| + |b|$). This inequality is also called the triangle inequality.

1.10.6 Under what conditions is it true that $|x + y| = |x| + |y|$?

1.10.7 Under what conditions is it true that

$$|x - y| + |y - z| = |x - z|?$$

1.10.8 Show that

$$|x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|$$

for any numbers $x_1, x_2, \ldots, x_n$.

1.10.9 Let E be a set of real numbers and let $A = \{|x| : x \in E\}$. What relations can you find between the infs and sups of the two sets?

1.10.10 Find the inf and sup of the set $\{x : |2x + \pi| < \sqrt{2}\}$.

1.11 Challenging Problems for Chapter 1

1.11.1 The complex numbers $\mathbb{C}$ are defined as equal to the set of all ordered pairs of real numbers subject to these operations:

$$(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2)$$

and

$$(a_1, b_1) \cdot (a_2, b_2) = (a_1 a_2 - b_1 b_2, a_1 b_2 + a_2 b_1).$$

(a) Show that $\mathbb{C}$ is a field.

(b) What are the additive and multiplicative identity elements?

(c) What are the additive and multiplicative inverses of an element (a, b)?

(d) Solve $(a, b)^2 = (1, 0)$ in $\mathbb{C}$.

(e) We identify $\mathbb{R}$ with a subset of $\mathbb{C}$ by identifying the elements $x \in \mathbb{R}$ with the element $(x, 0)$ in $\mathbb{C}$. Explain how this can be interpreted as saying that "$\mathbb{R}$ is a subfield of $\mathbb{C}$."

(f) Show that there is an element $i \in \mathbb{C}$ with $i^2 = -1$ so that every element $z \in \mathbb{C}$ can be written as $z = x + iy$ for $x, y \in \mathbb{R}$.

(g) Explain why the equation $x^2 + x + 1 = 0$ has no solution in $\mathbb{R}$ but two solutions in $\mathbb{C}$.

1.11.2 Can an order be defined on the field $\mathbb{C}$ of Exercise 1.11.1 in such a way so to make it an ordered field?

1.11.3 The statement that every complete ordered field "is" the real number system means the following. Suppose that F is a nonempty set with operations of addition "+" and multiplication "·" and an order relation "<" that satisfies all the axioms of an ordered field and also the axiom of completeness. Then there is a one-to-one onto function $f : \mathbb{R} \to F$ that has the following properties:

(a) $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$.

(b) $f(x \cdot y) = f(x) \cdot f(y)$ for all $x, y \in \mathbb{R}$.

(c) $f(x) < f(y)$ if and only if $x < y$ for $x, y \in \mathbb{R}$.

Thus, in a certain sense, F and $\mathbb{R}$ are essentially the same object. Attempt a proof of this statement. [Note that $x + y$ for $x, y \in \mathbb{R}$ refers to the addition in the reals whereas $f(x) + f(y)$ refers to the addition in the set F.]

1.11.4 We have assumed in the text that the set $\mathbb{N}$ is obviously contained in $\mathbb{R}$. After all, 1 is a real number (it's in the axioms), 2 is just $1 + 1$ and so real, 3 is $2 + 1$ etc. In that way we have been able to prove the material of Section 1.8. But there is a logical flaw here. We would need induction really to define $\mathbb{N}$ in this way (and not just say "etc."). Here is a set of exercises that would remedy that for students with some background in set manipulations.

(a) Define a set $S \subset \mathbb{R}$ to be *inductive* if $1 \in S$ and if $x \in S$ implies that $x + 1 \in S$. Show that $\mathbb{R}$ is inductive.

(b) Show that there is a smallest inductive set by showing that the intersection of the family of all inductive sets is itself inductive.

(c) Define $\mathbb{N}$ to be that smallest inductive set.

(d) Prove Theorem 1.13 now. (That is, show that any set S with the property stated there is inductive and conclude that $S = \mathbb{N}$.)

(e) Prove Theorem 1.12 now. (That is, with this definition of $\mathbb{N}$ prove the well-ordering property.)

1.11.5 Use this definition of "dense in a set" to answer the following questions:

A set E of real numbers is said to be *dense in a set* A if every interval (a, b) that contains a point of A also contains a point of E.

(a) Show that dense in the set of all reals is the same as dense.

(b) Give an example of a set E dense in $\mathbb{N}$ but with $E \cap \mathbb{N} = \emptyset$.

(c) Show that the irrationals are dense in the rationals. (A real number is *irrational* if it is not rational, that is if it belongs to $\mathbb{R}$ but not to $\mathbb{Q}$.)

(d) Show that the rationals are dense in the irrationals.

(e) What property does a set E have that is equivalent to the assertion that $\mathbb{R} \setminus E$ is dense in E?

1.11.6 Let G be a subgroup of the real numbers under addition (i.e., if x and y are in G, then $x + y \in G$ and $-x \in G$). Show that either G is a dense subset of $\mathbb{R}$ or else there is a real number α so that

$$G = \{n\alpha : n = 0, \pm 1, \pm 2, \pm 3, \dots\}.$$

SEE NOTE 8

Notes

[1]Exercise 1.3.3. Let F be the set of all numbers of the form $x + y\sqrt{2}$ where x, $y \in \mathbb{Q}$. Again to be sure that nine properties of a field hold it is enough to check, here, that $a + b$ and $a \cdot b$ are in F if both a and b are.

[2]Exercise 1.3.5. As a first step define what x^2 and $2x$ really mean. In fact, define 2. (It would be defined as $2 = 1 + 1$ since 1 and addition are defined in the field axioms.) Then multiply $(x + 1) \cdot (x + 1)$ using only the rules given here. Since your proof uses only the field axioms, it must be valid in any situation in which these axioms are true, not just for $\mathbb{R}$.

[3]Exercise 1.4.3. Suppose $a > 0$ and $b > 0$ and $a \neq b$. Establish that $\sqrt{a} \neq \sqrt{b}$. Establish that

$$(\sqrt{a} - \sqrt{b})^2 > 0.$$

Carry on. What have you proved? Now what if $a = b$?

[4]Exercise 1.6.4. You can use induction on the size of E, that is, prove for every natural number n that if E has n elements, then

$$\sup E = \max E.$$

[5]Exercise 1.7.3. Suppose not, then the set

$$\{1/n : n = 1, 2, 3, \dots\}$$

has a positive lower bound, etc. You will have to use the existence of a greatest lower bound.

[6]Exercise 1.7.7. Not that easy to show. Rule out the possibilities $\alpha^2 < 2$ and $\alpha^2 > 2$ using the archimedean property to assist.

[7]Exercise 1.9.8. To find a number in (x, y), find a rational in $(x//\sqrt{2}, y//\sqrt{2})$. Conclude from this that the set of all (irrational) numbers of the form $\pm m\sqrt{2}/n$ is dense.

[8]Exercise 1.11.6. If $G = \{0\}$, then take $\alpha = 0$. If not, let $\alpha = \inf G \cap (0, \infty)$. Case 1: If $\alpha = 0$ show that G is dense. Case 2: If $\alpha > 0$ show that

$$G = \{n\alpha : n = 0, \pm 1, \pm 2, \pm 3, \dots\}.$$

For case 1 consider an interval (r, s) with $r < s$. We wish to find a member of G in that interval. To keep the argument simple just consider, for the moment, the situation in which $0 < r < s$. Choose $g \in G$ with $0 < g < s - r$. The set

$$M = \{n \in \mathbb{N} : ng \geq s\}$$

is nonempty (why?) and so there is a minimal element m in M (why?). Now check that $(m-1)g$ is in G and inside the interval (r, s).

Chapter 2

SEQUENCES

2.1 Introduction

Let us start our discussion with a method for solving equations that originated with Newton in 1669. To solve an equation $f(x) = 0$ the method proposes the introduction of a new function

$$F(x) = x - \frac{f(x)}{f'(x)}.$$

We begin with a guess at a solution of $f(x) = 0$, say x_1 and compute $x_2 = F(x_1)$ in the hopes that x_2 is closer to a solution than x_1 was. The process is repeated so that $x_3 = F(x_2)$, $x_4 = F(x_3)$, $x_5 = F(x_4)$, ... and so on until the desired accuracy is reached. Processes of this type have been known for at least 3500 years although not in such a modern notation.

We illustrate by finding an approximate value for $\sqrt{2}$ this way. We solve the equation $f(x) = x^2 - 2 = 0$ by computing the function

$$F(x) = x - \frac{f(x)}{f'(x)} = x - \frac{x^2 - 2}{2x}$$

and using it to improve our guess. A first (very crude) guess of $x_1 = 1$ will produce the following list of values for our subsequent steps in the procedure. We have retained 60 digits in the decimal expansions to show how this is working:

$$
\begin{aligned}
x_1 &= 1.00000000000000000000000000000000000000000000000000000000000\\
x_2 &= 1.50000000000000000000000000000000000000000000000000000000000\\
x_3 &= 1.41666666666666666666666666666666666666666666666666666666667\\
x_4 &= 1.41421568627450980392156862745098039215686274509803921568628\\
x_5 &= 1.41421356237468991062629557889013491011655962211574404458490\\
x_6 &= 1.41421356237309504880168962350253024361498192577619742849829\\
x_7 &= 1.41421356237309504880168872420969807856967187537723400156101.
\end{aligned}
$$

To compare, here is the value of the true solution $\sqrt{2}$, computed in a different fashion to the same number of digits:

$$\sqrt{2} = 1.41421356237309504880168872420969807856967187537694807317668.$$

Note that after only four steps the procedure gives a value differing from the true value only in the sixth decimal place, and all subsequent values remain this close. A convenient way of expressing this is to write that

$$|x_n - \sqrt{2}| < 10^{-5} \text{ for all } n \geq 4.$$

By the seventh step, things are going even better and we can claim that

$$|x_n - \sqrt{2}| < 10^{-47} \text{ for all } n \geq 7.$$

It is inconceivable that anyone would require any further accuracy for any practical considerations. The error after the sixth step cannot exceed 10^{-47}, which is a tiny number. Even so, as mathematicians we can ask what may seem an entirely impractical sort of question. Can this accuracy of approximation continue forever? Is it possible that, if we wait long enough, we can find an approximation to $\sqrt{2}$ with *any* degree of accuracy?

Expressed more formally, if we are given a positive number ε (we call it epsilon to suggest that it measures an *error*) no matter how small, can we find a stage in this procedure so that the value computed and all subsequent values are closer to $\sqrt{2}$ than ε? In symbols, is there an integer n_0 (which will depend on just how small ε is) that is large enough so that

$$|x_n - \sqrt{2}| < \varepsilon \text{ for all } n \geq n_0?$$

If this is true then this sequence has a remarkable property. It is not merely in its first few terms a convenient way of computing $\sqrt{2}$ to some accuracy; the sequence truly represents the number $\sqrt{2}$ itself, and it cannot represent any other number. We shall say that the sequence converges to $\sqrt{2}$ and write

$$\lim_{n\to\infty} x_n = \sqrt{2}.$$

This is the beginning of the theory of convergence that is central to analysis. If mathematicians had never considered the ultimate behavior of such sequences and had contented themselves with using only the first few terms for practical computations, there would have been no subject known as analysis. These ideas lead, as you might imagine, to an ideal world of infinite precision, where sequences are not merely useful gadgets for getting good computations but are precise tools in discussing real numbers. From the theory of sequences and their convergence properties has developed a vast world of beautiful and useful mathematics.

For the student approaching this material for the first time this is a critical test. All of analysis, both pure and applied, rests on an understanding of limits. What you learn in this chapter will offer a foundation for all the rest that you will have to learn later.

2.2 Sequences

A sequence (of real numbers, of sets, of functions, of anything) is simply a list. There is a first element in the list, a second element, a third element, and so on continuing in an order forever. In mathematics a finite list is not called a sequence; a sequence must continue without interruption.

For a more formal definition notice that the natural numbers are playing a key role here. Every item in the sequence (the list) can be labeled by its position; label the first item with a "1," the second with a "2," and so on. Seen this way a sequence is merely then a function mapping the natural numbers $\mathbb{N}$ into some set. We state this as a definition. Since this chapter is exclusively about sequences of real numbers, the definition considers just this situation.

Definition 2.1: By a *sequence of real numbers* we mean a function

$$f : \mathbb{N} \to \mathbb{R}.$$

Thus the sequence *is* the function. Even so, we usually return to the list idea and write out the sequence f as

$$f(1), f(2), f(3), \dots, f(n), \dots$$

with the ellipsis (i.e., the three dots) indicating that the list is to continue in this fashion. The function values $f(1)$, $f(2)$, $f(3)$, ... are called the *terms* of the sequence. When it is not confusing we will refer to such a sequence using the expression

$$\{f(n)\}$$

(with the understanding that the index n ranges over all of the natural numbers).

If we need to return to the formality of functions we do, but try to keep the intuitive notion of a sequence as an unending list in mind. While computer scientists much prefer the function notation, mathematicians have become more accustomed to a subscript notation and would rather have the terms of the preceding sequence rendered as

$$f_1, f_2, f_3, \dots, f_n, \dots \quad \text{or} \quad \{f_n\}.$$

In this chapter we study sequences of real numbers. Later on we will encounter the same word applied to other lists of objects (e.g., sequences of intervals, sequences of sets, sequences of functions. In all cases the word sequence simply indicates a list of objects).

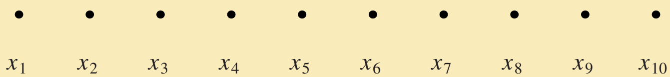

Figure 2.1. An arithmetic progression.

2.2.1 Sequence Examples

In order to specify some sequence we need to communicate what every term in the sequence is. For example, the sequence of even integers

$$2, 4, 6, 8, 10, \ldots$$

could be communicated in precisely that way: "Consider the sequence of even integers." Perhaps more direct would be to give a formula for all of the terms in the sequence: "Consider the sequence whose nth term is $x_n = 2n$." Or we could note that the sequence starts with 2 and then all the rest of the terms are obtained by adding 2 to the previous term: "Consider the sequence whose first term is 2 and whose nth term is 2 added to the $(n-1)$st term," that is,

$$x_n = 2 + x_{n-1}.$$

Often an explicit formula is best. Frequently though, a formula relating the nth term to some preceding term is preferable. Such formulas are called *recursion formulas* and would usually be more efficient if a computer is used to generate the terms.

Arithmetic Progressions The simplest types of sequences are those in which each term is obtained from the preceding by adding a fixed amount. These are called *arithmetic progressions*. The sequence

$$c, c + d, c + 2d, c + 3d, c + 4d, \ldots, c + (n-1)d, \ldots$$

is the most general arithmetic progression. The number d is called the *common difference*.

Every arithmetic progression could be given by a formula

$$x_n = c + (n-1)d$$

or a recursion formula

$$x_1 = c \qquad x_n = x_{n-1} + d.$$

Note that the explicit formula is of the form $x_n = f(n)$, where f is a linear function, $f(x) = dx + b$ for some b. Figure 2.1 shows the points of an arithmetic progression plotted on the line. If, instead, you plot the points (n, x_n) you will find that they all lie on a straight line with slope d.

Geometric Progressions. A variant on the arithmetic progression is obtained by replacing the addition of a fixed amount by the multiplication by a fixed amount. These sequences are called *geometric progressions*. The sequence

$$c, cr, cr^2, cr^3, cr^4, \dots, cr^{n-1}, \dots$$

is the most general geometric progression. The number r is called the *common ratio*.

Every geometric progression could be given by a formula

$$x_n = cr^{n-1}$$

or a recursion formula

$$x_1 = c \qquad x_n = rx_{n-1}.$$

Note that the explicit formula is of the form $x_n = f(n)$, where f is an exponential function $f(x) = br^x$ for some b. Figure 2.2 shows the points of a geometric progression plotted on the line. Alternatively, plot the points (n, x_n) and you will find that they all lie on the graph of an exponential function. If $c > 0$ and the common ratio r is larger than 1, the terms increase in size, becoming extremely large. If $0 < r < 1$, the terms decrease in size, getting smaller and smaller. (See Figure 2.2.)

Iteration The examples of an arithmetic progression and a geometric progression are special cases of a process called *iteration*. So too is the sequence generated by Newton's method in the introduction to this chapter.

Let f be some function. Start the sequence x_1, x_2, x_3, $\dots$ by assigning some value in the domain of f, say $x_1 = c$. All subsequent values are now obtained by feeding these values through the function repeatedly:

$$c, f(c), f(f(c)), f(f(f(c))), f(f(f(f(c)))), \dots.$$

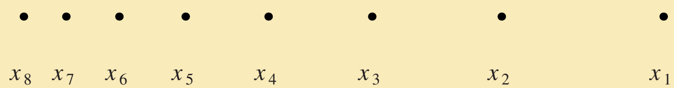

Figure 2.2. A geometric progression.

As long as all these values remain in the domain of the function f, the process can continue indefinitely and defines a sequence. If f is a function of the form $f(x) = x + b$, then the result is an arithmetic progression. If f is a function of the form $f(x) = ax$, then the result is a geometric progression.

A recursion formula best expresses this process and would offer the best way of writing a computer program to compute the sequence:

$$x_1 = c \qquad x_n = f(x_{n-1}).$$

Sequence of Partial Sums. If a sequence

$$x_1, x_2, x_3, x_4, \ldots$$

is given, we can construct a new sequence by adding the terms of the old one:

$$s_1 = x_1$$

$$s_2 = x_1 + x_2$$

$$s_3 = x_1 + x_2 + x_3$$

$$s_4 = x_1 + x_2 + x_3 + x_4$$

and continuing in this way. The process can also be described by a recursion formula:

$$s_1 = x_1\,, \qquad s_n = s_{n-1} + x_n.$$

The new sequence is called the *sequence of partial sums* of the old sequence $\{x_n\}$. We shall study such sequences in considerable depth in the next chapter.

For a particular example we could use $x_n = 1/n$ and the sequence of partial sums could be written as

$$s_n = 1 + 1/2 + 1/3 + \cdots + 1/n.$$

Is there a more attractive and simpler formula for s_n? The answer is no.

Example 2.2: The examples, given so far, are of a general nature and describe many sequences that we will encounter in analysis. But a sequence is just a list of numbers and need not be defined in any manner quite so systematic. For example, consider the sequence defined by $a_n = 1$ if n is divisible by three, $a_n = n$ if n is one more than a multiple of three, and $a_n = -2^n$ if n is two more than a multiple of three. The first few terms are evidently

$$1, -4, 1, 4, -32, 1, \ldots.$$

What would be the next three terms? ◀

Exercises

2.2.1 Let a sequence be defined by the phrase "consider the sequence of prime numbers $2, 3, 5, 7, 11, 13 \ldots$". Are you sure that this defines a sequence?

2.2.2 On IQ tests one frequently encounters statements such as "what is the next term in the sequence 3, 1, 4, 1, 5, . . . ?". In terms of our definition of a sequence is this correct usage? (By the way, what do you suppose the next term in the sequence might be?)

See Note 9

2.2.3 Give two different formulas (for two different sequences) that generate a sequence whose first four terms are 2, 4, 6, 8.

See Note 10

2.2.4 Give a formula that generates a sequence whose first five terms are 2, 4, 6, 8, π.

2.2.5 The examples listed here are the first few terms of a sequence that is either an arithmetic progression or a geometric progression. What is the next term in the sequence? Give a general formula for the sequence.

(a) 7, 4, 1, . . .

(b) .1, .01, .001, . . .

(c) 2, $\sqrt{2}$, 1, . . .

2.2.6 Consider the sequence defined recursively by

$$x_1 = \sqrt{2}\,, \quad x_n = \sqrt{2} + x_{n-1}.$$

Find an explicit formula for the nth term.

2.2.7 Consider the sequence defined recursively by

$$x_1 = \sqrt{2}\,, \quad x_n = \sqrt{2}x_{n-1}.$$

Find an explicit formula for the nth term.

2.2.8 Consider the sequence defined recursively by

$$x_1 = \sqrt{2}\,, \quad x_n = \sqrt{2 + x_{n-1}}.$$

Show, by induction, that $x_n < 2$ for all n.

2.2.9 Consider the sequence defined recursively by

$$x_1 = \sqrt{2}\,, \quad x_n = \sqrt{2 + x_{n-1}}.$$

Show, by induction, that $x_n < x_{n+1}$ for all n.

2.2.10 The sequence defined recursively by

$$f_1 = 1\,,\; f_2 = 1\,, \quad f_{n+2} = f_n + f_{n+1}$$

is called the *Fibonacci sequence*. It is possible to find an explicit formula for this sequence. Give it a try.

SEE NOTE 11

2.3 Countable Sets

Enrich.

A sequence of real numbers, formally, is a function whose domain is the set $\mathbb{N}$ of natural numbers and whose range is a subset of the reals $\mathbb{R}$. What sets might be the range of some sequence? To put it another way, what sets can have their elements arranged into an unending list? Are there sets that cannot be arranged into a list?

The arrangement of a collection of objects into a list is sometimes called an *enumeration*. Thus another way of phrasing this question is to ask what sets of real numbers can be *enumerated*?

The set of natural numbers is already arranged into a list in its natural order. The set of integers (including 0 and the negative integers) is not usually presented in the form of a list but can easily be so presented, as the following scheme suggests:

$$0, 1, -1, 2, -2, 3, -3, 4, -4, 5, -5, 6, -6, 7, -7, \ldots.$$

Example 2.3: The rational numbers can also be listed but this is quite remarkable, for at first sight no reasonable way of ordering them into a sequence seems likely to be possible. The usual order of the rationals in the reals is of little help.

To find such a scheme define the "rank" of a rational number m/n in its lowest terms (with $n \geq 1$) to be $|m| + n$. Now begin making a finite list of all the rational numbers at each rank; list these from smallest to largest. For example, at rank 1 we would have only the rational number $0/1$. At rank 2 we would have only the rational numbers $-1/1$, $1/1$. At rank 3 we would have only the rational numbers $-2/1$, $-1/2$, $1/2$, $2/1$. Carry on in this fashion through all the ranks. Now construct the final list by concatenating these shorter lists in order of the ranks:

$$0/1, -1/1, 1/1, -2/1, -1/2, 1/2, 2/1, \ldots.$$

The range of this sequence is the set of all rational numbers. ◀

Your first impression might be that few sets would be able to be the range of a sequence. But having seen in Example 2.3 that even the set of rational numbers $\mathbb{Q}$ that is seemingly so large can be listed, it might then appear that all sets can be so listed. After all, can you conceive of a set that is "larger" than the rationals in some way that would stop it being listed? The remarkable fact that there are sets that cannot be arranged to form the elements of some sequence was proved by Georg Cantor (1845–1918). This proof is essentially his original proof. (Note that this requires some familiarity with infinite decimal expansions; the exercises review what is needed.)

Theorem 2.4 (Cantor) *No interval (a, b) of real numbers can be the range of some sequence.*

Proof. It is enough to prove this for the interval $(0, 1)$ since there is nothing special about it (see Exercise 2.3.1). The proof is a proof by contradiction. We suppose that the theorem is false and that there is a sequence $\{s_n\}$ so that every number in the interval $(0, 1)$ appears at least once in the sequence. We obtain a contradiction by showing that this cannot be so. We shall use the sequence $\{s_n\}$ to find a number c in the interval $(0, 1)$ so that $s_n \neq c$ for all n.

Each of the points s_1, s_2, s_3 ... in our sequence is a number between 0 and 1 and so can be written as a decimal fraction. If we write this sequence out in decimal notation it might look like

$$s_1 = 0.x_{11}x_{12}x_{13}x_{14}x_{15}x_{16}\dots$$

$$s_2 = 0.x_{21}x_{22}x_{23}x_{24}x_{25}x_{26}\dots$$

$$s_3 = 0.x_{31}x_{32}x_{33}x_{34}x_{35}x_{36}\dots$$

etc. Now it is easy to find a number that is not in the list. Construct

$$c = 0.c_1c_2c_3c_4c_5c_6\dots$$

by choosing c_i to be either 5 or 6 whichever is different from x_{ii}. This number cannot be equal to any of the listed numbers s_1, s_2, s_3 ... since c and s_i differ in the ith position of their decimal expansions. This gives us our contradiction and so proves the theorem. ■

Definition 2.5: (Countable) A nonempty set S of real numbers is said to be *countable* if there is a sequence of real numbers whose range is the set S.

In the language of this definition then we can see that (1) any finite set is countable, (2) the natural numbers and the integers are countable, (3) the rational numbers are countable, and (4) no interval of real numbers is countable. By convention we also say that the empty set $\emptyset$ is countable.

Exercises

2.3.1 Show that, once it is known that the interval $(0, 1)$ cannot be expressed as the range of some sequence, it follows that any interval (a, b), $[a, b)$, $(a, b]$, or $[a, b]$ has the same property.

SEE NOTE 12

2.3.2 Some novices, on reading the proof of Cantor's theorem, say "Why can't you just put the number c that you found at the front of the list." What is your rejoinder?

2.3.3 A set (any set of objects) is said to be *countable* if it is either finite or there is an enumeration (list) of the set. Show that the following properties hold for arbitrary countable sets:

(a) All subsets of countable sets are countable.

(b) Any union of a pair of countable sets is countable.

(c) All finite sets are countable.

2.3.4 Show that the following property holds for countable sets: If

$$S_1,\ S_2,\ S_3,\ \ldots$$

is a sequence of countable sets of real numbers, then the set S formed by taking all elements that belong to at least one of the sets S_i is also a countable set.

SEE NOTE 13

2.3.5 Show that if a nonempty set is contained in the range of some sequence of real numbers, then there is a sequence whose range is precisely that set.

2.3.6 In Cantor's proof presented in this section we took for granted material about infinite decimal expansions. This is entirely justified by the theory of sequences studied later. Explain what it is that we need to prove about infinite decimal expansions to be sure that this proof is valid.

SEE NOTE 14

2.3.7 Define a relation on the family of subsets of $\mathbb{R}$ as follows. Say that $A \sim B$, where A and B are subsets of $\mathbb{R}$, if there is a function

$$f : A \to B$$

that is one-to-one and onto. (If $A \sim B$ we would say that A and B are "cardinally equivalent.") Show that this is an *equivalence relation*, that is, show that

(a) $A \sim A$ for any set A.

(b) If $A \sim B$ then $B \sim A$.

(c) If $A \sim B$ and $B \sim C$ then $A \sim C$.

2.3.8 Let A and B be finite sets. Under what conditions are these sets cardinally equivalent (in the language of Exercise 2.3.7)?

2.3.9 Show that an infinite set of real numbers that is countable is cardinally equivalent (in the language of Exercise 2.3.7) to the set $\mathbb{N}$. Give an example of an infinite set that is not cardinally equivalent to $\mathbb{N}$.

2.3.10 We define a real number to be *algebraic* if it is a solution of some polynomial equation

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0,$$

where all the coefficients are integers. Thus $\sqrt{2}$ is algebraic because it is a solution of $x^2 - 2 = 0$. The number π is not algebraic because no such polynomial equation can ever be found (although this is hard to prove). Show that the set of algebraic numbers is countable. A real number that is not algebraic is said to be *transcendental*. For example, it is known that e and π are transcendental. What can you say about the existence of other transcendental numbers?

See Note 15

2.4 Convergence

The sequence

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \dots$$

is getting closer and closer to the number 0. We say that this sequence *converges* to 0 or that the limit of the sequence is the number 0. How should this idea be properly defined?

The study of convergent sequences was undertaken and developed in the eighteenth century without any precise definition. The closest one might find to a definition in the early literature would have been something like

> *A sequence $\{s_n\}$ converges to a number L if the terms of the sequence get closer and closer to L.*

Apart from being too vague to be used as anything but a rough guide for the intuition, this is misleading in other respects. What about the sequence

$$.1, .01, .02, .001, .002, .0001, .0002, .00001, .00002, \ldots ?$$

Surely this should converge to 0 but the terms do not get steadily "closer and closer" but back off a bit at each second step. Also, the sequence

$$.1, .11, .111, .1111, .11111, .111111, \ldots$$

is getting "closer and closer" to .2, but we would not say the sequence converges to .2. A smaller number (1/9, which it is also getting closer and closer to) is the correct limit. We want not merely "closer and closer" but somehow a notion of "arbitrarily close."

The definition that captured the idea in the best way was given by Augustin Cauchy in the 1820s. He found a formulation that expressed the idea of "arbitrarily close" using inequalities. In this way the notion of limit is defined by a straightforward mathematical statement about inequalities.

Definition 2.6: (Limit of a Sequence) Let $\{s_n\}$ be a sequence of real numbers. We say that $\{s_n\}$ *converges* to a number L and write

$$\lim_{n\to\infty} s_n = L$$

or

$$s_n \to L \text{ as } n \to \infty$$

provided that for every number $\varepsilon > 0$ there is an integer N so that

$$|s_n - L| < \varepsilon$$

whenever $n \geq N$.

A sequence that converges is said to be *convergent*. A sequence that fails to converge is said to *diverge*. We are equally interested in both convergent and divergent sequences.

Note. In the definition the N depends on ε. If ε is particularly small, then N might have to be chosen large. In fact, then N is really a function of ε. Sometimes it is best to emphasize this and write $N(\varepsilon)$ rather than N.

Note, too, that if an N is found, then any larger N would also be able to be used. Thus the definition requires us to find some N but not necessarily the smallest N that would work.

While the definition does not say this, the real force of the definition is that the N can be determined *no matter how small a number ε is chosen.* If ε is given as rather large there may be no trouble finding the N value. If you find an N that works for $\varepsilon = .1$ that same N would work for all larger values of ε.

Example 2.7: Let us use the definition to prove that

$$\lim_{n\to\infty} \frac{n^2}{2n^2+1} = \frac{1}{2}.$$

It is by no means clear from the definition how to obtain that the limit is the number $L = \frac{1}{2}$. Indeed the definition is not intended as a method of finding limits. It assigns a precise meaning to the statement about the limit but offers no way of computing that limit. Fortunately most of us remember some calculus devices that can be used to first obtain the limit before attempting a proof of its validity.

$$\lim_{n\to\infty} \frac{n^2}{2n^2+1} = \lim_{n\to\infty} \frac{1}{2+1/n^2} = \frac{1}{\lim_{n\to\infty}(2+1/n^2)}$$

$$= \frac{1}{2+\lim_{n\to\infty}(1/n^2)} = \frac{1}{2}.$$

Indeed this would be a proof that the limit is $1/2$ provided that we could prove the validity of each of these steps. Later on we will prove this and so can avoid the ε, N arguments that we now use.

Let any positive ε be given. We need to find a number N [or $N(\varepsilon)$ if you prefer] so that every term in the sequence on and after the Nth term is closer to $1/2$ than ε, that is, so that

$$\left|\frac{n^2}{2n^2+1} - \frac{1}{2}\right| < \varepsilon$$

for $n = N$, $n = N + 1$, $n = N + 2$, ... It is easiest to work backward and discover just how large n should be for this. A little work shows that this will happen if

$$\frac{1}{2(2n^2+1)} < \varepsilon$$

or

$$4n^2 + 2 > \frac{1}{\varepsilon}.$$

The smallest n for which this statement is true could be our N. Thus we could use any integer N with

$$N^2 > \frac{1}{4}\left(\frac{1}{\varepsilon} - 2\right).$$

There is no obligation to find the smallest N that works and so, perhaps, the most convenient one here might be a bit larger, say take any integer N larger than

$$N > \frac{1}{2\sqrt{\varepsilon}}.$$

◀

The real lesson of the example, perhaps, is that we wish never to have to use the definition to check any limit computation. The definition offers a rigorous way to develop a theory of limits but an impractical method of computation of limits and a clumsy method of verification. Only rarely do we have to do a computation of this sort to verify a limit.

Uniqueness of Sequence Limits Let us take the first step in developing a theory of limits. This is to ensure that our definition has defined *limit* unambiguously. Is it possible that the definition allows for a sequence to converge to two different limits? If we have established that $s_n \to L$ is it possible that $s_n \to L_1$ for a different number L_1?

Theorem 2.8 (Uniqueness of Limits) *Suppose that*

$$\lim_{n\to\infty} s_n = L_1 \text{ and } \lim_{n\to\infty} s_n = L_2$$

are both true. Then $L_1 = L_2$.

Proof. Let ε be any positive number. Then, by definition, we must be able to find a number N_1 so that

$$|s_n - L_1| < \varepsilon$$

whenever $n \geq N_1$. We must also be able to find a number N_2 so that

$$|s_n - L_2| < \varepsilon$$

whenever $n \geq N_2$. Take m to be the maximum of N_1 and N_2. Then both assertions

$$|s_m - L_1| < \varepsilon \text{ and } |s_m - L_2| < \varepsilon$$

are true.

This allows us to conclude that

$$|L_1 - L_2| \leq |L_1 - s_m| + |s_m - L_2| < 2\varepsilon$$

so that

$$|L_1 - L_2| < 2\varepsilon.$$

But ε can be any positive number whatsoever. This could only be true if $L_1 = L_2$, which is what we wished to show. ■

Exercises

2.4.1 Give a precise ε, N argument to prove that $\lim_{n\to\infty} \frac{1}{n} = 0$.

2.4.2 Give a precise ε, N argument to prove the existence of

$$\lim_{n\to\infty} \frac{2n+3}{3n+4}.$$

2.4.3 Show that a sequence $\{s_n\}$ converges to a limit L if and only if the sequence $\{s_n - L\}$ converges to zero.

2.4.4 Show that a sequence $\{s_n\}$ converges to a limit L if and only if the sequence $\{-s_n\}$ converges to $-L$.

2.4.5 Show that Definition 2.6 is equivalent to the following slight modification:

> We write $\lim_{n\to\infty} s_n = L$ provided that for every positive integer m there is a real number N so that $|s_n - L| < 1/m$ whenever $n \geq N$.

2.4.6 Compute the limit

$$\lim_{n\to\infty} \frac{1+2+3+\cdots+n}{n^2}$$

and verify it by the definition.

See Note 16

2.4.7 Compute the limit

$$\lim_{n\to\infty} \frac{1^2+2^2+3^2+\cdots+n^2}{n^3}.$$

See Note 17

2.4.8 Suppose that $\{s_n\}$ is a convergent sequence. Prove that $\lim_{n\to\infty} 2s_n$ exists.

2.4.9 Prove that $\lim_{n\to\infty} n$ does not exist.

2.4.10 Prove that $\lim_{n\to\infty}(-1)^n$ does not exist.

2.4.11 The sequence $s_n = (-1)^n$ does not converge. For what values of $\varepsilon > 0$ is it nonetheless true that there is an integer N so that $|s_n - 1| < \varepsilon$ whenever $n \geq N$? For what values of $\varepsilon > 0$ is it nonetheless true that there is an integer N so that $|s_n - 0| < \varepsilon$ whenever $n \geq N$?

2.4.12 Let $\{s_n\}$ be a sequence that assumes only integer values. Under what conditions can such a sequence converge?

2.4.13 Let $\{s_n\}$ be a sequence and obtain a new sequence (sometimes called the "tail" of the sequence) by writing

$$t_n = s_{M+n} \quad \text{for } n = 1, 2, 3, \dots$$

where M is some integer (perhaps large). Show that $\{s_n\}$ converges if and only if $\{t_n\}$ converges.

2.4.14 Show that the statement "$\{s_n\}$ converges to L" is false if and only if there is a positive number c so that the inequality

$$|s_n - L| > c$$

holds for infinitely many values of n.

2.4.15 If $\{s_n\}$ is a sequence of positive numbers converging to 0, show that $\{\sqrt{s_n}\}$ also converges to zero.

2.4.16 If $\{s_n\}$ is a sequence of positive numbers converging to a positive number L, show that $\{\sqrt{s_n}\}$ converges to $\sqrt{L}$.

2.5 Divergence

A sequence that fails to converge is said to *diverge*. Some sequences diverge in a particularly interesting way, and it is worthwhile to have a language for this.

The sequence $s_n = n^2$ diverges because the terms get larger and larger. We are tempted to write

$$n^2 \to \infty \quad \text{or} \quad \lim_{n\to\infty} n^2 = \infty.$$

This conflicts with our definition of limit and so needs its own definition. We do not say that this sequence "converges to ∞" but rather that it "diverges to ∞."

Definition 2.9: (Divergence to ∞) Let $\{s_n\}$ be a sequence of real numbers. We say that $\{s_n\}$ *diverges to* ∞ and write

$$\lim_{n\to\infty} s_n = \infty$$

or

$$s_n \to \infty \text{ as } n \to \infty$$

provided that for every number M there is an integer N so that

$$s_n \geq M$$

whenever $n \geq N$.

Note. The definition does not announce this, but the force of the definition is that the choice of N is possible *no matter how large* M *is chosen.* There may be no difficulty in finding an N if the M given is not big.

Example 2.10: Let us prove that

$$\frac{n^2+1}{n+1} \to \infty$$

using the definition. If M is any positive number we need to find some point in the sequence after which all terms exceed M. Thus we need to consider the inequality

$$\frac{n^2+1}{n+1} \geq M.$$

After some arithmetic we see that this is equivalent to

$$n + \frac{1}{n+1} - \frac{n}{n+1} \geq M.$$

Since

$$\frac{n}{n+1} < 1$$

we see that, as long as $n \geq M+1$ this will be true. Thus take any integer $N \geq M+1$ and it will be true that

$$\frac{n^2+1}{n+1} \geq M$$

for all $n \geq N$. (Any larger value of N would work too.) ◀

Exercises

2.5.1 Formulate the definition of a sequence diverging to $-\infty$.

2.5.2 Show, using the definition, that $\lim_{n\to\infty} n^2 = \infty$.

2.5.3 Show, using the definition, that $\lim_{n\to\infty} \frac{n^3+1}{n^2+1} = \infty$.

2.5.4 Prove that if $s_n \to \infty$ then $-s_n \to -\infty$.

2.5.5 Prove that if $s_n \to \infty$ then $(s_n)^2 \to \infty$ also.

2.5.6 Prove that if $x_n \to \infty$ then the sequence $s_n = \frac{x_n}{x_n+1}$ is convergent. Is the converse true?

SEE NOTE 18

2.5.7 Suppose that a sequence $\{s_n\}$ of positive numbers satisfies $\lim_{n\to\infty} s_n = 0$. Show that $\lim_{n\to\infty} 1/s_n = \infty$. Is the converse true?

2.5.8 Suppose that a sequence $\{s_n\}$ of positive numbers satisfies the condition $s_{n+1} > \alpha s_n$ for all n where $\alpha > 1$. Show that $s_n \to \infty$.

2.5.9 The sequence $s_n = (-1)^n$ does not diverge to ∞. For what values of M is it nonetheless true that there is an integer N so that $s_n > M$ whenever $n \geq N$?

2.5.10 Show that the sequence

$$n^p + \alpha_1 n^{p-1} + \alpha_2 n^{p-2} + \cdots + \alpha_p$$

diverges to ∞, where here p is a positive integer and α_1, $\alpha_2, \ldots, \alpha_p$ are real numbers (positive or negative).

2.6 Boundedness Properties of Limits

A sequence is said to be *bounded* if its range is a bounded set. Thus a sequence $\{s_n\}$ is bounded if there is a number M so that every term in the sequence satisfies

$$|s_n| \leq M.$$

For such a sequence, every term belongs to the interval $[-M, M]$.

It is fairly evident that a sequence that is not bounded could not converge. This is important enough to state and prove as a theorem.

Theorem 2.11: *Every convergent sequence is bounded.*

Proof. Suppose that $s_n \to L$. Then for every number $\varepsilon > 0$ there is an integer N so that

$$|s_n - L| < \varepsilon$$

whenever $n \geq N$. In particular we could take just one value of ε, say $\varepsilon = 1$, and find a number N so that

$$|s_n - L| < 1$$

whenever $n \geq N$. From this we see that

$$|s_n| = |s_n - L + L| \leq |s_n - L| + |L| < |L| + 1$$

for all $n \geq N$. This number $|L| + 1$ would be an upper bound for all the numbers $|s_n|$ except that we have no indication of the values for $|s_1|$, $|s_2|$, $\dots$, $|s_{N-1}|$.

Thus if we write

$$M = \max\{|s_1|, |s_2|, \dots, |s_{N-1}|, |L| + 1\}$$

we must have

$$|s_n| \leq M$$

for *every* value of n. This is an upper bound, proving the theorem. ■

As a consequence of this theorem we can conclude that an unbounded sequence must diverge. Thus, even though it is a rather crude test, we can prove the divergence of a sequence if we are able somehow to show that it is unbounded. The next example illustrates this technique.

Example 2.12: We shall show that the sequence

$$s_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$$

diverges. The easiest proof of this is to show that it is unbounded and hence, by Theorem 2.11, could not converge.

We watch only at the steps 1, 2, 4, 8, …and make a rough lower estimate of s_1, s_2, s_4, s_8, …in order to show that there can be no bound on the sequence. After a bit of arithmetic we see that

$$s_1 = 1$$

$$s_2 = 1 + \frac{1}{2}$$

$$s_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{1}{2} + 2\left(\frac{1}{4}\right)$$

$$s_8 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)$$

$$\geq 1 + \frac{1}{2} + 2\left(\frac{1}{4}\right) + 4\left(\frac{1}{8}\right)$$

and, in general, that

$$s_{2^n} \geq 1 + n/2$$

for all $n = 0, 1, 2, \ldots$. Thus the sequence is not bounded and so must diverge. ◀

Example 2.13: As a variant of the sequence of the preceding example consider the sequence

$$t_n = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots + \frac{1}{n^p}$$

where p is any positive real number. The case $p = 1$ we have just found diverges.

For $p < 1$ the sequence is larger than it is for $p = 1$ and so the case is even stronger for divergence. For $p > 1$ the sequence is smaller and we cannot see immediately whether it is bounded or unbounded; in fact, with some effort we can show that such a sequence is bounded. What can we conclude? Nothing yet. An unbounded sequence diverges. A bounded sequence may converge or diverge. ◀

Exercises

2.6.1 Which statements are true?

(a) If $\{s_n\}$ is unbounded then it is true that either $\lim_{n\to\infty} s_n = \infty$ or else $\lim_{n\to\infty} s_n = -\infty$.

(b) If $\{s_n\}$ is unbounded then $\lim_{n\to\infty} |s_n| = \infty$.

(c) If $\{s_n\}$ and $\{t_n\}$ are both bounded then so is $\{s_n + t_n\}$.

(d) If $\{s_n\}$ and $\{t_n\}$ are both unbounded then so is $\{s_n + t_n\}$.

(e) If $\{s_n\}$ and $\{t_n\}$ are both bounded then so is $\{s_n t_n\}$.

(f) If $\{s_n\}$ and $\{t_n\}$ are both unbounded then so is $\{s_n t_n\}$.

(g) If $\{s_n\}$ is bounded then so is $\{1/s_n\}$.

(h) If $\{s_n\}$ is unbounded then $\{1/s_n\}$ is bounded.

2.6.2 If $\{s_n\}$ is bounded prove that $\{s_n/n\}$ is convergent.

2.6.3 State the converse of Theorem 2.11. Is it true?

2.6.4 State the contrapositive of Theorem 2.11. Is it true?

2.6.5 Suppose that $\{s_n\}$ is a sequence of positive numbers converging to a positive limit. Show that there is a positive number c so that $s_n > c$ for all n.

SEE NOTE 19

2.6.6 As a computer experiment compute the values of the sequence

$$s_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$$

for large values of n. Is there any indication in the numbers that you see that this sequence fails to converge or must be unbounded?

2.7 Algebra of Limits

Sequences can be combined by the usual arithmetic operations (addition, subtraction, multiplication, and division). Indeed most sequences we are likely to encounter can be seen to be composed of simpler sequences combined together in this way.

In Example 2.7 we suggested that the computations

$$\lim_{n\to\infty} \frac{n^2}{2n^2+1} = \lim_{n\to\infty} \frac{1}{2+1/n^2} = \frac{1}{\lim_{n\to\infty}(2+1/n^2)}$$

$$= \frac{1}{2 + \lim_{n\to\infty} 1/n^2} = \frac{1}{2}$$

could be justified. Note how this sequence has been obtained from simpler ones by ordinary processes of arithmetic. To justify such a method we need to investigate how the limit operation is influenced by algebraic operations.

Suppose that

$$s_n \to S \quad\text{and}\quad t_n \to T.$$

Then we would expect

$$Cs_n \to CS$$
$$s_n + t_n \to S + T$$
$$s_n - t_n \to S - T$$
$$s_n t_n \to ST$$

and

$$s_n/t_n \to S/T.$$

Each of these statements must be justified, however, solely on the basis of the definition of convergence, not on intuitive feelings that this should be the case. Thus we need to develop what could be called the "algebra of limits."

Theorem 2.14 (Multiples of Limits) *Suppose that $\{s_n\}$ is a convergent sequence and C a real number. Then*

$$\lim_{n\to\infty} Cs_n = C\left(\lim_{n\to\infty} s_n\right).$$

Proof. Let $S = \lim_{n\to\infty} s_n$. In order to prove that $\lim_{n\to\infty} Cs_n = CS$ we need to prove that, no matter what positive number ε is given, we can find an integer N so that, for all $n \geq N$,

$$|Cs_n - CS| < \varepsilon.$$

Note that

$$|Cs_n - CS| = |C|\,|s_n - S|$$

by properties of absolute values. This gives us our clue.

Suppose first that $C \neq 0$ and let $\varepsilon > 0$. Choose N so that

$$|s_n - S| < \varepsilon/|C|$$

if $n \geq N$. Then if $n \geq N$ we must have

$$|Cs_n - CS| = |C|\,|s_n - S| < |C|\,(\varepsilon/|C|) = \varepsilon.$$

This is precisely the statement that

$$\lim_{n\to\infty} Cs_n = CS$$

and the theorem is proved in the case $C \neq 0$. The case $C = 0$ is obvious. (Now we should probably delete our first paragraph since it does not contribute to the proof; it only serves to motivate us in finding the correct proof.) ■

Theorem 2.15 (Sums/Differences of Limits) *Suppose that the sequences $\{s_n\}$ and $\{t_n\}$ are convergent. Then*

$$\lim_{n\to\infty} (s_n + t_n) = \lim_{n\to\infty} s_n + \lim_{n\to\infty} t_n$$

and

$$\lim_{n\to\infty} (s_n - t_n) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} t_n.$$

Proof. Let $S = \lim_{n\to\infty} s_n$ and $T = \lim_{n\to\infty} t_n$. In order to prove that

$$\lim_{n\to\infty} (s_n + t_n) = S + T$$

we need to prove that no matter what positive number ε is given we can find an integer N so that

$$|(s_n + t_n) - (S + T)| < \varepsilon$$

if $n \geq N$. Note that

$$|(s_n + t_n) - (S + T)| \leq |s_n - S| + |t_n - T|$$

by the triangle inequality. Thus we can make this expression smaller than ε by making each of the two expressions on the right smaller than $\varepsilon/2$. This provides the method.

Suppose that $\varepsilon > 0$. Choose N_1 so that

$$|s_n - S| < \varepsilon/2$$

if $n \geq N_1$ and also choose N_2 so that

$$|t_n - T| < \varepsilon/2$$

if $n \geq N_2$. Then if n is greater than both N_1 and N_2 both of these inequalities will be true. Set

$$N = \max\{N_1, N_2\}$$

and note that if $n \geq N$ we must have

$$|(s_n + t_n) - (S + T)| \leq |s_n - S| + |t_n - T| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

This is precisely the statement that

$$\lim_{n\to\infty} (s_n + t_n) = S + T$$

and the first statement of the theorem is proved. The second statement is similar and is left as an exercise. (Once again, for a more formal presentation, we would delete the first paragraph.) ■

Theorem 2.16 (Products of Limits) *Suppose that $\{s_n\}$ and $\{t_n\}$ are convergent sequences. Then*

$$\lim_{n\to\infty} (s_n t_n) = \left(\lim_{n\to\infty} s_n\right)\left(\lim_{n\to\infty} t_n\right).$$

Proof. Let $S = \lim_{n\to\infty} s_n$ and $T = \lim_{n\to\infty} t_n$. In order to prove that $\lim_{n\to\infty}(s_n t_n) = ST$ we need to prove that no matter what positive number ε is given we can find an integer N so that, for all $n \geq N$,

$$|s_n t_n - ST| < \varepsilon.$$

It takes some experimentation with different ways of writing this to find the most useful version. Here is an inequality that offers the best approach:

$$|s_nt_n - ST| = |s_n(t_n - T) + s_nT - ST|$$
$$\leq |s_n|\,|t_n - T| + |T|\,|s_n - S|\,. \tag{1}$$

We can control the size of $|s_n - S|$ and $|t_n - T|$, T is constant, and $|s_n|$ cannot be too big. To control the size of $|s_n|$ we need to recall that convergent sequences are bounded (Theorem 2.11) and get a bound from there. With these preliminaries explained the rest of the proof should seem less mysterious. (Now this paragraph can be deleted for a more formal presentation.)

Suppose that $\varepsilon > 0$. Since $\{s_n\}$ converges it is bounded and hence, by Theorem 2.11, there is a positive number M so that $|s_n| \leq M$ for all n. Choose N_1 so that

$$|s_n - S| < \frac{\varepsilon}{2|T| + 1}$$

if $n \geq N_1$. [We did not use $\varepsilon/(2T)$ since there is a possibility that $T = 0$.] Also, choose N_2 so that

$$|t_n - T| < \frac{\varepsilon}{2M}$$

if $n \geq N_2$. Set $N = \max\{N_1, N_2\}$ and note that if $n \geq N$ we must have

$$|s_nt_n - ST| \leq |s_n|\,|t_n - T| + |T|\,|s_n - S|$$
$$\leq M\left(\frac{\varepsilon}{2M}\right) + |T|\left(\frac{\varepsilon}{2|T| + 1}\right) < \varepsilon.$$

This is precisely the statement that

$$\lim_{n\to\infty} s_nt_n = ST$$

and the theorem is proved. ■

Theorem 2.17 (Quotients of Limits) *Suppose that $\{s_n\}$ and $\{t_n\}$ are convergent sequences. Suppose further that $t_n \neq 0$ for all n and that the limit*

$$\lim_{n\to\infty} t_n \neq 0.$$

Then

$$\lim_{n\to\infty}\left(\frac{s_n}{t_n}\right) = \frac{\lim_{n\to\infty} s_n}{\lim_{n\to\infty} t_n}.$$

Proof. Rather than prove the theorem at once as it stands let us prove just a special case of the theorem, namely that

$$\lim_{n\to\infty}\left(\frac{1}{t_n}\right) = \frac{1}{\lim_{n\to\infty} t_n}.$$

Let $T = \lim_{n\to\infty} t_n$. We need to show that no matter what positive number ε is given we can find an integer N so that

$$\left|\frac{1}{t_n} - \frac{1}{T}\right| < \varepsilon$$

if $n \geq N$. To work with this inequality requires us to consider

$$\left|\frac{1}{t_n} - \frac{1}{T}\right| = \frac{|t_n - T|}{|t_n|\,|T|}.$$

It is only the $|t_n|$ in the denominator that offers any trouble since if it is too small we cannot control the size of the fraction. This explains the first step in the proof that we now give, which otherwise might have seemed strange.

Suppose that $\varepsilon > 0$. Choose N_1 so that

$$|t_n - T| < |T|/2$$

if $n \geq N_1$ and also choose N_2 so that

$$|t_n - T| < \varepsilon|T|^2/2$$

if $n \geq N_2$. From the first inequality we see that

$$|T| - |t_n| \leq |T - t_n| < |T|/2$$

and so

$$|t_n| \geq |T|/2$$

if $n \geq N_1$. Set $N = \max\{N_1, N_2\}$ and note that if $n \geq N$ we must have

$$\left|\frac{1}{t_n} - \frac{1}{T}\right| = \frac{|t_n - T|}{|t_n|\,|T|}$$

$$< \frac{\varepsilon |T|^2/2}{|T|^2/2} = \varepsilon.$$

This is precisely the statement that $\lim_{n\to\infty}(1/t_n) = 1/T$.

We now complete the proof of the theorem by applying the product theorem along with what we have just proved to obtain

$$\lim_{n\to\infty}\left(\frac{s_n}{t_n}\right) = \left(\lim_{n\to\infty} s_n\right)\left(\lim_{n\to\infty}\frac{1}{t_n}\right) = \frac{\lim_{n\to\infty} s_n}{\lim_{n\to\infty} t_n}$$

as required. ■

Exercises

2.7.1 By imitating the proof given for the first part of Theorem 2.15 show that

$$\lim_{n\to\infty}(s_n - t_n) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} t_n.$$

2.7.2 Show that $\lim_{n\to\infty}(s_n)^2 = (\lim_{n\to\infty} s_n)^2$ using the theorem on products and also directly from the definition of limit.

2.7.3 Explain which theorems are needed to justify the computation of the limit

$$\lim_{n\to\infty}\frac{n^2}{2n^2+1}$$

that introduced this section.

2.7.4 Prove Theorem 2.16 but verifying and using the inequality

$$|s_n t_n - ST| \leq |(s_n - S)(t_n - T)| + |S(t_n - T)| + |T(s_n - S)|$$

in place of the inequality (1). Which proof do you prefer?

2.7.5 Which statements are true?

(a) If $\{s_n\}$ and $\{t_n\}$ are both divergent then so is $\{s_n + t_n\}$.

(b) If $\{s_n\}$ and $\{t_n\}$ are both divergent then so is $\{s_n t_n\}$.

(c) If $\{s_n\}$ and $\{s_n + t_n\}$ are both convergent then so is $\{t_n\}$.

(d) If $\{s_n\}$ and $\{s_n t_n\}$ are both convergent then so is $\{t_n\}$.

(e) If $\{s_n\}$ is convergent so too is $\{1/s_n\}$.

(f) If $\{s_n\}$ is convergent so too is $\{(s_n)^2\}$.

(g) If $\{(s_n)^2\}$ is convergent so too is $\{s_n\}$.

2.7.6 Note that there are extra hypotheses in the quotient theorem (Theorem 2.17) that were not in the product theorem (Theorem 2.16). Explain why both of these hypotheses are needed.

2.7.7 A careless student gives the following as a proof of Theorem 2.16. Find the flaw:

"Suppose that $\varepsilon > 0$. Choose N_1 so that

$$|s_n - S| < \frac{\varepsilon}{2|T| + 1}$$

if $n \geq N_1$ and also choose N_2 so that

$$|t_n - T| < \frac{\varepsilon}{2|s_n| + 1}$$

if $n \geq N_2$. If $n \geq N = \max\{N_1, N_2\}$ then

$$|s_n t_n - ST| \leq |s_n|\,|t_n - T| + |T|\,|s_n - S|$$

$$\leq |s_n| \left(\frac{\varepsilon}{2|s_n| + 1} \right) + |T| \left(\frac{\varepsilon}{2|T| + 1} \right) < \varepsilon.$$

Well, that works!"

2.7.8 Why are Theorems 2.15 and 2.16 no help in dealing with the limits

$$\lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right)$$

and

$$\lim_{n\to\infty}\sqrt{n}\left(\sqrt{n+1}-\sqrt{n}\right)?$$

What else can you do?

2.7.9 In calculus courses one learns that a function $f:\mathbb{R}\to\mathbb{R}$ is continuous at y if for every $\varepsilon>0$ there is a $\delta>0$ so that $|f(x)-f(y)|<\varepsilon$ for all $|x-y|<\delta$. Show that if f is continuous at y and $s_n\to y$ then $f(s_n)\to f(y)$. Use this to prove that $\lim_{n\to\infty}(s_n)^2=(\lim_{n\to\infty}s_n)^2$.

2.8 Order Properties of Limits

In the preceding section we discussed the algebraic structure of limits. It is a natural mathematical question to ask how the algebraic operations are preserved under limits. As it happens, these natural mathematical questions usually are important in applications. We have seen that the algebraic properties of limits can be used to great advantage in computations of limits.

There is another aspect of structure of the real number system that plays an equally important role as the algebraic structure and that is the order structure. Does the limit operation preserve that order structure the same way that it preserves the algebraic structure? For example, if

$$s_n\le t_n$$

for all n, can we conclude that

$$\lim_{n\to\infty}s_n\le\lim_{n\to\infty}t_n?$$

In this section we solve this problem and several others related to the order structure. These results, too, will prove to be most useful in handling limits.

Theorem 2.18: *Suppose that* $\{s_n\}$ *and* $\{t_n\}$ *are convergent sequences and that*

$$s_n \leq t_n$$

for all n*. Then*

$$\lim_{n\to\infty} s_n \leq \lim_{n\to\infty} t_n.$$

Proof. Let $S = \lim_{n\to\infty} s_n$ and $T = \lim_{n\to\infty} t_n$ and suppose that $\varepsilon > 0$. Choose N_1 so that

$$|s_n - S| < \varepsilon/2$$

if $n \geq N_1$ and also choose N_2 so that

$$|t_n - T| < \varepsilon/2$$

if $n \geq N_2$. Set $N = \max\{N_1, N_2\}$ and note that if $n \geq N$ we must have

$$0 \leq t_n - s_n = T - S + (t_n - T) + (S - s_n) < T - S + \varepsilon/2 + \varepsilon/2.$$

This shows that

$$-\varepsilon < T - S.$$

This statement is true for *any* positive number ε. It would be false if $T - S$ is negative and hence $T - S$ is positive or zero (i.e., $T \geq S$ as required). ■

Note. There is a trap here that many students have fallen into. Since the condition $s_n \leq t_n$ implies

$$\lim_{n\to\infty} s_n \leq \lim_{n\to\infty} t_n$$

would it not follow "similarly" that the condition $s_n < t_n$ implies

$$\lim_{n\to\infty} s_n < \lim_{n\to\infty} t_n?$$

Be careful with this. It is false. See Exercise 2.8.1.

Corollary 2.19: *Suppose that* $\{s_n\}$ *is a convergent sequence and that*

$$\alpha \leq s_n \leq \beta$$

for all n*. Then*

$$\alpha \leq \lim_{n\to\infty} s_n \leq \beta.$$

Proof. Consider that the assumption here can be read as $\alpha_n \leq s_n \leq \beta_n$ where $\{\alpha_n\}$ and $\{\beta_n\}$ are constant sequences. Now apply the theorem. ■

Note. Again, don't forget the trap. The condition $\alpha < s_n < \beta$ for all n implies that

$$\alpha \leq \lim_{n\to\infty} s_n \leq \beta.$$

It would not imply that

$$\alpha < \lim_{n\to\infty} s_n < \beta.$$

The Squeeze Theorem The next theorem is another useful variant on these themes. Here an unknown sequence is sandwiched between two convergent sequences, allowing us to conclude that that sequence converges. This theorem is often taught as "the squeeze theorem," which seems a convenient label.

Theorem 2.20 (Squeeze Theorem) *Suppose that* $\{s_n\}$ *and* $\{t_n\}$ *are convergent sequences, that*

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n$$

and that

$$s_n \leq x_n \leq t_n$$

for all n*. Then* $\{x_n\}$ *is also convergent and*

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n.$$

Proof. Let L be the limit of the two sequences. Choose N_1 so that

$$|s_n - L| < \varepsilon$$

if $n \geq N_1$ and also choose N_2 so that

$$|t_n - L| < \varepsilon$$

if $n \geq N_2$. Set $N = \max\{N_1, N_2\}$. Note that

$$s_n - L \leq x_n - L \leq t_n - L$$

for all n and so

$$-\varepsilon < s_n - L \leq x_n - L \leq t_n - L < \varepsilon$$

if $n \geq N$. From this we see that

$$-\varepsilon < x_n - L < \varepsilon$$

or, to put it in a more familiar form,

$$|x_n - L| < \varepsilon$$

proving the statement of the theorem. ■

Example 2.21: Let θ be some real number and consider the computation of

$$\lim_{n\to\infty} \frac{\sin n\theta}{n}.$$

While this might seem hopeless at first sight since the values of $\sin n\theta$ are quite unpredictable, we recall that none of these values lies outside the interval $[-1, 1]$. Hence

$$-\frac{1}{n} \leq \frac{\sin n\theta}{n} \leq \frac{1}{n}.$$

The two outer sequences converge to the same value 0 and so the inside sequence (the "squeezed" one) must converge to 0 as well. ◀

Absolute Values A further theorem on the theme of order structure is often needed. The absolute value, we recall, is defined directly in terms of the order structure. Is absolute value preserved by the limit operation?

Theorem 2.22 (Limits of Absolute Values) *Suppose that $\{s_n\}$ is a convergent sequence. Then the sequence $\{|s_n|\}$ is also a convergent sequence and*

$$\lim_{n\to\infty} |s_n| = \left|\lim_{n\to\infty} s_n\right|.$$

Proof. Let $S = \lim_{n\to\infty} s_n$ and suppose that $\varepsilon > 0$. Choose N so that

$$|s_n - S| < \varepsilon$$

if $n \geq N$. Observe that, because of the triangle inequality, this means that

$$||s_n| - |S|| \leq |s_n - S| < \varepsilon$$

for all $n \geq N$. By definition

$$\lim_{n\to\infty} |s_n| = |S|$$

as required. ■

Maxima and Minima Since maxima and minima can be expressed in terms of absolute values, there is a corollary that is sometimes useful.

Corollary 2.23 (Max/Min of Limits) *Suppose that $\{s_n\}$ and $\{t_n\}$ are convergent sequences. Then the sequences*

$$\{\max\{s_n, t_n\}\} \quad \textit{and} \quad \{\min\{s_n, t_n\}\}$$

are also convergent and

$$\lim_{n\to\infty} \max\{s_n, t_n\} = \max\{\lim_{n\to\infty} s_n, \lim_{n\to\infty} t_n\}$$

and

$$\lim_{n\to\infty} \min\{s_n, t_n\} = \min\{\lim_{n\to\infty} s_n, \lim_{n\to\infty} t_n\}.$$

Proof. The first of these follows from the identity

$$\max\{s_n, t_n\} = \frac{s_n + t_n}{2} + \frac{|s_n - t_n|}{2}$$

and the theorem on limits of sums and the theorem on limits of absolute values. In the same way the second assertion follows from

$$\min\{s_n, t_n\} = \frac{s_n + t_n}{2} - \frac{|s_n - t_n|}{2}.$$

■

Exercises

2.8.1 Show that the condition $s_n < t_n$ does not imply that

$$\lim_{n\to\infty} s_n < \lim_{n\to\infty} t_n.$$

(If the proof of Theorem 2.18 were modified in an attempt to prove this false statement, where would the modifications fail?)

SEE NOTE 20

2.8.2 If $\{s_n\}$ is a sequence all of whose values lie inside an interval $[a, b]$ prove that $\{s_n/n\}$ is convergent.

2.8.3 A careless student gives the following as a proof of the squeeze theorem. Find the flaw:
"If $\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n = L$, then take limits in the inequality

$$s_n \le x_n \le t_n$$

to get $L \le \lim_{n\to\infty} x_n \le L$. This can only be true if $\lim_{n\to\infty} x_n = L$."

2.8.4 Suppose that $s_n \le t_n$ for all n and that $s_n \to \infty$. What can you conclude?

2.8.5 Suppose that $\lim_{n\to\infty} \frac{s_n}{n} > 0$ Show that $s_n \to \infty$.

2.8.6 Suppose that $\{s_n\}$ and $\{t_n\}$ are sequences of positive numbers, that

$$\lim_{n\to\infty} \frac{s_n}{t_n} = \alpha$$

and that $s_n \to \infty$. What can you conclude?

2.8.7 Suppose that $\{s_n\}$ and $\{t_n\}$ are sequences of positive numbers, that

$$\lim_{n\to\infty} \frac{s_n}{t_n} = \infty$$

and that $t_n \to \infty$. What can you conclude?

2.8.8 Suppose that $\{s_n\}$ and $\{t_n\}$ are sequences of positive numbers, that

$$\lim_{n\to\infty} \frac{s_n}{t_n} = \infty$$

and that $\{s_n\}$ is bounded. What can you conclude?

2.8.9 Let $\{s_n\}$ be a sequence of positive numbers. Show that the condition

$$\lim_{n\to\infty} \frac{s_{n+1}}{s_n} < 1$$

implies that $s_n \to 0$.

SEE NOTE 21

2.8.10 Let $\{s_n\}$ be a sequence of positive numbers. Show that the condition

$$\lim_{n\to\infty} \frac{s_{n+1}}{s_n} > 1$$

implies that $s_n \to \infty$.

SEE NOTE 22

2.9 Monotone Convergence Criterion

In many applications of sequence theory we find that the sequences that arise are going in one direction: The terms steadily get larger or steadily get smaller. The analysis of such sequences is much easier than for general sequences.

Definition 2.24: (Increasing) We say that a sequence $\{s_n\}$ is *increasing* if

$$s_1 < s_2 < s_3 < \cdots < s_n < s_{n+1} < \ldots .$$

Definition 2.25: (Decreasing) We say that a sequence $\{s_n\}$ is *decreasing* if

$$s_1 > s_2 > s_3 > \cdots > s_n > s_{n+1} > \ldots.$$

Often we encounter sequences that "increase" except perhaps occasionally successive values are equal rather than strictly larger. The following language is usually[1] used in this case.

Definition 2.26: (Nondecreasing) We say that a sequence $\{s_n\}$ is *nondecreasing* if

$$s_1 \leq s_2 \leq s_3 \leq \cdots \leq s_n \leq s_{n+1} \leq \ldots.$$

Definition 2.27: (Nonincreasing) We say that a sequence $\{s_n\}$ is *nonincreasing* if

$$s_1 \geq s_2 \geq s_3 \geq \cdots \geq s_n \geq s_{n+1} \geq \ldots.$$

Thus every increasing sequence is also nondecreasing but not conversely. A sequence that has any one of these four properties (increasing, decreasing, nondecreasing, or nonincreasing) is said to be *monotonic*. Monotonic sequences are often easier to deal with than sequences that can go both up and down.

The convergence issue for a monotonic sequence is particularly straightforward. We can imagine that an increasing sequence could increase up to some limit, or we could imagine that it could increase indefinitely and diverge to $+\infty$. It is impossible to imagine a third possibility. We express this as a theorem that will become our primary theoretical tool in investigating convergence of sequences.

[1] In some texts you will find that a nondecreasing sequence is said to be increasing and an increasing sequence is said to be *strictly* increasing. The way in which we intend these terms should be clear and intuitive. If your monthly salary occasionally rises but sometimes stays the same you would not likely say that it is increasing. You might, however, say "at least it never decreases" (i.e., it is nondecreasing).

Theorem 2.28 (Monotone Convergence Theorem) *Suppose that $\{s_n\}$ is a monotonic sequence. Then $\{s_n\}$ is convergent if and only if $\{s_n\}$ is bounded. More specifically,*

1. *If $\{s_n\}$ is nondecreasing then either $\{s_n\}$ is bounded and converges to $\sup\{s_n\}$ or else $\{s_n\}$ is unbounded and $s_n \to \infty$.*

2. *If $\{s_n\}$ is nonincreasing then either $\{s_n\}$ is bounded and converges to $\inf\{s_n\}$ or else $\{s_n\}$ is unbounded and $s_n \to -\infty$.*

Proof. If the sequence is unbounded then it diverges. This is true for any sequence, not merely monotonic sequences.

Thus the proof is complete if we can show that for any bounded monotonic sequence $\{s_n\}$ the limit is $\sup\{s_n\}$ in case the sequence is nondecreasing, or it is $\inf\{s_n\}$ in case the sequence is nonincreasing. Let us prove the first of these cases.

Let $\{s_n\}$ be assumed to be nondecreasing and bounded, and let

$$L = \sup\{s_n\}.$$

Then $s_n \leq L$ for all n and if $\beta < L$ there must be some term s_m say, with $s_m > \beta$. Let $\varepsilon > 0$. We know that there is an m so that

$$s_n \geq s_m > L - \varepsilon$$

for all $n \geq m$. But we already know that every term $s_n \leq L$. Putting these together we have that

$$L - \varepsilon < s_n \leq L < L + \varepsilon$$

or

$$|s_n - L| < \varepsilon$$

for all $n \geq m$. By definition then $s_n \to L$ as required. ■

How would we normally apply this theorem? Suppose a sequence $\{s_n\}$ were given that we recognize as increasing (or maybe just nondecreasing). Then to establish that $\{s_n\}$ converges we need only show that the sequence is bounded above, that is, we need to find just one number M with

$$s_n \leq M$$

for all n. Any crude upper estimate would verify convergence.

Example 2.29: Let us show that the sequence $s_n = 1/\sqrt{n}$ converges. This sequence is evidently decreasing. Can we find a lower bound? Yes, all of the terms are positive so that 0 is a lower bound. Consequently, the sequence must converge. If we wish to show that

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$

we need to do more. But to conclude convergence we needed only to make a crude estimate on how low the terms might go. ◀

Example 2.30: Let us examine the sequence

$$s_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}.$$

This sequence is evidently increasing. Can we find an upper bound? If we can then the series does converge. If we cannot then the series diverges. We have already (earlier) checked this sequence. It is unbounded and so $\lim_{n\to\infty} s_n = \infty$. ◀

Example 2.31: Let us examine the sequence

$$\sqrt{2},\ \sqrt{2+\sqrt{2}},\ \sqrt{2+\sqrt{2+\sqrt{2}}},\ \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}, \dots .$$

Handling such a sequence directly by the limit definition seems quite impossible. This sequence can be defined recursively by

$$x_1 = \sqrt{2} \quad x_n = \sqrt{2 + x_{n-1}}.$$

The computation of a few terms suggests that the sequence is increasing and so should be accessible by the methods of this section.

We prove this by induction. That $x_1 < x_2$ is just an easy computation (do it). Let us suppose that $x_{n-1} < x_n$ for some n and show that it must follow that $x_n < x_{n+1}$. But

$$x_n = \sqrt{2 + x_{n-1}} < \sqrt{2 + x_n} = x_{n+1}$$

where the middle step is the induction hypothesis (i.e., that $x_{n-1} < x_n$). It follows by induction that the sequence is increasing.

Now we show inductively that the sequence is bounded above. Any crude upper bound will suffice. It is clear that $x_1 < 10$. If $x_{n-1} < 10$ then

$$x_n = \sqrt{2 + x_{n-1}} < \sqrt{2 + 10} < 10$$

and so it follows, again by induction, that all terms of the sequence are smaller than 10. We conclude from the monotone convergence theorem that this sequence is convergent.

But to what? (Certainly it does not converges to 10 since that estimate was extremely crude.) That is not so easy to sort out, it seems. But perhaps it is, since we know that the sequence converges to something, say L. In the equation

$$(x_n)^2 = 2 + x_{n-1},$$

obtained by squaring the recursion formula given to us, we can take limits as $n \to \infty$. Since $x_n \to L$ so too does $x_{n-1} \to L$ and $(x_n)^2 \to L^2$. Hence

$$L^2 = 2 + L.$$

The only possibilities for L in this quadratic equation are $L = -1$ and $L = 2$. We know the limit L exists and we know that it is either -1 or 2. We can clearly rule out -1 as all of the numbers in our sequence were positive. Hence $x_n \to 2$. ◀

Exercises

2.9.1 Define a sequence $\{s_n\}$ recursively by setting $s_1 = \alpha$ and

$$s_n = \frac{(s_{n-1})^2 + \beta}{2s_{n-1}}$$

where α, $\beta > 0$.

(a) Show that for $n = 1, 2, 3, \dots$

$$\frac{(s_n - \sqrt{\beta})^2}{2s_n} = s_{n+1} - \beta.$$

(b) Show that $s_n > \sqrt{\beta}$ for all $n = 2, 3, 4, \dots$ unless $\alpha = \sqrt{\beta}$. What happens if $\alpha = \sqrt{\beta}$?

(c) Show that $s_2 > s_3 > s_4 > \dots s_n > \dots$ except in the case $\alpha = \sqrt{\beta}$.

(d) Does this sequence converge? To what?

(e) What is the relation of this sequence to the one introduced in Section 2.1 as Newton's method?

2.9.2 Define a sequence $\{t_n\}$ recursively by setting $t_1 = 1$ and

$$t_n = \sqrt{t_{n-1} + 1}.$$

Does this sequence converge? To what?

2.9.3 Consider the sequence $s_1 = 1$ and $s_n = \frac{2}{s_{n-1}^2}$. We argue that if $s_n \to L$ then $L = \frac{2}{L^2}$ and so $L^3 = 2$ or $L = \sqrt[3]{2}$. Our conclusion is that $\lim_{n\to\infty} s_n = \sqrt[3]{2}$. Do you have any criticisms of this argument?

2.9.4 Does the sequence

$$\frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \dots \cdot (2n)}$$

converge?

2.9.5 Does the sequence

$$\frac{2 \cdot 4 \cdot 6 \cdot \dots \cdot (2n) \cdot 1}{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1) \cdot n^2}$$

converge?

2.9.6 Several nineteenth-century mathematicians used, without proof, a principle in their proofs that has come to be known as the *nested interval property*:

Given a sequence of closed intervals

$$[a_1, b_1] \supset [a_2, b_2] \supset [a_3, b_3] \supset \ldots$$

arranged so that each interval is a subinterval of the one preceding it and so that the lengths of the intervals shrink to zero, then there is exactly one point that belongs to every interval of the sequence.

Prove this statement. Would it be true for a descending sequence of open intervals

$$(a_1, b_1) \supset (a_2, b_2) \supset (a_3, b_3) \supset \ldots?$$

2.10 Examples of Limits

The theory of sequence limits has now been developed far enough that we may investigate some interesting limits. Each of the limits in this section has some cultural interest. Most students would be expected to know and recognize these limits as they arise quite routinely. For us they are also an opportunity to show off our methods. Mostly we need to establish inequalities and use some of our theory. We do not need to use an ε, N argument since we now have more subtle and powerful tools at hand.

Example 2.32: (Geometric Progressions) Let r be a real number. What is the limiting behavior of the sequence

$$1, r, r^2, r^3, r^4, \ldots, r^n, \ldots$$

forming a geometric progression? If $r > 1$ then it is not hard to show that

$$r^n \to \infty.$$

If $r \leq -1$ the sequence certainly diverges. If $r = 1$ this is just a constant sequence.

The interesting case is

$$\lim_{n\to\infty} r^n = 0 \qquad \text{if } -1 < r < 1.$$

To prove this we shall use an easy inequality. Let $x > 0$ and n an integer. Then, using the binomial theorem (or induction if you prefer), we can show that

$$(1 + x)^n > nx.$$

Case (i): Let $0 < r < 1$. Then

$$r = \frac{1}{1+x}$$

(where $x = 1/r - 1 > 0$) and so

$$0 < r^n = \frac{1}{(1+x)^n} < \frac{1}{nx} \to 0$$

as $n \to \infty$. By the squeeze theorem we see that $r^n \to 0$ as required.

Case (ii): If $-1 < r < 0$ then $r = -t$ for $0 < t < 1$. Thus

$$-t^n \leq r^n \leq t^n.$$

By case (i) we know that $t^n \to 0$. By the squeeze theorem we see that $r^n \to 0$ again as required. ◀

Example 2.33: (Roots) An interesting and often useful limit is

$$\lim_{n\to\infty} \sqrt[n]{n} = 1.$$

To show this we once again derive an inequality from the binomial theorem. If $n \geq 2$ and $x > 0$ then

$$(1+x)^n > n(n-1)x^2/2.$$

For $n \geq 2$ write

$$\sqrt[n]{n} = 1 + x_n$$

(where $x_n = \sqrt[n]{n} - 1 > 0$) and so

$$n = (1+x_n)^n > n(n-1)x_n^2/2$$

or

$$0 < x_n^2 < \frac{2}{n-1} \to 0$$

as $n \to \infty$. By the squeeze theorem we see that $x_n \to 0$ and it follows that $\sqrt[n]{n} \to 1$ as required.

As a special case of this example note that

$$\sqrt[n]{C} \to 1$$

as $n \to \infty$ for any positive constant C. This is true because if $C > 1$ then

$$1 < \sqrt[n]{C} < \sqrt[n]{n}$$

for large enough n. By the squeeze theorem this shows that $\sqrt[n]{C} \to 1$. If, however, $0 < C < 1$ then

$$\sqrt[n]{C} = \frac{1}{\sqrt[n]{1/C}} \to 1$$

by the first case since $1/C > 1$. ◀

Example 2.34: (Sums of Geometric Progressions) For all values of x in the interval $(-1, 1)$ the limit

$$\lim_{n\to\infty} \left(1 + x + x^2 + x^3 + \cdots + x^n\right) = \frac{1}{1-x}.$$

While at first a surprising result, this is quite evident once we check the identity

$$(1-x)\left(1 + x + x^2 + x^3 + \cdots + x^n\right) = 1 - x^{n+1},$$

which just requires a straightforward multiplication. Thus

$$\lim_{n\to\infty} \left(1 + x + x^2 + x^3 + \cdots + x^n\right) = \lim_{n\to\infty} \frac{1 - x^{n+1}}{1-x} = \frac{1}{1-x}$$

where we have used the result we proved previously, namely that

$$x^{n+1} \to 0 \quad \text{if } |x| < 1.$$

One special case of this is useful to remember. Set $x = 1/2$. Then

$$\lim_{n\to\infty} \left(1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n}\right) = 2.$$

◀

Example 2.35: (Decimal Expansions) What meaning is assigned to the infinite decimal expansion

$$x = 0.d_1d_2d_3d_4\ldots d_n\ldots$$

where the choices of integers $0 \le d_i \le 9$ can be made in any way? Repeating decimals can always be converted into fractions and so the infinite process can be avoided. But if the pattern does not repeat, a different interpretation must be made.

The most obvious interpretation of this number x is to declare that it is the limit of the sequence

$$\lim_{n\to\infty} 0.d_1d_2d_3d_4\ldots d_n.$$

But how do we know that the limit exists? Our theory provides an immediate answer. Since this sequence is nondecreasing and every term is smaller than 1, by the monotone convergence theorem the sequence converges. This is true no matter what the choices of the decimal digits are. ◄

Example 2.36: (Expansion of e^x) Let $x > 0$ and consider the two closely related sequences

$$s_n = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$$

and

$$t_n = \left(1 + \frac{x}{n}\right)^n.$$

The relation between the two sequences becomes more apparent once the binomial theorem is used to expand the latter.

In more advanced mathematics it is shown that both sequences converge to e^x. Let us be content to prove that

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n.$$

The sequence $\{s_n\}$ is clearly increasing since each new term is the preceding term with a positive number added to it. To show convergence then we need only show that the sequence is bounded. This takes some arithmetic, but not too much.

Choose an integer N larger than $2x$. Note then that

$$\frac{x^{N+1}}{(N+1)!} < \frac{1}{2}\left(\frac{x^N}{N!}\right)$$

that

$$\frac{x^{N+2}}{(N+2)!} < \frac{1}{4}\left(\frac{x^N}{N!}\right)$$

and that

$$\frac{x^{N+3}}{(N+3)!} < \frac{1}{8}\left(\frac{x^N}{N!}\right).$$

Thus

$$s_n \leq \left[1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{N-1}}{(N-1)!}\right] + \frac{x^N}{N!}\left(1 + \frac{1}{2} + \frac{1}{4} \ldots\right)$$
$$\leq \left[1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{N-1}}{(N-1)!}\right] + 2\frac{x^N}{N!}.$$

Here we have used the limit for the sum of a geometric progression from Example 2.34 to make an upper estimate on how large this sum can get. Note that the N is fixed and so the number on the right-hand side of this inequality is just a number, and it is larger than every number in the sequence $\{s_n\}$.

It follows now from the monotone convergence theorem that $\{s_n\}$ converges. To handle $\{t_n\}$, first apply the binomial theorem to obtain

$$t_n = 1 + x + \frac{1 - 1/n}{2!}x^2 + \frac{(1 - 1/n)(1 - 2/n)}{3!}x^3 + \cdots \leq s_n.$$

From this we see that $\{t_n\}$ is increasing and that it is smaller than the convergent sequence $\{s_n\}$. It follows, again from the monotone convergence theorem, that $\{t_n\}$ converges. Moreover,

$$\lim_{n\to\infty} t_n \leq \lim_{n\to\infty} s_n.$$

If we can obtain the opposite inequality we will have proved our assertion. Let m be a fixed number and let $n > m$. Then, from the preceding expansion, we note that

$$t_n > 1 + x + \frac{1 - 1/n}{2!}x^2 + \frac{(1-1/n)(1-2/n)}{3!}x^3$$
$$+ \cdots + \frac{(1-1/n)(1-2/n) + \cdots + (1 - [m-1]/n)}{m!}x^m.$$

We can hold m fixed and allow $n \to \infty$ in this inequality and obtain that

$$\lim_{n\to\infty} t_n \geq s_m$$

for each m. From this it now follows that

$$\lim_{n\to\infty} t_n \geq \lim_{n\to\infty} s_n$$

and we have completed our task. ◀

Exercises

2.10.1 Since we know that

$$1 + x + x^2 + x^3 + \cdots + x^n \to \frac{1}{1-x}$$

this suggests the formula

$$1 + 2 + 4 + 8 + 16 + \cdots = \frac{1}{1-2} = -1.$$

Do you have any criticisms?

See Note 23

2.10.2 Let α and β be positive numbers. Discuss the convergence behavior of the sequence

$$\frac{\alpha^{\beta n}}{\beta^{\alpha n}}.$$

2.10.3 Define

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.$$

Show that $2 < e < 3$.

2.10.4 Show that

$$\lim_{n\to\infty}\left(1+\frac{1}{2n}\right)^n = \sqrt{e}.$$

2.10.5 Check the simple identity

$$\left(1+\frac{2}{n}\right) = \left(1+\frac{1}{n+1}\right)\left(1+\frac{1}{n}\right)$$

and use it to show that

$$\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n = e^2.$$

2.11 Subsequences

The sequence

$$1, -1, 2, -2, 3, -3, 4, -4, 5, -5, \dots$$

appears to contain within itself the two sequences

$$1, 2, 3, 4, 5, \dots$$

and

$$-1, -2, -3, -4, -5, \dots.$$

In order to have a language to express this we introduce the term *subsequence*. We would say that the latter two sequences are subsequences of the first sequence. Often a sequence is best studied by looking at some of its subsequences. But what is a proper definition of this term? We need a formal mathematical way of expressing the vague idea that a subsequence is obtained by crossing out some of the terms of the original sequence.

Definition 2.37: (Subsequences) Let

$$s_1, s_2, s_3, s_4, \ldots$$

be any sequence. Then by a *subsequence* of this sequence we mean any sequence

$$s_{n_1}, s_{n_2}, s_{n_3}, s_{n_4}, \ldots$$

where

$$n_1 < n_2 < n_3 < \ldots$$

is an increasing sequence of natural numbers.

Example 2.38: We can consider

$$1, 2, 3, 4, 5, \ldots$$

to be a subsequence of sequence

$$1, -1, 2, -2, 3, -3, 4, -4, 5, -5, \ldots$$

because it contains just the first, third, fifth, etc. terms of the original sequence. Here $n_1 = 1$, $n_2 = 3$, $n_3 = 5$, $\ldots$. ◀

In many applications of sequences it is the subsequences that need to be studied. For example, what can we say about the existence of monotonic subsequences, or bounded subsequences, or divergent subsequences, or convergent subsequences? The answers to these questions have important uses.

Existence of Monotonic Subsequences Our first question is easy to answer for any specific sequence, but harder to settle in general. Given a sequence can we always select a subsequence that is monotonic, either monotonic nondecreasing or monotonic nonincreasing?

Theorem 2.39: *Every sequence contains a monotonic subsequence.*

Proof. We construct first a nonincreasing subsequence if possible. We call the mth element x_m of the sequence $\{x_n\}$ a turn-back point if all later elements are less than or equal to it, in symbols if $x_m \geq x_n$

for all $n > m$. If there is an infinite subsequence of turn-back points $x_{m_1}, x_{m_2}, x_{m_3}, x_{m_4}, \dots$ then we have found our nonincreasing subsequence since

$$x_{m_1} \geq x_{m_2} \geq x_{m_3} \geq x_{m_4} \geq \dots .$$

This would not be possible if there are only finitely many turn-back points. Let us suppose that x_M is the last turn-back point so that any element x_n for $n > M$ is not a turn-back point. Since it is not there must be an element further on in the sequence greater than it, in symbols $x_m > x_n$ for some $m > n$. Thus we can choose $x_{m_1} > x_{M+1}$ with $m_1 > M + 1$, then $x_{m_2} > x_{m_1}$ with $m_2 > m_1$, and then $x_{m_3} > x_{m_2}$ with $m_3 > m_2$, and so on to obtain an increasing subsequence

$$x_{M+1} < x_{m_1} < x_{m_2} < x_{m_3} < x_{m_4} < \dots$$

as required. ■

Existence of Convergent Subsequences Having answered this question about the existence of monotonic subsequences, we can also now answer the question about the existence of convergent subsequences. This might, at first sight, seem just a curiosity, but it will give us later one of our most important tools in analysis.

The theorem is traditionally attributed to two major nineteenth-century mathematicians, Karl Theodor Wilhelm Weierstrass (1815-1897) and Bernhard Bolzano (1781–1848). These two mathematicians, the first German and the second Czech, rank with Cauchy among the founders of our subject.

Theorem 2.40 (Bolzano-Weierstrass) *Every bounded sequence contains a convergent subsequence.*

Proof. By Theorem 2.39 every sequence contains a monotonic subsequence. Here that subsequence would be both monotonic and bounded, and hence convergent. ■

Other (less important) questions of this type appear in the exercises.

Exercises

2.11.1 Show that, according to our definition, every sequence is a subsequence of itself. How would the definition have to be reworded to avoid this if, for some reason, this possibility were to have been avoided?

2.11.2 Show that every subsequence of a subsequence of a sequence $\{x_n\}$ is itself a subsequence of $\{x_n\}$.

2.11.3 If $\{s_{n_k}\}$ is a subsequence of $\{s_n\}$ and $\{t_{m_k}\}$ is a subsequence of $\{t_n\}$ then is it true that $\{s_{n_k} + t_{m_k}\}$ is a subsequence of $\{s_n + t_n\}$?

2.11.4 If $\{s_{n_k}\}$ is a subsequence of $\{s_n\}$ is $\{(s_{n_k})^2\}$ a subsequence of $\{(s_n)^2\}$?

2.11.5 Describe all sequences that have only finitely many different subsequences.

2.11.6 Establish which of the following statements are true.

(a) A sequence is convergent if and only if all of its subsequences are convergent.

(b) A sequence is bounded if and only if all of its subsequences are bounded.

(c) A sequence is monotonic if and only if all of its subsequences are monotonic.

(d) A sequence is divergent if and only if all of its subsequences are divergent.

2.11.7 Establish which of the following statements are true for an arbitrary sequence $\{s_n\}$.

(a) If all monotone subsequences of a sequence $\{s_n\}$ are convergent, then $\{s_n\}$ is bounded.

(b) If all monotone subsequences of a sequence $\{s_n\}$ are convergent, then $\{s_n\}$ is convergent.

(c) If all convergent subsequences of a sequence $\{s_n\}$ converge to 0, then $\{s_n\}$ converges to 0.

(d) If all convergent subsequences of a sequence $\{s_n\}$ converge to 0 and $\{s_n\}$ is bounded, then $\{s_n\}$ converges to 0.

2.11.8 Where possible find subsequences that are monotonic and subsequences that are convergent for the following sequences

(a) $\{(-1)^n n\}$

(b) $\{\sin(n\pi/8)\}$

(c) $\{n \sin(n\pi/8)\}$

(d) $\left\{\frac{n+1}{n}\sin(n\pi/8)\right\}$

(e) $\{1+(-1)^n\}$

(f) $\{r_n\}$ consists of all rational numbers in the interval $(0,1)$ arranged in some order.

2.11.9 Describe all subsequences of the sequence

$$1,0,1,0,1,0,1,0,1,0,1,0,\ldots.$$

Describe all convergent subsequences. Describe all monotonic subsequences.

2.11.10 If $\{s_{n_k}\}$ is a subsequence of $\{s_n\}$ show that $n_k \geq k$ for all $k=1,2,3,\ldots$.

2.11.11 Give an example of a sequence that contains subsequences converging to every natural number (and no other numbers).

2.11.12 Give an example of a sequence that contains subsequences converging to every number in $[0,1]$ (and no other numbers).

2.11.13 Show that there cannot exist a sequence that contains subsequences converging to every number in $(0,1)$ and no other numbers.

SEE NOTE 24

2.11.14 Show that if $\{s_n\}$ has no convergent subsequences, then $|s_n| \to \infty$ as $n \to \infty$.

2.11.15 If a sequence $\{x_n\}$ has the property that

$$\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} x_{2n+1} = L$$

show that the sequence $\{x_n\}$ converges to L.

2.11.16 If a sequence $\{x_n\}$ has the property that

$$\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} x_{2n+1} = \infty$$

show that the sequence $\{x_n\}$ diverges to ∞.

2.11.17 Let α and β be positive real numbers and define a sequence by setting $s_1=\alpha$, $s_2=\beta$ and $s_{n+2}=\frac{1}{2}(s_n+s_{n+1})$ for all $n=1,2,3,\ldots$. Show that the subsequences $\{s_{2n}\}$ and $\{s_{2n-1}\}$ are monotonic and convergent. Does the sequence $\{s_n\}$ converge? To what?

2.11.18 Without appealing to any of the theory of this section prove that every unbounded sequence has a strictly monotonic subsequence (i.e., either increasing or decreasing).

2.11.19 Show that if a sequence $\{x_n\}$ converges to a finite limit or diverges to $\pm\infty$ then every subsequence has precisely the same behavior.

2.11.20 Suppose a sequence $\{x_n\}$ has the property that every subsequence has a further subsequence convergent to L. Show that $\{x_n\}$ converges to L.

2.11.21 Let $\{x_n\}$ be a bounded sequence and let $x = \sup\{x_n : n \in \mathbb{N}\}$. Suppose that, moreover, $x_n < x$ for all n. Prove that there is a subsequence convergent to x.

2.11.22 Let $\{x_n\}$ be a bounded sequence, let

$$y = \inf\{x_n : n \in \mathbb{N}\} \quad \text{and} \quad x = \sup\{x_n : n \in \mathbb{N}\}.$$

Suppose that, moreover, $y < x_n < x$ for all n. Prove that there is a pair of convergent subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ so that

$$\lim_{k\to\infty} |x_{n_k} - x_{m_k}| = x - y.$$

2.11.23 Does every divergent sequence contain a divergent monotonic subsequence?

2.11.24 Does every divergent sequence contain a divergent bounded subsequence?

2.11.25 Construct a proof of the Bolzano-Weierstrass theorem for bounded sequences using the nested interval property and not appealing to the existence of monotonic subsequences.

2.11.26 Construct a direct proof of the assertion that every convergent sequence has a convergent, monotonic subsequence (i.e., without appealing to Theorem 2.39).

2.11.27 Let $\{x_n\}$ be a bounded sequence that we do not know converges. Suppose that it has the property that every one of its convergent subsequences converges to the same number L. What can you conclude?

2.11.28 Let $\{x_n\}$ be a bounded sequence that diverges. Show that there is a pair of convergent subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ so that

$$\lim_{k\to\infty} |x_{n_k} - x_{m_k}| > 0.$$

2.11.29 Let $\{x_n\}$ be a sequence. A number z with the property that for all $\varepsilon > 0$ there are infinitely many terms of the sequence in the interval $(z-\varepsilon, z+\varepsilon)$ is said to be a *cluster point* of the sequence. Show that z is a cluster point of a sequence if and only if there is a subsequence $\{x_{n_k}\}$ converging to z.

2.12 Cauchy Convergence Criterion

What property of a sequence characterizes convergence? As a "characterization" we would like some necessary and sufficient condition for a sequence to converge. We could simply write the definition and consider that that is a characterization. Thus the following technical statement would, indeed, be a characterization of the convergence of a sequence $\{s_n\}$.

A sequence $\{s_n\}$ is convergent if and only if $\exists L$ so that $\forall \varepsilon > 0$ $\exists N$ with the property that

$$|s_n - L| < \varepsilon$$

whenever $n \geq N$.

In mathematics when we ask for a characterization of a property we can expect to find many answers, some more useful than others. The limitation of this particular characterization is that it requires us to find the number L which is the limit of the sequence in advance. Compare this with a characterization of convergence of a monotonic sequence $\{s_n\}$.

A monotonic sequence $\{s_n\}$ is convergent if and only if it is bounded.

This is a wonderful and most useful characterization. But it applies only to monotonic sequences.

A correct and useful characterization, applicable to all sequences, was found by Cauchy. This is the content of the next theorem. Note that it has the advantage that it describes a convergent sequence with no reference whatsoever to the actual value of the limit. Loosely it asserts that a sequence converges if and only if the terms of the sequence are eventually arbitrarily close together.

Theorem 2.41 (Cauchy Criterion) *A sequence $\{s_n\}$ is convergent if and only if for each $\varepsilon > 0$ there exists an integer N with the property that*

$$|s_n - s_m| < \varepsilon$$

whenever $n \geq N$ and $m \geq N$.

Proof. This property of the theorem is so important that it deserves some terminology. A sequence is said to be a *Cauchy sequence* if it satisfies this property. Thus the theorem states that a sequence is convergent if and only if it is a Cauchy sequence. The terminology is most significant in more advanced situations where being a Cauchy sequence is not necessarily equivalent with being convergent.

Our proof is a bit lengthy and will require an application of the Bolzano-Weierstrass theorem.

The proof in one direction, however, is easy. Suppose that $\{s_n\}$ is convergent to a number L. Let $\varepsilon > 0$. Then there must be an integer N so that

$$|s_k - L| < \frac{\varepsilon}{2}$$

whenever $k \geq N$. Thus if both m and n are larger than N,

$$|s_n - s_m| \leq |s_n - L| + |L - s_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

which shows that $\{s_n\}$ is a Cauchy sequence.

Now let us prove the opposite (and more difficult) direction.

For the first step we show that every Cauchy sequence is bounded. Since the proof of this can be obtained by copying and modifying the proof of Theorem 2.11, we have left this as an exercise. (It is not really interesting that Cauchy sequences are bounded since after the proof is completed we know that all Cauchy sequences are convergent and so must, indeed, be bounded.)

For the second step we apply the Bolzano-Weierstrass theorem to the bounded sequence $\{s_n\}$ to obtain a convergent subsequence $\{s_{n_k}\}$.

The final step is a feature of Cauchy sequences. Once we know that $s_{n_k} \to L$ and that $\{s_n\}$ is Cauchy, we can show that $s_n \to L$ also. Let $\varepsilon > 0$ and choose N so that

$$|s_n - s_m| < \varepsilon/2$$

for all m, $n \geq N$. Choose K so that

$$|s_{n_k} - L| < \varepsilon/2$$

for all $k \geq K$. Suppose that $n \geq N$. Set m equal to any value of n_k that is larger than N and so that $k \geq K$. For this value $s_m = s_{n_k}$

$$|s_n - L| \leq |s_n - s_{n_k}| + |s_{n_k} - L| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

By definition, $\{s_n\}$ converges to L and so the proof is complete. ■

Example 2.42: The Cauchy criterion is most useful in theoretical developments rather than applied to concrete examples. Even so, occasionally it is the fastest route to a proof of convergence. For example, consider the sequence $\{x_n\}$ defined by setting $x_1 = 1$, $x_2 = 2$ and then, recursively,

$$x_n = \frac{x_{n-1} + x_{n-2}}{2}.$$

Each term after the second is the average of the preceding two terms. The distance between x_1 and x_2 is 1, that between x_2 and x_3 is 1/2, between x_3 and x_4 is 1/4, and so on. We see then that after the N stage all the distances are smaller than 2^{-N+1}, that is, that for all $n \geq N$ and $m \geq N$

$$|x_n - x_m| \leq \frac{1}{2^{N-1}}.$$

This is exactly the Cauchy criterion and so this sequence converges. Note that the Cauchy criterion offers no information on what the sequence is converging to. You must come up with another method to find out. ◄

Exercises

2.12.1 Show directly that the sequence $s_n = 1/n$ is a Cauchy sequence.

2.12.2 Show directly that any multiple of a Cauchy sequence is again a Cauchy sequence.

2.12.3 Show directly that the sum of two Cauchy sequences is again a Cauchy sequence.

2.12.4 Show directly that any Cauchy sequence is bounded.

2.12.5 The following criterion is weaker than the Cauchy criterion. Show that it is not equivalent:

For all $\varepsilon > 0$ there exists an integer N with the property that

$$|s_{n+1} - s_n| < \varepsilon$$

whenever $n \geq N$.

SEE NOTE 25

2.12.6 A careless student believes that the following statement is the Cauchy criterion.

For all $\varepsilon > 0$ and all positive integers p there exists an integer N with the property that

$$|s_{n+p} - s_n| < \varepsilon$$

whenever $n \geq N$.

Is this statement weaker, stronger, or equivalent to the Cauchy criterion?

2.12.7 Show directly that if $\{s_n\}$ is a Cauchy sequence then so too is $\{|s_n|\}$. From this conclude that $\{|s_n|\}$ converges whenever $\{s_n\}$ converges.

2.12.8 Show that every subsequence of a Cauchy sequence is Cauchy. (Do not use the fact that every Cauchy sequence is convergent.)

2.12.9 Show that every bounded monotonic sequence is Cauchy. (Do not use the monotone convergence theorem.)

2.12.10 Show that the sequence in Example 2.42 converges to 5/3.

SEE NOTE 26

2.13 Upper and Lower Limits

Adv.

If $\lim_{n\to\infty} x_n = L$ then, according to our definition, numbers α and β on either side of L, that is, $\alpha < L < \beta$, have the property that

$$\alpha < x_n \text{ and } x_n < \beta$$

for all sufficiently large n. In many applications only *half* of this information is used.

Example 2.43: Here is an example showing how half a limit is as good as a whole limit. Let $\{x_n\}$ be a sequence of positive numbers with the property that

$$\lim_{n\to\infty} \sqrt[n]{x_n} = L < 1.$$

Then we can prove that $x_n \to 0$. To see this pick numbers α and β so that

$$\alpha < L < \beta < 1.$$

There must be an integer N so that

$$\alpha < \sqrt[n]{x_n} < \beta < 1$$

for all $n \geq N$. Forget half of this and focus on

$$\sqrt[n]{x_n} < \beta < 1.$$

Then we have

$$x_n < \beta^n$$

for all $n \geq N$ and it is clear now why $x_n \to 0$. ◀

This example suggests that the definition of limit might be weakened to handle situations where less is needed. This way we have a tool to discuss the limiting behavior of sequences that may not necessarily converge. Even if the sequence does converge this often offers a tool that can be used without first finding a proof of convergence.

We break the definition of sequence limit into two half-limits as follows.

Definition 2.44: (Lim Sup) A *limit superior* of a sequence $\{x_n\}$, denoted as

$$\limsup_{n\to\infty} x_n,$$

is defined to be the infimum of all numbers β with the following property:

There is an integer N so that $x_n < \beta$ for all $n \geq N$.

Definition 2.45: (Lim Inf) A *limit inferior* of a sequence $\{x_n\}$, denoted as

$$\liminf_{n\to\infty} x_n,$$

is defined to be the supremum of all numbers α with the following property:

There is an integer N so that $\alpha < x_n$ for all $n \geq N$.

Note. In interpreting this definition note that, by our usual rules on infs and sups, the values $-\infty$ and ∞ are allowed. If there are *no* numbers β with the property of the definition, then the sequence is simply unbounded above. The infimum of the empty set is taken as ∞ and so

$$\limsup_{n\to\infty} x_n = \infty \Leftrightarrow \text{the sequence } \{x_n\} \text{ has no upper bound.}$$

On the other hand, if *every* number β has the property of the definition this means exactly that our sequence must be diverging to $-\infty$. The infimum of the set of *all* real numbers is taken as $-\infty$ and so

$$\limsup_{n\to\infty} x_n = -\infty \Leftrightarrow \text{the sequence } \{x_n\} \to -\infty.$$

The same holds in the other direction. A sequence that is unbounded below can be described by saying $\liminf_{n\to\infty} x_n = -\infty$. A sequence that diverges to ∞ can be described by saying $\liminf_{n\to\infty} x_n = \infty$.

We refer to these concepts as "upper limits" and "lower limits" or "extreme limits." They extend our theory describing the limiting behavior of sequences to allow precise descriptions of divergent sequences. Obviously, we should establish very quickly that the upper limit is indeed greater than or equal to the lower limit since our language suggests this.

Theorem 2.46: *Let $\{x_n\}$ be a sequence of real numbers. Then*

$$\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n.$$

Proof. If $\limsup_{n\to\infty} x_n = \infty$ or if $\liminf_{n\to\infty} x_n = -\infty$ we have nothing to prove. If not then take any number β larger than $\limsup_{n\to\infty} x_n$ and any number α smaller than $\liminf_{n\to\infty} x_n$. By definition then there is an integer N so that $x_n < \beta$ for all $n \geq N$ and an integer M so that $\alpha < x_n$ for all $n \geq M$. It must be true that $\alpha < \beta$. But β is *any* number larger than $\limsup_{n\to\infty} x_n$. Hence

$$\alpha \leq \limsup_{n\to\infty} x_n.$$

Similarly, α is *any* number smaller than $\liminf_{n\to\infty} x_n$. Hence

$$\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n$$

as required. ■

How shall we use the limit superior of a sequence $\{x_n\}$? If

$$\limsup_{n\to\infty} x_n = L$$

then every number $\beta > L$ has the property that $x_n < \beta$ for all n large enough. This is because L is the infimum of such numbers β. On the other hand, any number $b < L$ cannot have this property so $x_n \geq b$ for infinitely many indices n. Thus numbers slightly larger than L must be upper bounds for the sequence eventually. Numbers slightly less than L are not upper bounds eventually. To express this a little more precisely, the number L is the limit superior of a sequence $\{x_n\}$ exactly when the following holds:

> For every $\varepsilon > 0$ there is an integer N so that $x_n < L + \varepsilon$ for all $n \geq N$ and $x_n > L - \varepsilon$ for infinitely many $n \geq N$.

The next theorem gives another characterization which is sometimes easier to apply. This version also better explains why we describe this notion as a "lim sup" and "lim inf."

Theorem 2.47: *Let $\{x_n\}$ be a sequence of real numbers. Then*

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \sup\{x_n, x_{n+1}, x_{n+2}, x_{n+3}, \dots\}$$

and

$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} \inf\{x_n, x_{n+1}, x_{n+2}, x_{n+3}, \dots\}.$$

Proof. Let us prove just the statement for lim sups as the lim inf statement can be proved similarly.

Write
$$y_n = \sup\{x_n, x_{n+1}, x_{n+2}, x_{n+3}, \dots\}.$$
Then $x_n \leq y_n$ for all n and so, using the inequality promised in Exercise 2.13.5,
$$\limsup_{n\to\infty} x_n \leq \limsup_{n\to\infty} y_n.$$
But $\{y_n\}$ is a nonincreasing sequence and so
$$\limsup_{n\to\infty} y_n = \lim_{n\to\infty} y_n.$$
From this it follows that
$$\limsup_{n\to\infty} x_n \leq \lim_{n\to\infty} \sup\{x_n, x_{n+1}, x_{n+2}, x_{n+3}, \dots\}.$$

Let us now show the reverse inequality. If $\limsup_{n\to\infty} x_n = \infty$ then the sequence is unbounded above. Thus for all n
$$\sup\{x_n, x_{n+1}, x_{n+2}, x_{n+3}, \dots\} = \infty$$
and so, in this case,
$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \sup\{x_n, x_{n+1}, x_{n+2}, x_{n+3}, \dots\}$$
must certainly be true.

If
$$\limsup_{n\to\infty} x_n < \infty$$
then take any number β larger than $\limsup_{n\to\infty} x_n$. By definition then there is an integer N so that $x_n < \beta$ for all $n \geq N$. It follows that
$$\lim_{n\to\infty} \sup\{x_n, x_{n+1}, x_{n+2}, x_{n+3}, \dots\} \leq \beta.$$
But β is *any* number larger than $\limsup_{n\to\infty} x_n$. Hence
$$\lim_{n\to\infty} \sup\{x_n, x_{n+1}, x_{n+2}, x_{n+3}, \dots\} \leq \limsup_{n\to\infty} x_n.$$

We have proved both inequalities, the equality follows, and the theorem is proved. ■

The connection between limits and extreme limits is close. If a limit exists then the upper and lower limits must be the same.

Theorem 2.48: *Let $\{x_n\}$ be a sequence of real numbers. Then $\{x_n\}$ is convergent if and only if $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n$ and these are finite. In this case*

$$\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = \lim_{n\to\infty} x_n.$$

Proof. Let $\varepsilon > 0$. If $\limsup_{n\to\infty} x_n = L$ then there is an integer N_1 so that $x_n < L + \varepsilon$ for all $n \geq N_1$. If it is also true that $\liminf_{n\to\infty} x_n = L$ then there is an integer N_2 so that $x_n > L - \varepsilon$ for all $n \geq N_2$. Putting these together we have

$$L - \varepsilon < x_n < L + \varepsilon$$

for all

$$n \geq N = \max\{N_1, N_2\}.$$

By definition then $\lim_{n\to\infty} x_n = L$.

Conversely, if $\lim_{n\to\infty} x_n = L$ then for some N,

$$L - \varepsilon < x_n < L + \varepsilon$$

for all $n \geq N$. Thus

$$L - \varepsilon \leq \liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n \leq L + \varepsilon.$$

Since ε is an arbitrary positive number we must have

$$L = \liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n$$

as required. ■

In the exercises you will be asked to compute several lim sups and lim infs. This is just for familiarity with the concepts. Computations are not so important. What is important is the use of these ideas in theoretical developments. More critical is how these limit operations relate to arithmetic or order properties. The limit of a sum is the sum of the two limits. Is this true for lim sups and lim infs? (See Exercise 2.13.9.) Do not skip these exercises.

Exercises

2.13.1 Complete Example 2.43 by showing that if $\{x_n\}$ is a sequence of positive numbers with the property that $\limsup_{n\to\infty} \sqrt[n]{x_n} < 1$ then $x_n \to 0$. Show that if

$$\liminf_{n\to\infty} \sqrt[n]{x_n} > 1$$

then $x_n \to \infty$. What can you conclude if $\limsup_{n\to\infty} \sqrt[n]{x_n} > 1$? What can you conclude if $\liminf_{n\to\infty} \sqrt[n]{x_n} < 1$?

2.13.2 Compute lim sups and lim infs for the following sequences

(a) $\{(-1)^n n\}$
(b) $\{\sin(n\pi/8)\}$
(c) $\{n\sin(n\pi/8)\}$
(d) $\{[(n+1)\sin(n\pi/8)]/n\}$
(e) $\{1+(-1)^n\}$
(f) $\{r_n\}$ consists of all rational numbers in the interval $(0,1)$ arranged in some order.

2.13.3 Give examples of sequences of rational numbers $\{a_n\}$ with

(a) upper limit $\sqrt{2}$ and lower limit $-\sqrt{2}$,
(b) upper limit $+\infty$ and lower limit $\sqrt{2}$,
(c) upper limit π and lower limit e.

2.13.4 Show that $\limsup_{n\to\infty}(-x_n) = -(\liminf_{n\to\infty} x_n)$.

2.13.5 If two sequences $\{a_n\}$ and $\{b_n\}$ satisfy the inequality $a_n \le b_n$ for all sufficiently large n, show that

$$\limsup_{n\to\infty} a_n \le \limsup_{n\to\infty} b_n \quad \text{and} \quad \liminf_{n\to\infty} a_n \le \liminf_{n\to\infty} b_n.$$

2.13.6 Show that $\lim_{n\to\infty} x_n = \infty$ if and only if

$$\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = \infty.$$

2.13.7 Show that if $\limsup_{n\to\infty} a_n = L$ for a finite real number L and $\varepsilon > 0$, then $a_n > L + \varepsilon$ for only finitely many n and $a_n > L - \varepsilon$ for infinitely many n.

2.13.8 Show that for any monotonic sequence $\{x_n\}$

$$\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = \lim_{n\to\infty} x_n$$

(including the possibility of infinite limits).

2.13.9 Show that for any bounded sequences $\{a_n\}$ and $\{b_n\}$

$$\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

Give an example to show that the equality need not occur.

2.13.10 What is the correct version for the lim inf of Exercise 2.13.9?

2.13.11 Show that for any bounded sequences $\{a_n\}$ and $\{b_n\}$ of positive numbers

$$\limsup_{n\to\infty}(a_n b_n) \leq (\limsup_{n\to\infty} a_n)(\limsup_{n\to\infty} b_n).$$

Give an example to show that the equality need not occur.

2.13.12 Correct the careless student proof in Exercise 2.8.3 for the squeeze theorem by replacing lim with limsup and liminf in the argument.

2.13.13 What relation, if any, can you state for the lim sups and lim infs of a sequence $\{a_n\}$ and one of its subsequences $\{a_{n_k}\}$?

2.13.14 If a sequence $\{a_n\}$ has no convergent subsequences, what can you state about the lim sups and lim infs of the sequence?

2.13.15 Let S denote the set of all real numbers t with the property that some subsequence of a given sequence $\{a_n\}$ converges to t. What is the relation between the set S and the lim sups and lim infs of the sequence $\{a_n\}$?

SEE NOTE 27

2.13.16 Prove the following assertion about the upper and lower limits for any sequence $\{a_n\}$ of positive real numbers:

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} \le \liminf_{n\to\infty} \sqrt[n]{a_n} \le \limsup_{n\to\infty} \sqrt[n]{a_n} \le \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

Give an example to show that each of these inequalities may be strict.

2.13.17 For any sequence $\{a_n\}$ write $s_n = (a_1 + a_2 + \cdots + a_n)/n$. Show that

$$\liminf_{n\to\infty} a_n \le \liminf_{n\to\infty} s_n \le \limsup_{n\to\infty} s_n \le \limsup_{n\to\infty} a_n.$$

Give an example to show that each of these inequalities may be strict.

2.14 Challenging Problems for Chapter 2

2.14.1 Let α and β be positive numbers. Show that

$$\lim_{n\to\infty} \sqrt[n]{\alpha^n + \beta^n} = \max\{\alpha, \beta\}.$$

2.14.2 For any convergent sequence $\{a_n\}$ write $s_n = (a_1 + a_2 + \cdots + a_n)/n$, the sequence of averages. Show that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} s_n.$$

Give an example to show that $\{s_n\}$ could converge even if $\{a_n\}$ diverges.

2.14.3 Let $a_1 = 1$ and define a sequence recursively by

$$a_{n+1} = \sqrt{a_1 + a_2 + \cdots + a_n}.$$

Show that $\lim_{n\to\infty} \frac{a_n}{n} = 1/2$.

2.14.4 Let $x_1 = \theta$ and define a sequence recursively by

$$x_{n+1} = \frac{x_n}{1 + x_n/2}.$$

For what values of θ is it true that $x_n \to 0$?

2.14.5 Let $\{a_n\}$ be a sequence of numbers in the interval $(0, 1)$ with the property that

$$a_n < \frac{a_{n-1} + a_{n+1}}{2}$$

for all $n = 2, 3, 4, \ldots$. Show that this sequence is convergent.

2.14.6 For any convergent sequence $\{a_n\}$ write

$$s_n = \sqrt[n]{(a_1 a_2 \dots a_n)},$$

the sequence of geometric averages. Show that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} s_n$. Give an example to show that $\{s_n\}$ could converge even if $\{a_n\}$ diverges.

2.14.7 If

$$\lim_{n\to\infty} \frac{s_n - \alpha}{s_n + \alpha} = 0$$

what can you conclude about the sequence $\{s_n\}$?

2.14.8 A function f is defined by

$$f(x) = \lim_{n\to\infty} \left(\frac{1-x^2}{1+x^2}\right)^n$$

at every value x for which this limit exists. What is the domain of the function?

2.14.9 A function f is defined by

$$f(x) = \lim_{n\to\infty} \frac{1}{x^n + x^{-n}}$$

at every value x for which this limit exists. What is the domain of the function?

2.14.10 Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a positive function with a derivative f' that is everywhere continuous and negative. Apply Newton's method to obtain a sequence

$$x_1 = \theta\,, \quad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Show that $x_n \to \infty$ for any starting value θ. [This problem assumes some calculus background.]

2.14.11 Let $f(x) = x^3 - 3x + 3$. Apply Newton's method to obtain a sequence

$$x_1 = \theta\,, \quad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Show that for any positive integer p there is a starting value θ such that the sequence $\{x_n\}$ is periodic with period p.

See Note 28

2.14.12 Determine all subsequential limit points of the sequence $x_n = \cos n$.

SEE NOTE 29

2.14.13 A sequence $\{s_n\}$ is said to be *contractive* if there is a positive number $0 < r < 1$ so that

$$|s_{n+1} - s_n| \leq r|s_n - s_{n-1}|$$

for all $n = 2, 3, 4, \ldots$.

(a) Show that the sequence defined by $s_1 = 1$ and $s_n = (4 + s_{n-1})^{-1}$ for $n = 2, 3, \ldots$ is contractive.

(b) Show that every contractive sequence is Cauchy.

(c) Show that a sequence can satisfy the condition

$$|s_{n+1} - s_n| < |s_n - s_{n-1}|$$

for all $n = 2, 3, 4, \ldots$ and not be contractive, nor even convergent.

(d) Is every convergent sequence contractive?

SEE NOTE 30

2.14.14 The sequence defined recursively by

$$f_1 = 1,\ f_2 = 1 \quad f_{n+2} = f_n + f_{n+1}$$

is called the *Fibonacci sequence.* Let

$$r_n = f_{n+1}/f_n$$

be the sequence of ratios of successive terms of the Fibonacci sequence.

(a) Show that $r_1 < r_3 < r_5 < \cdots < r_6 < r_4 < r_2$.

(b) Show that $r_{2n} - r_{2n-1} \to 0$.

(c) Deduce that the sequence $\{r_n\}$ converges. Can you find a way to determine that limit? (This is related to the roots of the equation $x^2 - x - 1 = 0$.)

2.14.15 A sequence of real numbers $\{x_n\}$ has the property that

$$(2 - x_n)x_{n+1} = 1.$$

Show that $\lim_{n\to\infty} x_n = 1$.

SEE NOTE 31

2.14.16 Let $\{a_n\}$ be an arbitrary sequence of positive real numbers. Show that

$$\limsup_{n\to\infty}\left(\frac{a_1+a_{n+1}}{a_n}\right)^n \geq e.$$

SEE NOTE 32

2.14.17 Suppose that the sequence whose nth term is

$$s_n + 2s_{n+1}$$

is convergent. Show that $\{s_n\}$ is also convergent.

SEE NOTE 33

2.14.18 Show that the sequence

$$\sqrt{7}, \sqrt{7-\sqrt{7}}, \sqrt{7-\sqrt{7+\sqrt{7}}}, \sqrt{7-\sqrt{7+\sqrt{7-\sqrt{7}}}}, \dots$$

converges and find its limit.

SEE NOTE 34

2.14.19 Let a_1 and a_2 be positive numbers and suppose that the sequence $\{a_n\}$ is defined recursively by

$$a_{n+2} = \sqrt{a_n} + \sqrt{a_{n+1}}.$$

Show that this sequence converges and find its limit.

SEE NOTE 35

Notes

[9]Exercise 2.2.2. For the next term in the sequence some people might expect a 1. Most mathematicians would expect a 9.

[10]Exercise 2.2.3. Here is a formula that generates the first five terms of the sequence 0, 0, 0, 0, c, ...

$$f(n) = \frac{c(n-1)(n-2)(n-3)(n-4)}{4!}.$$

[11]Exercise 2.2.10. The formula is

$$f_n = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\}.$$

It can be verified by induction.

[12]Exercise 2.3.1. Find a function $f : (a, b) \to (0, 1)$ one-to-one onto and consider the sequence $f(s_n)$, where $\{s_n\}$ is a sequence that is claimed to have all of (a, b) as its range.

[13]Exercise 2.3.4. We can consider that the elements of each of the sets S_i can be listed, say,

$$S_1 = \{x_{11}, x_{12}, x_{13}, \dots\}$$

$$S_2 = \{x_{21}, x_{22}, x_{23}, \dots\}$$

and so on. Now try to think of a way of listing all of these items, that is, making one big list that contains them all.

[14]Exercise 2.3.6. We need (i) every number has a decimal expansion; (ii) the decimal expansion is unique except in the case of expansions that terminate in a string of zeros or nines [e.g., $1/2 = 0.5000000\cdots = .49999999\dots$], thus if a and b are numbers such that in the nth decimal place one has a 5 (or a 6) and the other does not then either $a \neq b$, or perhaps one ends in a string of zeros and the other in a string of nines; and (iii) every string of 5's and 6's defines a real number with that decimal expansion.

[15]Exercise 2.3.10. Try to find a way of ranking the algebraic numbers in the same way that the rational numbers were ranked.

[16]Exercise 2.4.6. You will need the identity

$$1 + 2 + 3 + \cdots + n = n(n+1)/2.$$

[17]Exercise 2.4.7. You will need to find an identity for the sum of the squares similar to the identity $1+2+3+\cdots+n = n(n+1)/2$.

[18]Exercise 2.5.6. To establish a correct converse, reword: If all $x_n > 0$ and $\frac{x_n}{x_n+1} \to 1$, then $x_n \to \infty$. Prove that this is true. The converse of the statement in the exercise is false (e.g., $x_n = 1/n$).

[19]Exercise 2.6.5. Use the same method as used in the proof of Theorem 2.11.

[20]Exercise 2.8.1. Give a counterexample. Perhaps find two sequences so that $s_n < 0 < t_n$ for all n and yet $\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n = 0$.

[21]Exercise 2.8.9. Take any number r strictly between 1 and that limit. Show that for some N, $s_{n+1} < rs_n$ if $n \geq N$. Deduce that

$$s_{N+2} < r^2 s_N$$

and

$$s_{N+3} < r^3 s_N.$$

Carry on.

[22]Exercise 2.8.10. Take any number r strictly between 1 and that limit. Show that for some N, $s_{n+1} > rs_n$ if $n \geq N$. Deduce that

$$s_{N+2} > r^2 s_N$$

and

$$s_{N+3} > r^3 s_N.$$

Carry on.

[23]Exercise 2.10.1. In terms of our theory of convergence this statement has no meaning since (as you should show) the sequence diverges. Even so, many great mathematicians, including Euler, would have accepted and used this formula. The fact that it is useful suggests that there are ways of interpreting such statements other than as convergence assertions.

[24]Exercise 2.11.13. If a sequence contains subsequences converging to every number in $(0,1)$ show that it also contains a subsequence converging to 0.

[25]Exercise 2.12.5. Consider the sequence

$$s_n = 1 + 1/2 + 1/3 + \cdots + 1/n.$$

[26]Exercise 2.12.10. Compare to

$$1 + 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{6} - \ldots$$

which is the sum of a geometric progression.

[27]Exercise 2.13.15. Consider separately the cases where the sequence is bounded or not.

[28]Exercise 2.14.11. A sequence $\{x_n\}$ is periodic with period p if $x_{n+p} = x_n$ for all values of n and no smaller value of p will work. (Note that if $\{x_n\}$ is periodic with period p, then $x_n = x_{n+p} = x_{n+2p} = x_{n+3p} = \ldots$.)

[29]Exercise 2.14.12. Clearly, no number larger than 1 or less than -1 could be such a limit. Show that in fact the interval $[-1, 1]$ is the set of all such limit points. If $x \in [-1, 1]$ there must be a number y so that $\cos y = x$ (why?). Now consider the set of numbers

$$G = \{n + 2m\pi : n, m \in \mathbb{Z}\}.$$

Using Exercise 1.11.6 or otherwise, show that this is dense. Hence there are pairs of integers n, m so that

$$|y - n + 2m\pi| < \varepsilon.$$

From this deduce that

$$|\cos y - \cos(n + 2m\pi)| < \varepsilon$$

and so $|x - \cos n| < \varepsilon$.

[30]Exercise 2.14.13. For (a) show that

$$|s_{n+1} - s_n| \leq \frac{1}{17}|s_n - s_{n-1}|$$

for all $n = 2, 3, 4, \ldots$. For (b) you will need to use the fact that the sum of geometric progressions is bounded, in fact that

$$1 + r + r^2 + \cdots + r^n < (1 - r)^{-1}$$

if $0 < r < 1$. Express for $m > n$,

$$|s_m - s_n| \leq |s_{n+1} - s_n|$$
$$+|s_{n+2} - s_{n+1}| + \cdots + |s_m - s_{m-1}|$$

and then use the contractive hypothesis. Note that

$$|s_4 - s_3| \leq r|s_3 - s_2| \leq r^2|s_2 - s_1|.$$

For (d) you might have to wait for the study of series in order to find an appropriate example of a convergent sequence that is not contractive.

[31]Exercise 2.14.15. This is from the 1947 Putnam Mathematical Competition.

[32]Exercise 2.14.16. This is from the 1949 Putnam Mathematical Competition.

[33]Exercise 2.14.17. This is from the 1950 Putnam Mathematical Competition.

[34]Exercise 2.14.18. This is from the 1953 Putnam Mathematical Competition.

[35]Exercise 2.14.19. Problem posed by A. Emerson in the *Amer. Math. Monthly*, **85** (1978), p. 496.

Chapter 3

INFINITE SUMS

✂ This chapter on infinite sums and series may be skipped over in designing a course or covered later as the need arises. The basic material in Sections 3.4, 3.5, and parts of 3.6 will be needed, but not before the study of series of functions in Chapter 14. All of the enrichment or advanced sections may be omitted and are not needed in the sequel.

3.1 Introduction

The use of infinite sums goes back in time much further, apparently, than the study of sequences. The sum

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \cdots = 2$$

has been long known. It is quite easy to convince oneself that this must be valid by arithmetic or geometric "reasoning." After all, just start adding and keeping track of the sum as you progress:

$$1,\ 1\tfrac{1}{2},\ 1\tfrac{3}{4},\ 1\tfrac{7}{8},\ 1\tfrac{15}{16},\ \ldots.$$

Figure 3.1 makes this seem transparent.

But there is a serious problem of meaning here. A finite sum is well defined, an infinite sum is not. Neither humans nor computers can add an infinite column of numbers.

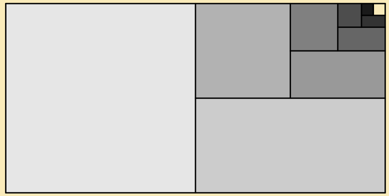

Figure 3.1. $1 + 1/2 + 1/4 + 1/8 + 1/16 + \cdots = 2.$

The meaning that is commonly assigned to the preceding sum appears in the following computations:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \lim_{n\to\infty} \left\{ 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} \right\}$$

$$= \lim_{n\to\infty} \left\{ 2 \left[1 - \frac{1}{2^{n+1}} \right] \right\} = 2.$$

This reduces the computation of an infinite sum to that of a finite sum followed by a limit operation. Indeed this is exactly what we were doing when we computed 1, $1\frac{1}{2}$, $1\frac{3}{4}$, $1\frac{7}{8}$, $1\frac{15}{16}$, … and felt that this was a compelling reason for thinking of the sum as 2.

In terms of the development of the theory of this textbook this seems entirely natural and hardly surprising. We have mastered sequences in Chapter 2 and now pass to infinite sums in Chapter 3 using the methods of sequences. Historically this was not the case. Infinite summations appear to have been studied and used long before any development of sequences and sequence limits. Indeed, even to form the notion of an infinite sum as previously, it would seem that we should already have some concept of sequences, but this is not the way things developed.

It was only by the time of Cauchy that the modern theory of infinite summation was developed using sequence limits as a basis for the theory. We can transfer a great deal of our expertise in sequential limits to the problem of infinite sums. Even so, the study in this chapter has its own character and charm. In many ways infinite sums are much more interesting and important to analysis than sequences.

3.2 Finite Sums

We should begin our discussion of infinite sums with finite sums. There is not much to say about finite sums. Any finite collection of real numbers may be summed in any order and any grouping. That is not to say that we shall not encounter *practical* problems in this. For example, what is the sum of the first 10^{100} prime numbers? No computer or human could find this within the time remaining in this universe. But there is no *mathematical* problem in saying that it is defined; it is a sum of a finite number of real numbers.

There are a number of notations and a number of skills that we shall need to develop in order to succeed at the study of infinite sums that is to come. The notation of such summations may be novel. How best to write out a symbol indicating that some set of numbers

$$\{a_1, a_2, a_3, \dots, a_n\}$$

has been summed? Certainly

$$a_1 + a_2 + a_3 + \cdots + a_n$$

is too cumbersome a way of writing such sums. The following have proved to communicate much better:

$$\sum_{i \in I} a_i$$

where I is the set $\{1, 2, 3, \dots, n\}$ or

$$\sum_{1 \le i \le n} a_i \quad \text{or} \quad \sum_{i=1}^{n} a_i.$$

Here the Greek letter Σ, corresponding to an uppercase "S," is used to indicate a sum.

It is to Leonhard Euler (1707–1783) that we owe this sigma notation for sums (first used by him in 1755). The notations $f(x)$ for functions, e and π, i for $\sqrt{-1}$ are also his. These alone indicate the level of influence he has left. In his lifetime he wrote 886 papers and books and is considered the most prolific writer of mathematics that has lived.

The usual rules of elementary arithmetic apply to finite sums. The commutative, associative, and distributive rules assume a different look when written in Euler's notation:

$$\sum_{i\in I} a_i + \sum_{i\in I} b_i = \sum_{i\in I} (a_i + b_i)\,,$$

$$\sum_{i\in I} ca_i = c\sum_{i\in I} a_i,$$

and

$$\left(\sum_{i\in I} a_i\right) \times \left(\sum_{j\in J} b_j\right) = \sum_{i\in I}\left(\sum_{j\in J} a_i b_j\right) = \sum_{j\in J}\left(\sum_{i\in I} a_i b_j\right).$$

Each of these can be checked mainly by determining the meanings and seeing that the notation produces the correct result.

Occasionally in applications of these ideas one would like a simplified expression for a summation. The best known example is perhaps

$$\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2},$$

which is easily proved. When a sum of n terms for a general n has a simpler expression such as this it is usual to say that it has been expressed in *closed form*. Novices, seeing this, usually assume that any summation with some degree of regularity should allow a closed form expression and that it is always important to get a closed form expression. If not, what can you do with a sum that cannot be simplified?

One of the simplest of sums

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots + \tfrac{1}{n} = \textstyle\sum_{k=1}^{n} \tfrac{1}{k}$$

does not allow any convenient formula, expressing the sum as some simple function of n. This is typical.

It is only the rarest of summations that will allow simple formulas. Our work is mostly in *estimating* such expressions; we hardly ever succeed in computing them exactly.

Even so, there are a few special cases that should be remembered and which make our task in some cases much easier.

Telescoping Sums. If a sum can be rewritten in the special form below, a simple computation (canceling s_1, s_2, etc.) gives the following closed form:

$$(s_1 - s_0) + (s_2 - s_1) + (s_3 - s_2) + (s_4 - s_3) + \cdots + (s_n - s_{n-1}) = s_n - s_0.$$

It is convenient to call such a sum "telescoping" as an indication of the method that can be used to compute it.

Example 3.1: For a specific example of a sum that can be handled by considering it as telescoping, consider the sum

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \cdots + \frac{1}{(n-1) \cdot n}.$$

A closed form is available since, using partial fractions, each term can be expressed as

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}.$$

Thus

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} =$$

$$\sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1} \right) = 1 - \frac{1}{n+1}.$$

The exercises contain a number of other examples of this type. ◀

Geometric Progressions. If the terms of a sum are in a geometric progression (i.e., if each term is some constant factor times the previous term), then a closed form for any such sum is available:

$$1 + r + r^2 + \cdots + r^{n-1} + r^n = \frac{1 - r^{n+1}}{1 - r}. \tag{1}$$

This assumes that $r \neq 1$; if $r = 1$ the sum is easily seen to be just $n + 1$. The formula in (1) can be proved by converting to a telescoping sum. Consider instead $(1 - r)$ times the preceding sum:

$$(1 - r)(1 + r + r^2 + \cdots + r^{n-1} + r^n) = (1 - r) + (r - r^2) + \cdots + (r^n - r^{n+1}).$$

Now add this up as a telescoping sum to obtain the formula stated in (1).

Any geometric progression assumes the form

$$A + Ar + Ar^2 + \cdots + Ar^n = A(1 + r + r^2 + \cdots + r^n)$$

and formula (1) (which should be memorized) is then applied.

Summation By Parts. Sums are frequently given in a form such as

$$\sum_{k=1}^{n} a_k b_k$$

for sequences $\{a_k\}$ and $\{b_k\}$. If a formula happens to be available for

$$s_n = a_1 + a_2 + \cdots + a_n,$$

then there is a frequently useful way of rewriting this sum (using $s_0 = 0$ for convenience):

$$\sum_{k=1}^{n} a_k b_k = \sum_{k=1}^{n} (s_k - s_{k-1}) b_k$$

$$= s_1(b_1 - b_2) + s_2(b_2 - b_3) + \cdots + s_{n-1}(b_{n-1} - b_n) + s_n b_n.$$

Usually some extra knowledge about the sequences $\{s_k\}$ and $\{b_k\}$ can then be used to advantage. The computation is trivial (it is all contained in the preceding equation which is easily checked). Sometimes this summation formula is referred to as Abel's transformation after the Norwegian mathematician Niels Henrik Abel (1802–1829), who was one of the founders of the rigorous theory of infinite sums. It is the analog for finite sums of the integration by parts formula of calculus.

Abel's most important contributions are to analysis but he is forever immortalized in group theory (to which he made a small contribution) by the fact that commutative groups are called "Abelian."

Exercises

3.2.1 Prove the formula

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

3.2.2 Give a formal definition of $\sum_{i\in I} a_i$ for any finite set I and any function $a : I \to \mathbb{R}$ that uses induction on the number of elements of I.

SEE NOTE 36

Your definition should be able to handle the case $I = \emptyset$.

3.2.3 Check the validity of the formulas given in this section for manipulating finite sums. Are there any other formulas you can propose and verify?

3.2.4 Is the formula

$$\sum_{i\in I\cup J} a_i = \sum_{i\in I} a_i + \sum_{i\in J} a_i$$

valid?

SEE NOTE 37

3.2.5 Let $I = \{(i,j) : 1 \le i \le m,\ 1 \le j \le n\}$. Show that

$$\sum_{(i,j)\in I} a_{ij} = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}.$$

3.2.6 Give a formula for the sum of n terms of an arithmetic progression. (An arithmetic progression is a list of numbers, each of which is obtained by adding a fixed constant to the previous one in the list.) For the purposes of infinite sums (our concern in this chapter) such a formula will be of little use. Explain why.

3.2.7 Obtain formulas (or find a source for such formulas) for the sums

$$\sum_{k=1}^{n} k^p = 1^p + 2^p + 3^p + \cdots + n^p$$

of the pth powers of the natural numbers where $p = 1, 2, 3, 4, \ldots$. Again, for the purposes of infinite sums such formulas will be of little use.

3.2.8 Explain the (vague) connection between integration by parts and summation by parts.

See Note 38

3.2.9 Obtain a formula for $\sum_{k=1}^{n}(-1)^k$.

3.2.10 Obtain a formula for

$$2 + 2\sqrt{2} + 4 + 4\sqrt{2} + 8 + 8\sqrt{2} + \cdots + 2^m.$$

3.2.11 Obtain the formula

$$\sin\theta + \sin 2\theta + \sin 3\theta + \sin 4\theta + \cdots + \sin n\theta = \frac{\cos\theta/2 - \cos(2n+1)\theta/2}{2\sin\theta/2}.$$

How should the formula be interpreted if the denominator of the fraction is zero?

See Note 39

3.2.12 Obtain the formula

$$\cos\theta + \cos 3\theta + \cos 5\theta + \cos 7\theta + \cdots + \cos(2n-1)\theta = \frac{\sin 2n\theta}{2\sin\theta}.$$

3.2.13 If

$$s_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{n+1}\frac{1}{n}$$

show that $1/2 \le s_n \le 1$ for all n.

3.2.14 If

$$s_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$$

show that $s_{2^n} \geq 1 + n/2$ for all n.

3.2.15 Obtain a closed form for

$$\sum_{k=1}^{n} \frac{1}{k(k+2)(k+4)}.$$

3.2.16 Obtain a closed form for

$$\sum_{k=1}^{n} \frac{\alpha r + \beta}{k(k+1)(k+2)}.$$

3.2.17 Let $\{a_k\}$ and $\{b_k\}$ be sequences with $\{b_k\}$ decreasing and

$$|a_1 + a_2 + \cdots + a_k| \leq K$$

for all k. Show that

$$\left| \sum_{k=1}^{n} a_k b_k \right| \leq K b_1$$

for all n.

3.2.18 If r is the interest rate (e.g., $r = .06$) over a period of years, then

$$P(1+r)^{-1} + P(1+r)^{-2} + \cdots + P(1+r)^{-n}$$

is the present value of an annuity of P dollars paid every year, starting next year and for n years. Give a shorter formula for this. (A perpetuity has nearly the same formula but the payments continue forever. See Exercise 3.4.12.)

3.2.19 Define a finite product (product of a finite set of real numbers) by writing

$$\prod_{k=1}^{n} a_k = a_1 a_2 a_3 \ldots a_n.$$

What elementary properties can you determine for products?

3.2.20 Find a closed form expression for

$$\prod_{k=1}^{n} \frac{k^3-1}{k^3+1}.$$

3.3 Infinite Unordered sums

We now pass to the study of infinite sums. We wish to interpret

$$\sum_{i \in I} a_i$$

for an index set I that is infinite. The study of finite sums involves no analysis, no limits, no ε's, in short none of the processes that are special to analysis. To define and study infinite sums requires many of our skills in analysis.

To begin our study imagine that we are given a collection of numbers a_i indexed over an infinite set I (i.e., there is a function $a : I \to \mathbb{R}$) and we wish the sum of the totality of these numbers. If the set I has some structure, then we can use that structure to decide how to start adding the numbers. For example, if a is a sequence so that $I = \mathbb{N}$, then we should likely start adding at the beginning of the sequence:

$$a_1,\ a_1 + a_2,\ a_1 + a_2 + a_3,\ a_1 + a_2 + a_3 + a_4, \ldots$$

and so defining the sum as the limit of this sequence of partial sums.

Another set I would suggest a different order. For example, if $I = \mathbb{Z}$ (the set of all integers), then a popular method of adding these up would be to start off:

$$a_0,\ a_{-1} + a_0 + a_1,$$

$$a_{-2} + a_{-1} + a_0 + a_1 + a_2,$$

$$a_{-3} + a_{-2} + a_{-1} + a_0 + a_1 + a_2 + a_3,\ \ldots$$

once again defining the sum as the limit of this sequence.

It seems that the method of summation and hence defining the meaning of the expression

$$\sum_{i \in I} a_i$$

for infinite sets I must depend on the nature of the set I and hence on the particular problems of the subject one is studying. This is true to some extent. But it does not stop us from inventing a method that will apply to *all* infinite sets I. We must make a definition that takes account of no extra structure or ordering for the set I and just treats it as a set. This is called the unordered sum and the notation $\sum_{i \in I} a_i$ is always meant to indicate that an unordered sum is being considered. The key is just how to pass from finite sums to infinite sums. Both of the previous examples used the idea of taking some finite sums (in a systematic way) and then passing to a limit.

Definition 3.2: Let I be an infinite set and a a function $a : I \to \mathbb{R}$. Then we write

$$\sum_{i \in I} a_i = c$$

and say that the sum *converges* if for every $\varepsilon > 0$ there is a finite set $I_0 \subset I$ so that, for every finite set J, $I_0 \subset J \subset I$,

$$\left| \sum_{i \in J} a_i - c \right| < \varepsilon.$$

A sum that does not converge is said to *diverge*.

Note that we never form a sum of infinitely many terms. The definition always computes finite sums.

Example 3.3: Let us show, directly from the definition, that

$$\sum_{i \in \mathbb{Z}} 2^{-|i|} = 3.$$

If we first sum

$$\sum_{-N\leq i\leq N} 2^{-|i|}$$

by rearranging the terms into the sum

$$1+2(2^{-1}+2^{-2}+\cdots+2^{-N})$$

we can see why the sum is likely to be 3. Let $\varepsilon>0$ and choose N so that $2^{-N}<\varepsilon/4$. From the formula for a finite geometric progression we have

$$\left|\sum_{-N\leq i\leq N} 2^{-|i|}-3\right| = 2|(2^{-1}+2^{-2}+\cdots+2^{-N})-1| = 2(2^{-N})<\varepsilon/2.$$

Also, if $K\subset\mathbb{Z}$ with K finite and $|k|>N$ for all $k\in K$, then

$$\sum_{k\in K} 2^{-|k|}<2(2^{-N})<\varepsilon/2$$

again from the formula for a finite geometric progression. Let

$$I_0=\{i\in\mathbb{Z}: -N\leq i\leq N\}.$$

If $I_0\subset J\subset\mathbb{Z}$ with J finite then

$$\left|\sum_{i\in J} 2^{-|i|}-3\right| = \left|\sum_{-N\leq i\leq N} 2^{-|i|}-3\right| + \sum_{i\in J\setminus I_0} 2^{-|i|}<\varepsilon$$

as required. ◀

3.3.1 Cauchy Criterion

In most theories of convergence one asks for a necessary and sufficient condition for convergence. We saw in studying sequences that the Cauchy criterion provided such a condition for the convergence of a sequence.

There is usually in any theory of this kind a type of Cauchy criterion. Here is the Cauchy criterion for sums.

Theorem 3.4: *A necessary and sufficient condition that the sum $\sum_{i\in I} a_i$ converges is that for every $\varepsilon > 0$ there is a finite set I_0 so that*

$$\left|\sum_{i\in J} a_i\right| < \varepsilon$$

for every finite set $J \subset I$ that contains no elements of I_0 (i.e., for all finite sets $J \subset I \setminus I_0$).

Proof. As usual in Cauchy criterion proofs, one direction is easy to prove. Suppose that $\sum_{i\in I} a_i = C$ converges. Then for every $\varepsilon > 0$ there is a finite set I_0 so that

$$\left|\sum_{i\in K} a_i - C\right| < \varepsilon/2$$

for every finite set $I_0 \subset K \subset I$. Let J be a finite subset of $I \setminus I_0$ and consider taking a sum over $K = I_0 \cup J$. Then

$$\left|\sum_{i\in I_0\cup J} a_i - C\right| < \varepsilon/2$$

and

$$\left|\sum_{i\in I_0} a_i - C\right| < \varepsilon/2.$$

By subtracting these two inequalities and remembering that

$$\sum_{i\in I_0\cup J} a_i = \sum_{i\in J} a_i + \sum_{i\in I_0} a_i$$

(since I_0 and J are disjoint) we obtain

$$\left|\sum_{i\in J} a_i\right| < \varepsilon.$$

This is exactly the Cauchy criterion.

Conversely, suppose that the sum does satisfy the Cauchy criterion. Then, applying that criterion to $\varepsilon = 1, 1/2, 1/3, \dots$ we can choose a sequence of finite sets $\{I_n\}$ so that

$$\left|\sum_{i\in J} a_i\right| < 1/n$$

for every finite set $J \subset I \setminus I_n$. We can arrange our choices to make

$$I_1 \subset I_2 \subset I_3 \subset \dots$$

so that the sequence of sets is increasing.

Let

$$c_n = \sum_{i\in I_n} a_i$$

Then for any $m > n$,

$$|c_n - c_m| = \left|\sum_{i\in I_m\setminus I_n} a_i\right| < 1/n.$$

It follows from this that $\{c_n\}$ is a Cauchy sequence of real numbers and hence converges to some real number c. Let $\varepsilon > 0$ and choose an integer N larger than $2/\varepsilon$ and so that $|c_N - c| < \varepsilon/2$. Then, for any $n > N$ and any finite set J with $I_N \subset J \subset I$,

$$\left|\sum_{i\in J} a_i - c\right| \leq \left|\sum_{i\in I_N} a_i - c_N\right| + |c_N - c| + \left|\sum_{i\in J\setminus I_N} a_i\right| < 0 + \varepsilon/2 + 1/N < \varepsilon.$$

By definition, then,

$$\sum_{i\in I} a_i = c$$

and the theorem is proved. ■

All But Countably Many Terms in a Convergent Sum Are Zero. Our next theorem shows that having "too many" numbers to add up causes problems. If the set I is not countable then most of the a_i that we are to add up should be zero if the sum is to exist. This shows too that the theory of sums is in an essential way limited to taking sums over countable sets. It is notationally possible to have a sum

$$\sum_{x\in[0,1]} f(x)$$

but that sum cannot be defined unless $f(x)$ is mostly zero with only countably many exceptions.

Theorem 3.5: *Suppose that $\sum_{i\in I} a_i$ converges. Then $a_i = 0$ for all $i \in I$ except for a countable subset of I.*

Proof. We shall use Exercise 3.3.2, where it is proved that for any convergent sum there is a positive integer M so that all the sums

$$\left|\sum_{i\in I_0} a_i\right| \leq M$$

for any finite set $I_0 \subset I$. Let m be an integer. We ask how many elements a_i are there such that $a_i > 1/m$? It is easy to see that there are at most Mm of them since if there were any more our sum would exceed M. Similarly, there are at most Mm terms such that $-a_i > 1/m$. Thus each element of $\{a_i : i \in I\}$ that is not zero can be given a "rank" m depending on whether

$$1/m < a_i \leq 1/(m-1) \text{ or } 1/m < -a_i \leq 1/(m-1).$$

As there are only finitely many elements at each rank, this gives us a method for listing all of the nonzero elements in $\{a_i : i \in I\}$ and so this set is countable. ■

The elementary properties of unordered sums are developed in the exercises. These sums play a small role in analysis, a much smaller role than the ordered sums we shall consider in the next sections. The methods of proof, however, are well worth studying since they are used in some form or other in many parts of analysis. These exercises offer an interesting setting in which to test your skills in analysis, skills that will play a role in all of your subsequent study.

Exercises

3.3.1 Show that if $\sum_{i\in I} a_i$ converges, then the sum is unique.

See Note 40

3.3.2 Show that if $\sum_{i\in I} a_i$ converges, then there is a positive number M so that all the sums

$$\left|\sum_{i\in I_0} a_i\right| \leq M$$

for any finite set $I_0 \subset I$.

See Note 41

3.3.3 Suppose that all the terms in the sum $\sum_{i\in I} a_i$ are nonnegative and that there is a positive number M so that all the sums

$$\sum_{i\in I_0} a_i \leq M$$

for any finite set $I_0 \subset I$. Show that $\sum_{i\in I} a_i$ must converge.

See Note 42

3.3.4 Show that if $\sum_{i\in I} a_i$ converges so too does $\sum_{i\in J} a_i$ for every subset $J \subset I$.

3.3.5 Show that if $\sum_{i\in I} a_i$ converges and each $a_i \geq 0$, then

$$\sum_{i\in I} a_i = \sup\left\{\sum_{i\in J} a_i :\ J\subset I,\ J \text{ finite}\right\}.$$

3.3.6 Each of the rules for manipulation of the finite sums of Section 3.2 can be considered for infinite unordered sums. Formulate the correct statement and prove what you think to be the analog of these statements that we know hold for finite sums:

$$\sum_{i\in I} a_i + \sum_{i\in I} b_i = \sum_{i\in I} (a_i + b_i)$$

$$\sum_{i\in I} ca_i = c\sum_{i\in I} a_i$$

$$\sum_{i\in I} a_i \times \sum_{i\in J} b_j = \sum_{i\in I}\sum_{j\in J} a_i b_j = \sum_{j\in J}\sum_{i\in I} a_i b_j.$$

3.3.7 Prove that

$$\sum_{i\in I\cup J} a_i + \sum_{i\in I\cap J} a_i = \sum_{i\in I} a_i + \sum_{i\in J} a_i$$

under appropriate convergence assumptions.

3.3.8 Let $\sigma : I \to J$ one-to-one and onto. Establish that

$$\sum_{j\in J} a_j = \sum_{i\in I} a_{\sigma(i)}$$

under appropriate convergence assumptions.

3.3.9 Find the sum

$$\sum_{i\in\mathbb{N}} \frac{1}{2^i}.$$

SEE NOTE 43

3.3.10 Show that

$$\sum_{i\in\mathbb{N}} \frac{1}{i}$$

diverges. Are there any infinite subsets $J \subset \mathbb{N}$ such that

$$\sum_{i\in J} \frac{1}{i}$$

converges?

3.3.11 Show that $\sum_{i\in I} a_i$ converges if and only if both $\sum_{i\in I}[a_i]^+$ and $\sum_{i\in I}[a_i]^-$ converge and that

$$\sum_{i\in I} a_i = \sum_{i\in I}[a_i]^+ - \sum_{i\in I}[a_i]^-$$

and

$$\sum_{i\in I} |a_i| = \sum_{i\in I}[a_i]^+ + \sum_{i\in I}[a_i]^-.$$

See Note 44

3.3.12 Compute

$$\sum_{(i,j)\in\mathbb{N}\times\mathbb{N}} 2^{-i-j}.$$

What kind of *ordered* sum would seem natural here (in the way that ordered sums over $\mathbb{N}$ and $\mathbb{Z}$ were considered in this section)?

See Note 45

3.4 Ordered Sums: Series

For the vast majority of applications, one wishes to sum not an arbitrary collection of numbers but most commonly some sequence of numbers:

$$a_1 + a_2 + a_3 + \dots .$$

The set $\mathbb{N}$ of natural numbers has an order structure, and it is not in our best interests to ignore that order since that is the order in which the sequence is presented to us.

The most compelling way to add up a sequence of numbers is to begin accumulating:

$$a_1,\ a_1 + a_2,\ a_1 + a_2 + a_3,\ a_1 + a_2 + a_3 + a_4, \dots$$

and to define the sum as the limit of this sequence. This is what we shall do.

If you studied Section 3.3 on unordered summation you should also compare this "ordered" method with the unordered method. The ordered sum of a sequence is called a *series* and the notation

$$\sum_{k=1}^{\infty} a_k$$

is used exclusively for this notion.

Definition 3.6: Let $\{a_k\}$ be a sequence of real numbers. Then we write

$$\sum_{k=1}^{\infty} a_k = c$$

and say that the series *converges* if the sequence

$$s_n = \sum_{k=1}^{n} a_k$$

(called the *sequence of partial sums of the series*) converges to c. If the series does not converge it is said to be *divergent*.

This definition reduces the study of series to the study of sequences. We already have a highly developed theory of convergent sequences in Chapter 2 that we can apply to develop a theory of series. Thus we can rapidly produce a fairly deep theory of series from what we already know. As the theory develops, however, we shall see that it begins to take a character of its own and stops looking like a mere application of sequence ideas.

3.4.1 Properties

The following short harvest of theorems we obtain directly from our sequence theory. The convergence or divergence of a series $\sum_{k=1}^{\infty} a_k$ depends on the convergence or divergence of the sequence of partial sums

$$s_n = \sum_{k=1}^{n} a_k$$

and the value of the series is the limit of the sequence. To prove each of the theorems we now list requires only to find the correct theorem on sequences from Chapter 2. This is left as Exercise 3.4.2.

Theorem 3.7: *If a series $\sum_{k=1}^{\infty} a_k$ converges, then the sum is unique.*

Theorem 3.8: *If both series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ converge, then so too does the series*

$$\sum_{k=1}^{\infty}(a_k + b_k)$$

and

$$\sum_{k=1}^{\infty}(a_k + b_k) = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k.$$

Theorem 3.9: *If the series $\sum_{k=1}^{\infty} a_k$ converges, then so too does the series $\sum_{k=1}^{\infty} ca_k$ for any real number c and*

$$\sum_{k=1}^{\infty} ca_k = c\sum_{k=1}^{\infty} a_k.$$

Theorem 3.10: *If both series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ converge and $a_k \leq b_k$ for each k, then*

$$\sum_{k=1}^{\infty} a_k \leq \sum_{k=1}^{\infty} b_k.$$

Theorem 3.11: *Let $M \geq 1$ be any integer. Then the series*

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + a_4 + \dots$$

converges if and only if the series

$$\sum_{k=1}^{\infty} a_{M+k} = a_{M+1} + a_{M+2} + a_{M+3} + a_{M+4} + \dots$$

converges.

Note. If we call $\sum_p^{\infty} a_i$ a "tail" for the series $\sum_1^{\infty} a_i$, then we can say that this last theorem asserts that it is the behavior of the tail that determines the convergence or divergence of the series. Thus in questions of convergence we can easily ignore the first part of the series—however many terms we like. Naturally, the actual sum of the series will depend on having all the terms.

3.4.2 Special Series

Telescoping Series Any series for which we can find a closed form for the partial sums we should probably be able to handle by sequence methods. Telescoping series are the easiest to deal with.

If the sequence of partial sums of a series can be computed in some closed form $\{s_n\}$, then the series can be rewritten in the telescoping form

$$(s_1) + (s_2 - s_1) + (s_3 - s_2) + (s_4 - s_3) + \dots + (s_n - s_{n-1}) + \dots$$

and the series studied by means of the sequence $\{s_n\}$.

Example 3.12: Consider the series

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1} \right) = \lim_{n\to\infty} \left(1 - \frac{1}{n+1} \right) = 1$$

with an easily computable sequence of partial sums. ◀

Do not be too encouraged by the apparent ease of the method illustrated by the example. In practice we can hardly ever do anything but make a crude estimate on the size of the partial sums. An exact expression, as we have here, would be rarely available. Even so, it is entertaining and instructive to handle a number of series by such a method (as we do in the exercises).

Geometric Series Geometric series form another convenient class of series that we can handle simply by sequence methods. From the elementary formula

$$1 + r + r^2 + \cdots + r^{n-1} + r^n = \frac{1 - r^{n+1}}{1 - r} \quad (r \neq 1)$$

we see immediately that the study of such a series reduces to the computation of the limit

$$\lim_{n\to\infty} \frac{1 - r^{n+1}}{1 - r} = \frac{1}{1 - r}$$

which is valid for $-1 < r < 1$ (which is usually expressed as $|r| < 1$) and invalid for all other values of r. Thus, for $|r| < 1$ the series

$$\sum_{k=1}^{\infty} r^{k-1} = 1 + r + r^2 + \cdots = \frac{1}{1 - r} \tag{2}$$

and is convergent and for $|r| \geq 1$ the series diverges. It is well worthwhile to memorize this fact and formula (2) for the sum of the series.

Harmonic Series As a first taste of an elementary looking series that presents a new challenge to our methods, consider the series

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \dots,$$

which is called the *harmonic series*. Let us show that this series diverges.

This series has no closed form for the sequence of partial sums $\{s_n\}$ and so there seems no hope of merely computing $\lim_{n\to\infty} s_n$ to determine the convergence or divergence of the harmonic series. But we can make estimates on the size of s_n even if we cannot compute it directly. The sequence of partial sums increases at each step, and if we watch only at the steps 1, 2, 4, 8, ... and make a rough lower estimate of s_1, s_2, s_4, s_8, ... we see that $s_{2^n} \geq 1 + n/2$ for all n (see Exercise 3.2.14). From this we see that $\lim_{n\to\infty} s_n = \infty$ and so the series diverges.

Alternating Harmonic Series A variant on the harmonic series presents immediately a new challenge. Consider the series

$$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots,$$

which is called the *alternating harmonic series*.

The reason why this presents a different challenge is that the sequence of partial sums is no longer increasing. Thus estimates as to how big that sequence get may be of no help. We can see that the sequence is bounded, but that does not imply convergence for a non monotonic sequence. Once again, we have no closed form for the partial sums so that a routine computation of a sequence limit is not available.

By computing the partial sums s_2, s_4, s_6, ... we see that the subsequence $\{s_{2n}\}$ is increasing. By computing the partial sums s_1, s_3, s_5, ... we see that the subsequence $\{s_{2n-1}\}$ is decreasing. A few more observations show us that

$$1/2 = s_2 < s_4 < s_6 < \dots < s_5 < s_3 < s_1 = 1. \tag{3}$$

Our theory of sequences now allows us to assert that both limits

$$\lim_{n\to\infty} s_{2n} \quad \text{and} \quad \lim_{n\to\infty} s_{2n-1}$$

exist. Finally, since

$$s_{2n} - s_{2n-1} = \frac{-1}{2n} \to 0$$

we can conclude that $\lim_{n\to\infty} s_n$ exists. [It is somewhere between $\frac{1}{2}$ and 1 because of the inequalities (3) but exactly what it is would take much further analysis.] Thus we have proved that the alternating harmonic series converges (which is in contrast to the divergence of the harmonic series).

p-Harmonic Series The series

$$\sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \dots$$

for any parameter $0 < p < \infty$ is called the *p-harmonic series.* The methods we have used in the study of the harmonic series can be easily adapted to handle this series. As a first observation note that if $0 < p < 1$, then

$$\frac{1}{k^p} > \frac{1}{k}.$$

Thus the p-harmonic series for $0 < p < 1$ is larger than the harmonic series itself. Since the latter series has unbounded partial sums it is easy to argue that our series does too and, hence, diverges for all $0 < p \leq 1$.

What about $p > 1$? Now the terms are smaller than the harmonic series, small enough it turns out that the series converges. To show this we can group the terms in the same manner as before for the harmonic series and obtain

$$1 + \left[\frac{1}{2^p} + \frac{1}{3^p}\right] + \left[\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right] + \left[\frac{1}{8^p} + \dots + \frac{1}{15^p}\right] + \dots$$

$$\leq 1 + \frac{2}{2^p} + \frac{4}{4^p} + \frac{8}{8^p} + \dots \leq \frac{1}{1 - 2^{1-p}}$$

since we recognize the latter series as a convergent geometric series with ratio 2^{1-p}. In this way we obtain an upper bound for the partial sums of the series

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$

for all $p > 1$. Since the partial sums are increasing and bounded above, the series must converge.

Size of the Terms It should seem apparent from the examples we have seen that a convergent series must have ultimately small terms. If $\sum_{k=1}^{\infty} a_k$ converges, then it seems that a_k must tend to 0 as k gets large. Certainly for the geometric series that idea precisely described the situation:

$$\sum_{k=1}^{\infty} r^{k-1}$$

converges if $|r| < 1$, which is exactly when the terms tend to zero and diverges when $|r| \geq 1$, which is exactly when the terms do not tend to zero.

A reasonable conjecture might be that this is always the situation: A series $\sum_{k=1}^{\infty} a_k$ converges if and only if $a_k \to 0$ as $k \to \infty$. But we have already seen the harmonic series diverges even though its terms do get small; they simply don't get small fast enough. Thus the correct observation is simple and limited.

If $\sum_{k=1}^{\infty} a_k$ converges, then $a_k \to 0$ as $k \to \infty$.

To check this is easy. If $\{s_n\}$ is the sequence of partial sums of a convergent series $\sum_{k=1}^{\infty} a_k = C$, then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = C - C = 0.$$

The converse, as we just noted, is false. To obtain convergence of a series it is not enough to know that the terms tend to zero. We shall see, though, that many of the tests that follow discuss the *rate* at which the terms tend to zero.

Exercises

3.4.1 Let $\{s_n\}$ be any sequence of real numbers. Show that this sequence converges to a number S if and only if the series

$$s_1 + \sum_{k=2}^{\infty} (s_k - s_{k-1})$$

converges and has sum S.

3.4.2 State which theorems from Chapter 2 would be used to prove Theorems 3.7–3.11.

3.4.3 If $\sum_{k=1}^{\infty}(a_k + b_k)$ converges, what can you say about the series

$$\sum_{k=1}^{\infty} a_k \text{ and } \sum_{k=1}^{\infty} b_k?$$

3.4.4 If $\sum_{k=1}^{\infty}(a_k + b_k)$ diverges, what can you say about the series

$$\sum_{k=1}^{\infty} a_k \text{ and } \sum_{k=1}^{\infty} b_k?$$

3.4.5 If the series $\sum_{k=1}^{\infty}(a_{2k} + a_{2k-1})$ converges, what can you say about the series $\sum_{k=1}^{\infty} a_k$?

3.4.6 If the series $\sum_{k=1}^{\infty} a_k$ converges, what can you say about the series

$$\sum_{k=1}^{\infty}(a_{2k} + a_{2k-1})?$$

3.4.7 If both series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ converge, what can you say about the series $\sum_{k=1}^{\infty} a_k b_k$?

3.4.8 How should we interpret

$$\sum_{k=0}^{\infty} a_{k+1}, \sum_{k=-5}^{\infty} a_{k+6} \text{ and } \sum_{k=5}^{\infty} a_{k-4}?$$

3.4.9 If s_n is a strictly increasing sequence of positive numbers, show that it is the sequence of partial sums of some series with positive terms.

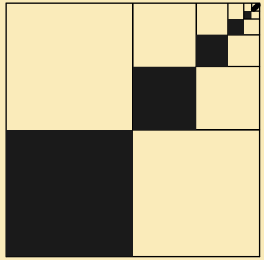

Figure 3.2. What is the area of the black region?

3.4.10 If $\{a_{n_k}\}$ is a subsequence of $\{a_n\}$, is there anything you can say about the relation between the convergence behavior of the series $\sum_{k=1}^{\infty} a_k$ and its "subseries" $\sum_{k=1}^{\infty} a_{n_k}$?

See Note 46

3.4.11 Express the infinite repeating decimal

$$.123451234512345123451234512345\ldots$$

as the sum of a convergent geometric series and compute its sum (as a rational number) in this way.

3.4.12 Using your result from Exercise 3.2.18, obtain a formula for a *perpetuity* of P dollars a year paid every year, starting next year and for every after. You most likely used a geometric series; can you find an argument that avoids this?

3.4.13 Suppose that a bird flying 100 miles per hour (mph) travels back and forth between a train and the railway station, where the train and the bird start off together 1 mile away and the train is approaching the station at a fixed rate of 60 mph. How far has the bird traveled when the train arrives? You most likely did not use a geometric series; can you find an argument that does?

3.4.14 What proportion of the area of the square in Figure 3.2 is black?

3.4.15 Does the series

$$\sum_{k=1}^{\infty} \log\left(\frac{k+1}{k}\right)$$

converge or diverge?

SEE NOTE 47

3.4.16 Show that

$$\frac{1}{r-1} = \frac{1}{r+1} + \frac{2}{r^2+1} + \frac{4}{r^4+1} + \frac{8}{r^8+1} + \dots$$

for all $r > 1$.

SEE NOTE 48

3.4.17 Obtain a formula for the sum

$$2 + \frac{2}{\sqrt{2}} + 1 + \frac{1}{\sqrt{2}} + \frac{1}{2} + \frac{1}{2\sqrt{2}} + \dots.$$

3.4.18 Obtain a formula for the sum

$$\sum_{k=1}^{\infty} \frac{1}{k(k+2)(k+4)}.$$

3.4.19 Obtain a formula for the sum

$$\sum_{k=1}^{\infty} \frac{\alpha r + \beta}{k(k+1)(k+2)}.$$

3.4.20 Find all values of x for which the the following series converges and determine the sum:

$$x + \frac{x}{1+x} + \frac{x}{(1+x)^2} + \frac{x}{(1+x)^3} + \frac{x}{(1+x)^4} + \dots.$$

3.4.21 Determine whether the series

$$\sum_{k=1}^{\infty} \frac{1}{a+kb}$$

converges or diverges where a and b are positive real numbers.

3.4.22 We have proved that the harmonic series diverges. A computer experiment seems to show otherwise. Let s_n be the sequence of partial sums and, using a computer and the recursion formula

$$s_{n+1} = s_n + \frac{1}{n+1},$$

compute s_1, s_2, s_3, ... and stop when it appears that the sequence is no longer changing. This does happen! Explain why this is not a contradiction.

3.4.23 Let M be any integer. In Theorem 3.11 we saw that the series $\sum_{k=1}^{\infty} a_k$ converges if and only if the series $\sum_{k=1}^{\infty} a_{M+k}$ converges. What is the exact relation between the sums of the two series?

3.4.24 Write up a formal proof that the p-harmonic series

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$

converges for $p > 1$ using the method sketched in the text.

See Note 49

3.4.25 With a short argument using what you know about the harmonic series, show that the p-harmonic series for $0 < p \leq 1$ is divergent.

3.4.26 Obtain the divergence of the improper calculus integral

$$\int_0^{\infty} \frac{|\sin x|}{x}\, dx$$

by comparing with the harmonic series.

See Note 50

3.4.27 We have seen that the condition $a_n \to 0$ is a necessary, but not sufficient, condition for convergence of the series $\sum_{k=1}^{\infty} a_k$. Is the condition $na_n \to 0$ either necessary or sufficient for the convergence? This says terms are going to zero *faster* than $1/k$.

3.4.28 Let p be an integer greater or equal to 2 and let x be a real number in the interval $[0, 1)$. Construct a sequence of integers $\{k_n\}$ as follows: Divide the interval $[0, 1)$ into p intervals of equal length

$$[0, 1/p),\ [1/p, 2/p),\ \ldots,\ [(p-1)/p, 1)$$

and label them from left to right as 0, 1, ..., $p-1$. Then k_1 is chosen so that x belongs to the k_1th interval. Repeat the process applying it now to the interval $[(k_1-1)/p, k_1/p)$ in which x lies, dividing it into p intervals of equal length and choose k_2 so that x belongs to the k_2th interval of the new subintervals. Continue this process inductively to define the sequence $\{k_n\}$. Show that

$$x = \sum_{i=1}^{\infty} \frac{k_i}{p^i}.$$

[This is called the *p-adic representation* of the number x.]

See Note 51

3.5 Criteria for Convergence

How do we determine the convergence or divergence of a series? The meaning of convergence or divergence is directly given in terms of the sequence of partial sums. But usually it is very difficult to say much about that sequence. Certainly we hardly ever get a closed form for the partial sums.

For a successful theory of series we need some criteria that will enable us to assert the convergence or divergence of a series without much bothering with an intimate acquaintance with the sequence of partial sums. The following material begins the development of these criteria.

3.5.1 Boundedness Criterion

If a series $\sum_{k=1}^{\infty} a_k$ consists entirely of nonnegative terms, then it is clear that the sequence of partial sums forms a monotonic sequence. It is strictly increasing if all terms are positive.

We have a well-established fundamental principle for the investigation of all monotonic sequences:

A monotonic sequence is convergent if and only if it is bounded.

Applied to the study of series, this principle says that a series $\sum_{k=1}^{\infty} a_k$ consisting entirely of nonnegative terms will converge if the sequence of partial sums is bounded and will diverge if the sequence of partial sums is unbounded.

This reduces the study of the convergence/divergence behavior of such series to inequality problems:

Is there or is there not a number M so that

$$s_n = \sum_{k=1}^{n} a_k \leq M$$

for all integers n?

This is both good news and bad. Theoretically it means that convergence problems for this special class of series reduce to another problem: one of boundedness. That is good news, reducing an apparently difficult problem to one we already understand. The bad news is that inequality problems may still be difficult.

Note. A word of warning. The boundedness of the partial sums of a series is not of as great an interest for series where the terms can be both positive and negative. For such series the boundedness of the partial sums does not guarantee convergence.

3.5.2 Cauchy Criterion

One of our main theoretical tools in the study of convergent sequences is the Cauchy criterion describing (albeit somewhat technically) a necessary and sufficient condition for a sequence to be convergent.

If we translate that criterion to the language of series we shall then have a necessary and sufficient condition for a series to be convergent. Again it is rather technical and mostly useful in developing a theory rather than in testing specific series. The translation is nearly immediate.

Definition 3.13: The series
$$\sum_{k=1}^{\infty} a_k$$
is said to satisfy the *Cauchy criterion for convergence* provided that for every $\varepsilon > 0$ there is an integer N so that all of the finite sums
$$\left|\sum_{k=n}^{m} a_k\right| < \varepsilon$$
for any $N \le n < m < \infty$.

Now we have a principle that can be applied in many theoretical situations:

A series $\sum_{k=1}^{\infty} a_k$ converges if and only if it satisfies the Cauchy criterion for convergence.

Note. It may be useful to think of this conceptually. The criterion asserts that convergence is equivalent to the fact that blocks of terms
$$\sum_{k=N}^{M} a_k$$
added up and taken from far on in the series must be small. Loosely we might describe this by saying that a convergent series has a "small tail."

Note too that if the series converges, then this criterion implies that for every $\varepsilon > 0$ there is an integer N so that
$$\left|\sum_{k=n}^{\infty} a_k\right| < \varepsilon$$
for every $n \ge N$.

3.5.3 Absolute Convergence

If a series consists of nonnegative terms only, then we can obtain convergence or divergence by estimating the size of the partial sums. If the partial sums remain bounded, then the series converges; if not, the series diverges.

No such conclusion can be made for a series $\sum_{k=1}^{\infty} a_k$ of positive and negative numbers. Boundedness of the partial sums does not allow us to conclude anything about convergence or divergence since the sequence of partial sums would not be monotonic. What we can do is ask whether there is any relation between the two series

$$\sum_{k=1}^{\infty} a_k \quad \text{and} \quad \sum_{k=1}^{\infty} |a_k|$$

where the latter series has had the negative signs stripped from it. We shall see that convergence of the series of absolute values ensures convergence of the original series. Divergence of the series of absolute values gives, however, no information.

This gives us a useful test that will prove the convergence of a series $\sum_{k=1}^{\infty} a_k$ by investigating instead the related series $\sum_{k=1}^{\infty} |a_k|$ without the negative signs.

Theorem 3.14: *If the series $\sum_{k=1}^{\infty} |a_k|$ converges, then so too does the series $\sum_{k=1}^{\infty} a_k$.*

Proof. The proof takes two applications of the Cauchy criterion. If $\sum_{k=1}^{\infty} |a_k|$ converges, then for every $\varepsilon > 0$ there is an integer N so that all of the finite sums

$$\sum_{k=n}^{m} |a_k| < \varepsilon$$

for any $N \leq n < m < \infty$. But then

$$\left| \sum_{k=n}^{m} a_k \right| \leq \sum_{k=n}^{m} |a_k| < \varepsilon.$$

It follows, by the Cauchy criterion applied to the series $\sum_{k=1}^{\infty} a_k$, that this series is convergent. ■

Note. Note that there is no claim in the statement of this theorem that the two series have the same sum, just that the convergence of one implies the convergence of the other.

For theoretical reasons it is important to know when the series $\sum_{k=1}^{\infty} |a_k|$ of absolute values converges. Such series are "more" than convergent. They are convergent in a way that allows more manipulations than would otherwise be available. They can be thought of as more robust; a series that converges, but whose absolute series does not converge is in some ways fragile. This leads to the following definitions.

Definition 3.15: A series $\sum_{k=1}^{\infty} a_k$ is said to be *absolutely convergent* if the related series $\sum_{k=1}^{\infty} |a_k|$ converges.

Definition 3.16: A series $\sum_{k=1}^{\infty} a_k$ is said to be *nonabsolutely convergent* if the series $\sum_{k=1}^{\infty} a_k$ converges but the series $\sum_{k=1}^{\infty} |a_k|$ diverges.

Note that every *absolutely* convergent series is also convergent. We think of it as "more than convergent." Fortunately, the terminology preserves the meaning even though the "absolutely" refers to the absolute value, not to any other implied meaning. This play on words would not be available in all languages.

Example 3.17: Using this terminology, applied to series we have already studied, we can now assert the following:

> Any geometric series $1 + r + r^2 + r^3 + \dots$ is absolutely convergent if $|r| < 1$ and divergent if $|r| \geq 1$.

and

> The alternating harmonic series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$ is nonabsolutely convergent.

◀

Exercises

3.5.1 Suppose that $\sum_{k=1}^{\infty} a_k$ is a convergent series of positive terms. Show that $\sum_{k=1}^{\infty} a_k^2$ is convergent. Does the converse hold?

3.5.2 Suppose that $\sum_{k=1}^{\infty} a_k$ is a convergent series of positive terms. Show that $\sum_{k=1}^{\infty} \sqrt{a_k a_{k+1}}$ is convergent. Does the converse hold?

3.5.3 Suppose that both series

$$\sum_{k=1}^{\infty} a_k \text{ and } \sum_{k=1}^{\infty} b_k$$

are absolutely convergent. Show that then so too is the series $\sum_{k=1}^{\infty} a_k b_k$. Does the converse hold?

3.5.4 Suppose that both series

$$\sum_{k=1}^{\infty} a_k \text{ and } \sum_{k=1}^{\infty} b_k$$

are nonabsolutely convergent. Show that it does not follow that the series $\sum_{k=1}^{\infty} a_k b_k$ is convergent.

3.5.5 Alter the harmonic series $\sum_{k=1}^{\infty} 1/k$ by deleting all terms in which the denominator contains a specified digit (say 3). Show that the new series converges.

SEE NOTE 52

3.5.6 Show that the geometric series $\sum_{n=1}^{\infty} r^n$ is convergent for $|r| < 1$ by using directly the Cauchy convergence criterion.

3.5.7 Show that the harmonic series is divergent by using directly the Cauchy convergence criterion.

3.5.8 Obtain a proof that every series $\sum_{k=1}^{\infty} a_k$ for which $\sum_{k=1}^{\infty} |a_k|$ converges must itself be convergent without using the Cauchy criterion.

SEE NOTE 53

3.5.9 Show that a series $\sum_{k=1}^{\infty} a_k$ is absolutely convergent if and only if two at least of the series

$$\sum_{k=1}^{\infty} a_k \text{ , } \sum_{k=1}^{\infty} [a_k]^+ \text{ and } \sum_{k=1}^{\infty} [a_k]^-$$

converge. (If two converge, then all three converge.)

3.5.10 The sum rule for convergent series

$$\sum_{k=1}^{\infty}(a_k + b_k) = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k$$

can be expressed by saying that if any two of these series converges so too does the third. What kind of statements can you make for absolute convergence and for nonabsolute convergence?

3.5.11 Show that a series $\sum_{k=1}^{\infty} a_k$ is absolutely convergent if and only if every subseries $\sum_{k=1}^{\infty} a_{n_k}$ converges.

3.5.12 A sequence $\{x_n\}$ of real numbers is said to be of *bounded variation* if the series

$$\sum_{k=2}^{\infty} |x_k - x_{k-1}|$$

converges.

(a) Show that every sequence of bounded variation is convergent.

(b) Show that not every convergent sequence is of bounded variation.

(c) Show that all monotonic convergent sequences are of bounded variation.

(d) Show that any linear combination of two sequences of bounded variation is of bounded variation.

(e) Is the product of of two sequences of bounded variation also of bounded variation?

3.5.13 Establish the Cauchy-Schwarz inequality: For any finite sequences

$$\{a_1, a_2, \dots, a_n\} \text{ and } \{b_1, b_2, \dots, b_n\}$$

the inequality

$$\left|\sum_{k=1}^{n} a_k b_k\right| \leq \left(\sum_{k=1}^{n} (a_k)^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{n} (b_k)^2\right)^{\frac{1}{2}}$$

must hold.

3.5.14 Using the Cauchy-Schwarz inequality (Exercise 3.5.13), show that if $\{a_n\}$ is a sequence of nonnegative numbers for which $\sum_{n=1}^{\infty} a_n$ converges, then the series

$$\sum_{n=0}^{\infty} \frac{\sqrt{a_n}}{n^p}$$

also converges for any $p > \frac{1}{2}$. Without the Cauchy-Schwarz inequality what is the best you can prove for convergence?

3.5.15 Suppose that $\sum_{n=1}^{\infty} a_n^2$ converges. Show that

$$\limsup_{n\to\infty} \frac{a_1 + \sqrt{2}a_2 + \sqrt{3}a_3 + \sqrt{4}a_4 + \cdots + \sqrt{n}a_n}{n} < \infty.$$

See Note 54

3.5.16 Let x_1, x_2, x_3 be a sequence of positive numbers and write

$$s_n = \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}$$

and

$$t_n = \frac{\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \cdots + \frac{1}{x_n}}{n}.$$

If $s_n \to S$ and $t_n \to T$, show that $ST \geq 1$.

See Note 55

3.6 Tests for Convergence

In many investigations and applications of series it is important to recognize that a given series converges, converges absolutely, or diverges. Frequently the sum of the series is not of much interest, just the convergence behavior. Over the years a battery of tests have been developed to make this task easier.

There are only a few basic principles that we can use to check convergence or divergence and we have already discussed these in Section 3.5. One of the most basic is that a series of nonnegative terms is convergent if and only if the sequence of partial sums is bounded. Most of the tests in the sequel are just clever ways of checking that the partial sums are bounded without having to do the computations involved in finding that upper bound.

3.6.1 Trivial Test

The first test is just an observation that we have already made about series: If a series $\sum_{k=1}^{\infty} a_k$ converges, then $a_k \to 0$. We turn this into a divergence test. For example, some novices will worry for a long time over a series such as

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt[k]{k}}$$

applying a battery of tests to it to determine convergence. The simplest way to see that this series diverges is to note that the terms tend to 1 as $k \to \infty$. Perhaps this is the first thing that should be considered for any series. If the terms do not get small there is no point puzzling whether the series converges. It does not.

3.18 (Trivial Test) *If the terms of the series $\sum_{k=1}^{\infty} a_k$ do not converge to 0, then the series diverges.*

Proof. We have already proved this, but let us prove it now as a special case of the Cauchy criterion. For all $\varepsilon > 0$ there is an N so that

$$|a_n| = \left| \sum_{k=n}^{n} a_k \right| < \varepsilon$$

for all $n \geq N$ and so, by definition, $a_k \to 0$. ■

3.6.2 Direct Comparison Tests

A series $\sum_{k=1}^{\infty} a_k$ with all terms nonnegative can be handled by estimating the size of the partial sums. Rather than making a direct estimate it is sometimes easier to find a bigger series that converges. This larger series provides an upper bound for our series without the need to compute one ourselves.

Note. Make sure to apply these tests only for series with nonnegative terms since, for arbitrary series, this information is useless.

3.19 (Direct Comparison Test I) *Suppose that the terms of the series*

$$\sum_{k=1}^{\infty} a_k$$

are each smaller than the corresponding terms of the series

$$\sum_{k=1}^{\infty} b_k,$$

that is, that

$$0 \leq a_k \leq b_k$$

for all k. If the larger series converges, then so does the smaller series.

Proof. If $0 \leq a_k \leq b_k$ for all k, then

$$\sum_{k=1}^{n} a_k \leq \sum_{k=1}^{n} b_k \leq \sum_{k=1}^{\infty} b_k.$$

Thus the number $B = \sum_{k=1}^{\infty} b_k$ is an upper bound for the sequence of partial sums of the series $\sum_{k=1}^{\infty} a_k$. It follows that $\sum_{k=1}^{\infty} a_k$ must converge. ■

Note. In applying this and subsequent tests that demand that all terms of a series satisfy some requirement, we should remember that convergence and divergence of a series $\sum_{k=1}^{\infty} a_k$ depends only on the behavior of a_k for large values of k. Thus this test (and many others) could be reformulated so as to apply only for k greater than some integer N.

3.20 (Direct Comparison Test II) *Suppose that the terms of the series*

$$\sum_{k=1}^{\infty} a_k$$

are each larger than the corresponding terms of the series

$$\sum_{k=1}^{\infty} c_k,$$

that is, that

$$0 \leq c_k \leq a_k$$

for all k. If the smaller series diverges, then so does the larger series.

Proof. This follows from Test 3.19 since if the larger series did not diverge, then it must converge and so too must the smaller series. ■

Here are two examples illustrating how these tests may be used.

Example 3.21: Consider the series

$$\sum_{k=1}^{\infty} \frac{k+5}{k^3+k^2+k+1}.$$

While the partial sums might seem hard to estimate at first, a fast glance suggests that the terms (crudely) are similar to $1/k^2$ for large values of k and we know that the series $\sum_{k=1}^{\infty} 1/k^2$ converges. Note that

$$\frac{k+5}{k^3+k^2+k+1} = \frac{1+5/k}{k^2(1+1/k+1/k^2+1/k^3)} \leq \frac{C}{k^2}$$

for some choice of C (e.g., $C=6$ will work). We now claim that our given series converges by a direct comparison with the convergent series $\sum_{k=1}^{\infty} C/k^2$. (This is a p-harmonic series with $p=2$.) ◀

Example 3.22: Consider the series

$$\sum_{k=1}^{\infty}\sqrt{\frac{k+5}{k^2+k+1}}.$$

Again, a fast glance suggests that the terms (crudely) are similar to $1/\sqrt{k}$ for large values of k and we know that the series $\sum_{k=1}^{\infty} 1/\sqrt{k}$ diverges. Note that

$$\frac{k+5}{k^2+k+1} = \frac{1+5/k}{k(1+1/k+1/k^2)} \geq \frac{C}{k}$$

for some choice of C (e.g., $C = \frac{1}{4}$ will work). We now claim that our given series diverges by a direct comparison with the divergent series $\sum_{k=1}^{\infty}\sqrt{C}/\sqrt{k}$. (This is a p-harmonic series with $p = 1/2$.) ◀

The examples show both advantages and disadvantages to the method. We must invent the series that is to be compared and we must do some amount of inequality work to show that comparison. The next tests replace the inequality work with a limit operation, which is occasionally easier to perform.

3.6.3 Limit Comparison Tests

We have seen that a series $\sum_{k=1}^{\infty} a_k$ with all terms nonnegative can be handled by comparing with a larger convergent series or a smaller divergent series. Rather than check all the terms of the two series being compared, it is convenient sometimes to have this checked automatically by the computation of a limit. In this section, since the tests involve a fraction, we must be sure not only that all terms are nonnegative, but also that we have not divided by zero.

3.23 (Limit Comparison Test I) *Let each $a_k \geq 0$ and $b_k > 0$. If the terms of the series $\sum_{k=1}^{\infty} a_k$ can be compared to the terms of the series $\sum_{k=1}^{\infty} b_k$ by computing*

$$\lim_{k\to\infty}\frac{a_k}{b_k} < \infty$$

and if the latter series converges, then so does the former series.

Proof. The proof is easy. If the stated limit exists and is finite then there are numbers M and N so that

$$\frac{a_k}{b_k} < M$$

for all $k \geq N$. This shows that $a_k \leq Mb_k$ for all $k \geq N$. Consequently, applying the direct comparison test, we find that the series $\sum_{k=N}^{\infty} a_k$ converges by comparison with the convergent series $\sum_{k=N}^{\infty} Mb_k$. ∎

3.24 (Limit Comparison Test II) *Let each $a_k > 0$ and $c_k > 0$. If the terms of the series $\sum_{k=1}^{\infty} a_k$ can be compared to the terms of the series $\sum_{k=1}^{\infty} c_k$ by computing*

$$\lim_{k\to\infty} \frac{a_k}{c_k} > 0$$

and if the latter series diverges, then so does the original series.

Proof. Since the limit exists and is not zero there are numbers $\varepsilon > 0$ and N so that

$$\frac{a_k}{c_k} > \varepsilon$$

for all $k \geq N$. This shows that, for all $k \geq N$,

$$a_k \geq \varepsilon c_k.$$

Consequently, by the direct comparison test the series $\sum_{k=N}^{\infty} a_k$ diverges by comparison with the divergent series $\sum_{k=N}^{\infty} \varepsilon c_k$. ∎

We repeat our two examples, Example 3.21 and 3.22, where we previously used the direct comparison test to check for convergence.

Example 3.25: We look again at the series

$$\sum_{k=1}^{\infty} \frac{k+5}{k^3+k^2+k+1},$$

comparing it, as before, to the convergent series $\sum_{k=1}^{\infty} 1/k^2$. This now requires computing the limit

$$\lim_{k\to\infty} \frac{k^2(k+5)}{k^3+k^2+k+1},$$

which elementary calculus arguments show is 1. Since it is not infinite, the original series can now be claimed to converge by a limit comparison. ◀

Example 3.26: Again, consider the series

$$\sum_{k=1}^{\infty} \sqrt{\frac{k+5}{k^2+k+1}}$$

by comparing with the divergent series $\sum_{k=1}^{\infty} 1/\sqrt{k}$. We are required to compute the limit

$$\lim_{k\to\infty} \sqrt{k}\sqrt{\frac{k+5}{k^2+k+1}},$$

which elementary calculus arguments show is 1. Since it is not zero, the original series can now be claimed to diverge by a limit comparison. ◀

3.6.4 Ratio Comparison Test

Again we wish to compare two series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ composed of positive terms. Rather than directly comparing the size of the terms we compare the ratios of the terms. The inspiration for this test rests on attempts to compare directly a series with a convergent geometric series. If $\sum_{k=1}^{\infty} b_k$ is a geometric series with common ratio r, then evidently

$$\frac{b_{k+1}}{b_k} = r.$$

This suggests that perhaps a comparison of ratios of successive terms would indicate how fast a series might be converging.

3.27 (Ratio Comparison Test) *If the ratios satisfy*

$$\frac{a_{k+1}}{a_k} \leq \frac{b_{k+1}}{b_k}$$

for all k (or just for all k sufficiently large) and the series $\sum_{k=1}^{\infty} b_k$, with the larger ratio is convergent, then the series $\sum_{k=1}^{\infty} a_k$ is also convergent.

Proof. As usual, we assume all terms are positive in both series. If the ratios satisfy

$$\frac{a_{k+1}}{a_k} \leq \frac{b_{k+1}}{b_k}$$

for $k > N$, then they also satisfy

$$\frac{a_{k+1}}{b_{k+1}} \leq \frac{a_k}{b_k},$$

which means that the sequence $\{a_k/b_k\}$ is decreasing for $k > N$. In particular, that sequence is bounded above, say by C, and so

$$a_k \leq C b_k.$$

Thus an application of the direct comparison test shows that the series $\sum_{k=1}^{\infty} a_k$ converges. ■

3.6.5 d'Alembert's Ratio Test

The ratio comparison test requires selecting a series for comparison. Often a geometric series $\sum_{k=1}^{\infty} r^k$ for some $0 < r < 1$ may be used. How do we compute a number r that will work? We would wish to use $b_k = r^k$ with a choice of r so that

$$\frac{a_{k+1}}{a_k} \leq \frac{b_{k+1}}{b_k} = \frac{r^{k+1}}{r^k} = r.$$

One useful and easy way to find whether there will be such an r is to compute the limit of the ratios.

3.28 (Ratio Test) *If terms of the series $\sum_{k=1}^{\infty} a_k$ are all positive and the ratios satisfy*

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} < 1$$

then the series $\sum_{k=1}^{\infty} a_k$ is convergent.

Proof. The proof is easy. If

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} < 1,$$

then there is a number $\beta < 1$ so that

$$\frac{a_{k+1}}{a_k} < \beta$$

for all sufficiently large k. Thus the series $\sum_{k=1}^{\infty} a_k$ converges by the ratio comparison test applied to the convergent geometric series $\sum_{k=1}^{\infty} \beta^k$. ■

Note. The ratio test can also be pushed to give a divergence answer: If

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} > 1 \tag{4}$$

then the series $\sum_{k=1}^{\infty} a_k$ is divergent. But it is best to downplay this test or you might think it gives an answer as useful as the convergence test. From (4) it follows that there must be an N and β so that

$$\frac{a_{k+1}}{a_k} > \beta > 1$$

for all $k \geq N$. Then

$$a_{N+1} > \beta a_N,$$

$$a_{N+2} > \beta a_{N+1} > \beta^2 a_N,$$

and

$$a_{N+3} > \beta a_{N+2} > \beta^3 a_N.$$

We see that the terms a_k of the series are growing large at a geometric rate. Not only is the series diverging, but it is diverging in a dramatic way.

We can summarize how this test is best applied. If terms of the series $\sum_{k=1}^{\infty} a_k$ are all positive, compute

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = L.$$

1. If $L < 1$, then the series $\sum_{k=1}^{\infty} a_k$ is convergent.
2. If $L > 1$, then the series $\sum_{k=1}^{\infty} a_k$ is divergent; moreover, the terms $a_k \to \infty$.
3. If $L = 1$, then the series $\sum_{k=1}^{\infty} a_k$ may diverge or converge, the test being inconclusive.

Example 3.29: The series

$$\sum_{k=0}^{\infty} \frac{(k!)^2}{(2k)!}$$

is particularly suited for an application of the ratio test since the ratio is easily computed and a limit taken: If we write $a_k = (k!)^2/(2k)!$, then

$$\frac{a_{k+1}}{a_k} = \frac{((k+1)!)^2}{(2k+2)!}\frac{(2k)!}{(k!)^2} = \frac{(k+1)^2}{(2k+2)(2k+1)} \to \frac{1}{4}.$$

Consequently, this is a convergent series. More than that, it is converging faster than any geometric series

$$\sum_{k=0}^{\infty} \left(\frac{1}{4}+\varepsilon\right)^k$$

for any positive ε. (To make this expression "converging faster" more precise, see Exercise 3.12.5.) ◀

3.6.6 Cauchy's Root Test

There is yet another way to achieve a comparison with a convergent geometric series. We suspect that a series $\sum_{k=1}^{\infty} a_k$ can be compared to some geometric series $\sum_{k=1}^{\infty} r^k$ but do not know how to compute the value of r that might work. The limiting values of the ratios

$$\frac{a_{k+1}}{a_k}$$

provide one way of determining what r might work but often are difficult to compute. Instead we recognize that a comparison of the form

$$a_k \leq Cr^k$$

would mean that

$$\sqrt[k]{a_k} \leq \sqrt[k]{C} r.$$

For large k the term $\sqrt[k]{C}$ is close to 1, and this motivates our next test, usually attributed to Cauchy.

3.30 (Root Test) *If terms of the series $\sum_{k=1}^{\infty} a_k$ are all nonnegative and if the roots satisfy*

$$\lim_{k\to\infty} \sqrt[k]{a_k} < 1,$$

then that series converges.

Proof. This is almost trivial. If

$$(a_k)^{1/k} < \beta < 1$$

for all $k \geq N$, then

$$a_k < \beta^k$$

and so $\sum_{k=1}^{\infty} a_k$ converges by direct comparison with the convergent geometric series $\sum_{k=1}^{\infty} \beta^k$. ■

Again we can summarize how this test is best applied. The conclusions are nearly identical with those for the ratio test. Compute

$$\lim_{k\to\infty} (a_k)^{1/k} = L.$$

1. If $L < 1$, then the series $\sum_{k=1}^{\infty} a_k$ is convergent.
2. If $L > 1$, then the series $\sum_{k=1}^{\infty} a_k$ is divergent; moreover, the terms $a_k \to \infty$.
3. If $L = 1$, then the series $\sum_{k=1}^{\infty} a_k$ may diverge or converge, the test being inconclusive.

Example 3.31: In Example 3.29 we found the series

$$\sum_{k=0}^{\infty} \frac{(k!)^2}{(2k)!}$$

to be handled easily by the ratio test. It would be extremely unpleasant to attempt a direct computation using the root test. On the other hand, the series

$$\sum_{k=0}^{\infty} kx^k = x + 2x^2 + 3x^3 + 4x^4 + \dots$$

for $x > 0$ can be handled by either of these tests. You should try the ratio test while we try the root test:

$$\lim_{k\to\infty} \left(kx^k\right)^{1/k} = \lim_{k\to\infty} \sqrt[k]{k}x = x$$

and so convergence can be claimed for all $0 < x < 1$ and divergence for all $x > 1$. The case $x = 1$ is inconclusive for the root test, but the trivial test shows instantly that the series diverges for $x = 1$. ◀

3.6.7 Cauchy's Condensation Test

Occasionally a method that is used to study a specific series can be generalized into a useful test. Recall that in studying the sequence of partial sums of the harmonic series it was convenient to watch only at the steps 1, 2, 4, 8, ... and make a rough lower estimate. The reason this worked was simply that the terms in the harmonic series decrease and so estimates of s_1, s_2, s_4, s_8, ... were easy to obtain using just that fact. This turns quickly into a general test.

3.32 (Cauchy's Condensation Test) *If the terms of a series $\sum_{k=1}^{\infty} a_k$ are nonnegative and decrease monotonically to zero, then that series converges if and only if the related series*

$$\sum_{j=1}^{\infty} 2^j a_{2^j}$$

converges.

Proof. Since all terms are nonnegative, we need only compare the size of the partial sums of the two series. Computing first the sum of $2^{p+1}-1$ terms of the original series, we have

$$a_1 + (a_2 + a_3) + \cdots + (a_{2^p} + a_{2^p+1} + \cdots + a_{2^{p+1}-1})$$
$$\leq a_1 + 2a_2 + \cdots + 2^p a_{2^p}.$$

And, with the inequality sign in the opposite direction, we compute the sum of 2^p terms of the original series to obtain

$$a_1 + a_2 + (a_3 + a_4) + \cdots + (a_{2^{p-1}+1} + a_{2^{p-1}+2} + \cdots + a_{2^p})$$
$$\geq \frac{1}{2}\left(a_1 + 2a_2 + \cdots + 2^p a_{2^p}\right).$$

If either series has a bounded sequence of partial sums so too then does the other series. Thus both converge or else both diverge. ■

Example 3.33: Let us use this test to study the p-harmonic series:

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$

for $p > 0$. The terms decrease to zero and so the convergence of this series is equivalent to the convergence of the series

$$\sum_{j=1}^{\infty} 2^j \left(\frac{1}{2^j}\right)^p$$

and this series is a geometric series

$$\sum_{j=1}^{\infty} \left(2^{1-p}\right)^j .$$

This converges precisely when $2^{1-p} < 1$ or $p > 1$ and diverges when $2^{1-p} \geq 1$ or $p \leq 1$. Thus we know exactly the convergence behavior of the p-harmonic series for all values of p. (For $p \leq 0$ we have divergence just by the trivial test.) ◀

It is worth deriving a simple test from the Cauchy condensation test as a corollary. This is an improvement on the trivial test. The trivial test requires that $\lim_{k\to\infty} a_k = 0$ for a convergent series $\sum_{k=1}^{\infty} a_k$. This next test, which is due to Abel, shows that slightly more can be said if the terms form a monotonic sequence. The sequence $\{a_k\}$ must go to zero faster than $\{1/k\}$.

Corollary 3.34: *If the terms of a convergent series $\sum_{k=1}^{\infty} a_k$ decrease monotonically, then*

$$\lim_{k\to\infty} k a_k = 0.$$

Proof. By the Cauchy condensation test we know that

$$\lim_{j\to\infty} 2^j a_{2^j} = 0.$$

If $2^j \leq k \leq 2^{j+1}$, then $a_k \leq a_{2^j}$ and so

$$k a_k \leq 2\left(2^j a_{2^j}\right),$$

which is small for large j. Thus $k a_k \to 0$ as required. ■

3.6.8 Integral Test

ch.

To determine the convergence of a series $\sum_{k=1}^{\infty} a_k$ of nonnegative terms it is often necessary to make some kind of estimate on the size of the sequence of partial sums. Most of our tests have done this automatically, saving us the labor of computing such estimates. Sometimes those estimates can be obtained by calculus methods. The integral test allows us to estimate the partial sums $\sum_{k=1}^{n} f(k)$ by computing instead $\int_1^n f(x)\,dx$ in certain circumstances. This is more than a convenience; it also shows a close relation between series and infinite integrals, which is of much importance in analysis.

3.35 (Integral Test) *Let f be a nonnegative decreasing function on $[1,\infty)$ such that the integral $\int_1^X f(x)\,dx$ can be computed for all $X>1$. If*

$$\lim_{X\to\infty}\int_1^X f(x)\,dx < \infty$$

exists, then the series $\sum_{k=1}^{\infty} f(k)$ converges. If

$$\lim_{X\to\infty}\int_1^X f(x)\,dx = \infty,$$

then the series $\sum_{k=1}^{\infty} f(k)$ diverges.

Proof. Since the function f is decreasing we must have

$$\int_k^{k+1} f(x)\,dx \le f(k) \le \int_{k-1}^{k} f(x)\,dx.$$

Applying these inequalities for $k=2,3,4,\dots,n$ we obtain

$$\int_1^{n+1} f(x)\,dx \le \sum_{k=1}^{n} f(k) \le f(1) + \int_1^n f(x)\,dx. \tag{5}$$

The series converges if and only if the partial sums are bounded. But we see from the inequalities (5) that if the limit of the integral is finite, then these partial sums are bounded. If the limit of the integral is infinite, then these partial sums are unbounded. ■

Note. The convergence of the integral yields the convergence of the series. There is no claim that the sum of the series $\sum_{k=1}^{\infty} f(k)$ and the value of the infinite integral $\int_1^{\infty} f(x)\,dx$ are the same. In this regard, however, see Exercise 3.6.21.

Example 3.36: According to this test the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ can be studied by computing

$$\lim_{X\to\infty} \int_1^X \frac{dx}{x} = \lim_{X\to\infty} \log X = \infty.$$

For the same reasons the p-harmonic series

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$

for $p > 1$ can be studied by computing

$$\lim_{X\to\infty} \int_1^X \frac{dx}{x^p} = \lim_{X\to\infty} \frac{1}{p-1}\left(1 - \frac{1}{X^{p-1}}\right) = \frac{1}{p-1}.$$

In both cases we obtain the same conclusion as before. The harmonic series diverges and, for $p > 1$, the p-harmonic series converges. ◀

3.6.9 Kummer's Tests

The ratio test requires merely taking the limit of the ratios

$$\frac{a_{k+1}}{a_k}$$

but often fails. We know that if this tends to 1, then the ratio test says nothing about the convergence or divergence of the series $\sum_{k=1}^{\infty} a_k$.

Kummer's tests provide a collection of ratio tests that can be designed by taking different choices of sequence $\{D_k\}$. The choices $D_k = 1$, $D_k = k$ and $D_k = k \ln k$ are used in the following tests. Ernst Eduard

Kummer (1810–1893) is probably most famous for his contributions to the study of Fermat's last theorem; his tests arose in his study of hypergeometric series.

3.37 (Kummer's Tests) *The series* $\sum_{k=1}^{\infty} a_k$ *can be tested by the following criteria. Let* $\{D_k\}$ *denote any sequence of positive numbers and compute*

$$L = \liminf_{k\to\infty}\left[D_k \frac{a_k}{a_{k+1}} - D_{k+1}\right].$$

If $L > 0$ *the series* $\sum_{k=1}^{\infty} a_k$ *converges. On the other hand, if*

$$\left[D_k \frac{a_k}{a_{k+1}} - D_{k+1}\right] \leq 0$$

for all sufficiently large k *and if the series*

$$\sum_{k=1}^{\infty} \frac{1}{D_k}$$

diverges, then the series $\sum_{k=1}^{\infty} a_k$ *diverges.*

Proof. If $L > 0$, then we can choose a positive number $\alpha < L$. By the definition of a liminf this means there must exist an integer N so that for all $k \geq N$,

$$\alpha < \left[D_k \frac{a_k}{a_{k+1}} - D_{k+1}\right].$$

Rewriting this, we find that

$$\alpha a_{k+1} < D_k a_k - D_{k+1} a_{k+1}.$$

We can write this inequality for $k = N,\ N+1,\ N+2,\ \ldots,\ N+p$ to obtain

$$\alpha a_{N+1} < D_N a_N - D_{N+1} a_{N+1}$$

$$\alpha a_{N+2} < D_{N+1} a_{N+1} - D_{N+2} a_{N+2}$$

and so on. Adding these (note the telescoping sums), we find that

$$\alpha(a_{N+1} + a_{N+2} + \cdots + a_{N+p+1})$$
$$< D_{N+1}a_{N+1} - D_{N+p+1}a_{N+p+1} < D_{N+1}a_{N+1}.$$

(The final inequality just uses the fact that all the terms here are positive.)

From this inequality we can determine that the partial sums of the series $\sum_{k=1}^{\infty} a_k$ are bounded. By our usual criterion, this proves that this series converges.

The second part of the theorem requires us to establish divergence. Suppose now that

$$D_k \frac{a_k}{a_{k+1}} - D_{k+1} \leq 0$$

for all $k \geq N$. Then

$$D_k a_k \leq D_{k+1} a_{k+1}.$$

Thus the sequence $\{D_k a_k\}$ is increasing after $k = N$. In particular,

$$D_k a_k \geq C$$

for some C and all $k \geq N$ and so

$$a_k \geq \frac{C}{D_k}.$$

It follows by a direct comparison with the divergent series $\sum C/D_k$ that our series also diverges. ■

Note. In practice, for the divergence part of the test, it may be easier to compute

$$L = \limsup_{k\to\infty} \left[D_k \frac{a_k}{a_{k+1}} - D_{k+1}\right].$$

If $L < 0$, then we would know that

$$\left[D_k \frac{a_k}{a_{k+1}} - D_{k+1}\right] \leq 0$$

for all sufficiently large k and so, if the series $\sum_{k=1}^{\infty} \frac{1}{D_k}$ diverges, then the series $\sum_{k=1}^{\infty} a_k$ diverges.

Example 3.38: What is Kummer's test if the sequence used is the simplest possible $D_k = 1$ for all k? In this case it is simply the ratio test. For example, suppose that

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = r.$$

Then, replacing $D_k = 1$, we have

$$\lim_{k\to\infty} \left[D_k \frac{a_k}{a_{k+1}} - D_{k+1} \right] = \lim_{k\to\infty} \left[\frac{a_k}{a_{k+1}} - 1 \right] = \frac{1}{r} - 1.$$

Thus, by Kummer's test, if $1/r - 1 < 0$ we have divergence while if $1/r - 1 > 0$ we have convergence. These are just the cases $r > 1$ and $r < 1$ of the ratio test. ◀

3.6.10 Raabe's Ratio Test

✂ *Adv.*

A simple variant on the ratio test is known as Raabe's test. Suppose that

$$\lim_{k\to\infty} \frac{a_k}{a_{k+1}} = 1$$

so that the ratio test is inconclusive. Then instead compute

$$\lim_{k\to\infty} k \left(\frac{a_k}{a_{k+1}} - 1 \right).$$

The series $\sum_{k=1}^{\infty} a_k$ converges or diverges depending on whether this limit is greater than or less than 1.

3.39 (Raabe's Test) *The series $\sum_{k=1}^{\infty} a_k$ can be tested by the following criterion. Compute*

$$L = \lim_{k\to\infty} k\left(\frac{a_k}{a_{k+1}} - 1\right).$$

Then

1. *If $L > 1$, the series $\sum_{k=1}^{\infty} a_k$ converges.*
2. *If $L < 1$, the series $\sum_{k=1}^{\infty} a_k$ diverges.*
3. *If $L = 1$, the test is inconclusive.*

Proof. This is Kummer's test using the sequence $D_k = k$. ■

Example 3.40: Consider the series

$$\sum_{k=0}^{\infty} \frac{k^k}{e^k k!}.$$

An attempt to apply the ratio test to this series will fail since the ratio will tend to 1, the inconclusive case. But if instead we consider the limit

$$\lim_{k\to\infty} k\left(\left(\frac{k^k}{e^k k!}\right)\left(\frac{e^{k+1}(k+1)!}{(k+1)^{k+1}}\right) - 1\right)$$

as called for in Raabe's test, we can use calculus methods (L'Hôpital's rule) to obtain a limit of $\frac{1}{2}$. Consequently, this series diverges. ◀

3.6.11 Gauss's Ratio Test

Raabe's test can be replaced by a closely related test due to Gauss. We might have discovered while using Raabe's test that

$$\lim_{k\to\infty} k\left(\frac{a_k}{a_{k+1}} - 1\right) = L.$$

This suggests that in any actual computation we will have discovered, perhaps by division, that

$$\frac{a_k}{a_{k+1}} = 1 + \frac{L}{k} + \text{ terms involving } \frac{1}{k^2} \text{ etc.}$$

The case $L > 1$ corresponds to convergence and the case $L < 1$ to divergence, both by Raabe's test. What if $L = 1$, which is considered inconclusive in Raabe's test?

Gauss's test offers a different way to look at Raabe's test and also has an added advantage that it handles this case that was left as inconclusive in Raabe's test.

3.41 (Gauss's Test) *The series $\sum_{k=1}^{\infty} a_k$ can be tested by the following criterion. Suppose that*

$$\frac{a_k}{a_{k+1}} = 1 + \frac{L}{k} + \frac{\phi(k)}{k^2}$$

where $\phi(k)$ $(k = 1, 2, 3, \dots)$ forms a bounded sequence. Then

1. *If $L > 1$ the series $\sum_{k=1}^{\infty} a_k$ converges.*
2. *If $L \leq 1$ the series $\sum_{k=1}^{\infty} a_k$ diverges.*

Proof. As we noted, for $L > 1$ and $L < 1$ this is precisely Raabe's test. Only the case $L = 1$ is new! Let us assume that

$$\frac{a_k}{a_{k+1}} = 1 + \frac{1}{k} + \frac{x_k}{k^2}$$

where $\{x_k\}$ is a bounded sequence.

To prove this case (that the series diverges) we shall use Kummer's test with the sequence $D_k = k \log k$. We consider the expression

$$\left[D_k \frac{a_k}{a_{k+1}} - D_{k+1} \right],$$

which now assumes the form

$$k \log k \frac{a_k}{a_{k+1}} - (k+1) \log(k+1)$$

$$= k\log k\left(1+\frac{1}{k}+\frac{x_k}{k^2}\right)-(k+1)\log(k+1).$$

We need to compute the limit of this expression as $k\to\infty$. It takes only a few manipulations (which you should try) to see that the limit is -1. For this use the facts that

$$(\log k)/k \to 0$$

and

$$(k+1)\log(1+1/k)\to 1$$

as $k\to\infty$.

We are now in a position to claim, by Kummer's test, that our series $\sum_{k=1}^{\infty} a_k$ diverges. To apply this part of the test requires us to check that the series

$$\sum_{k=2}^{\infty}\frac{1}{k\log k}$$

diverges. Several tests would work for this. Perhaps Cauchy's condensation test is the easiest to apply, although the integral test can be used too [see Exercise 3.6.2(c)]. ■

Note. In Gauss's test you may be puzzling over how to obtain the expression

$$\frac{a_k}{a_{k+1}} = 1+\frac{L}{k}+\frac{\phi(k)}{k^2}.$$

In practice often the fraction a_k/a_{k+1} is a ratio of polynomials and so usual algebraic procedures will supply this. In theory, though, there is no problem. For any L we could simply write

$$\phi(k)=k^2\left(\frac{a_k}{a_{k+1}}-1+\frac{L}{k}\right).$$

Thus the real trick is whether it can be done in such a way that the $\phi(k)$ do not grow too large.

Also, in some computations you might prefer to leave the ratio as a_{k+1}/a_k the way it was for the ratio test. In that case Gauss's test would assume the form

$$\frac{a_{k+1}}{a_k} = 1 - \frac{L}{k} + \frac{\phi(k)}{k^2}.$$

(Note the minus sign.) The conclusions are exactly the same.

Example 3.42: The series

$$1 + mx + \frac{m(m-1)}{2!}x^2 +$$
$$\frac{m(m-1)(m-2)}{3!}x^3 + \frac{m(m-1)\dots(m-k+1)}{k!}x^k + \dots$$

is called the *binomial series.* When m is a positive integer all terms for $k > m$ are zero and the series reduces to the binomial formula for $(1+x)^m$. Here now m is any real number and the hope remains that the formula might still be valid, but using a series rather than a finite sum. This series plays an important role in many applications. Let us check for absolute convergence at $x = 1$. We can assume that $m \neq 0$ since that case is trivial.

If we call the absolute value of the $k+1$–st term a_k so

$$a_{k+1} = \left| \frac{m(m-1)\dots(m-k+1)}{k!} \right|,$$

then a simple calculation shows that for large values of k

$$\frac{a_{k+1}}{a_k} = 1 - \frac{m+1}{k}.$$

Here we are using the version a_{k+1}/a_k rather than the reciprocal; see the preceding note.

There are no higher-order terms to worry about in Gauss's test here and so the series $\sum a_k$ converges if $m+1 > 1$ and diverges if $m+1 < 1$. Thus the binomial series converges absolutely for $x = 1$ if $m > 0$. For $m = 0$ the series certainly converges since all terms except for the first one are identically zero. For $m < 0$

we know so far only that it does not converge absolutely. A closer analysis, for those who might care to try, will show that the series is nonabsolutely convergent for $-1 < m < 0$ and divergent for $m \leq -1$. ◄

3.6.12 Alternating Series Test

We pass now to a number of tests that are needed for studying series of terms that may change signs. The simplest first step in studying a series $\sum_{i=1}^{\infty} a_i$, where the a_i are both negative and positive, is to apply one from our battery of tests to the series $\sum_{i=1}^{\infty} |a_i|$. If any test shows that this converges, then we know that our original series converges absolutely. This is even better than knowing it converges.

But what shall we do if the series is not absolutely convergent or if such attempts fail? One method applies to special series of positive and negative terms. Recall how we handled the series

$$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

(called the alternating harmonic series). We considered separately the partial sums s_2, s_4, s_6, ... and s_1, s_3, s_5, ... The special pattern of + and − signs alternating one after the other allowed us to see that each subsequence $\{s_{2n}\}$ and $\{s_{2n-1}\}$ was monotonic. All the features of this argument can be put into a test that applies to a wide class of series, similar to the alternating harmonic series.

3.43 (Alternating Series Test) *The series*

$$\sum_{k=1}^{\infty} (-1)^{k-1} a_k,$$

whose terms alternate in sign, converges if the sequence $\{a_k\}$ decreases monotonically to zero. Moreover, the value of the sum of such a series lies between the values of the partial sums at any two consecutive stages.

Proof. The proof is just exactly the same as for the alternating harmonic series. Since the a_k are nonnegative and decrease, we compute that

$$a_1 - a_2 = s_2 \leq s_4 \leq s_6 \leq \cdots \leq s_5 \leq s_3 \leq s_1 = a_1.$$

These subsequences then form bounded monotonic sequences and so

$$\lim_{n\to\infty} s_{2n} \quad\text{and}\quad \lim_{n\to\infty} s_{2n-1}$$

exist. Finally, since

$$s_{2n} - s_{2n-1} = -a_{2n} \to 0$$

we can conclude that $\lim_{n\to\infty} s_n = L$ exists. From the proof it is clear that the value L lies in each of the intervals $[s_2, s_1]$, $[s_2, s_3]$, $[s_4, s_3]$, $[s_4, s_5]$, ... and so, as stated, the sum of the series lies between the values of the partial sums at any two consecutive stages. ■

3.6.13 Dirichlet's Test

✂ *Adv.*

Our next test derives from the summation by parts formula

$$\sum_{k=1}^{n} a_k b_k = s_1(b_1 - b_2) + s_2(b_2 - b_3) + \cdots + s_{n-1}(b_{n-1} - b_n) + s_n b_n$$

that we discussed in Section 3.2. We can see that if there is some special information available about the sequences $\{s_n\}$ and $\{b_n\}$ here, then the convergence of the series $\sum_{k=1}^{n} a_k b_k$ can be proved. The test gives one possibility for this. The next section gives a different variant.

The test is named after Lejeune Dirichlet[1] (1805–1859) who is most famous for his work on Fourier series, in which this test plays an important role.

[1] One of his contemporaries described him thus: "He is a rather tall, lanky-looking man, with moustache and beard about to turn grey with a somewhat harsh voice and rather deaf. He was unwashed, with his cup of coffee and cigar. One of his failings is forgetting time, he pulls his watch out, finds it past three, and runs out without even finishing the sentence." (From *http://www-history.mcs.st-and.ac.uk/history*.

3.44 (Dirichlet Test) *If $\{b_n\}$ is a sequence decreasing to zero and the partial sums of the series $\sum_{k=1}^{\infty} a_k$ are bounded, then the series $\sum_{k=1}^{\infty} a_k b_k$ converges.*

Proof. Write $s_n = \sum_{k=1}^{n} a_k$. By our assumptions on the series $\sum_{k=1}^{\infty} a_k$ there is a positive number M so that $|s_n| \leq M$ for all n. Let $\varepsilon > 0$ and choose N so large that $b_n < \varepsilon/(2M)$ if $n \geq N$.

The summation by parts formula shows that for $m > n \geq N$

$$\left|\sum_{k=n}^{m} a_k b_k\right| = |a_n b_n + a_{n+1} b_{n+1} + \cdots + a_m b_m|$$

$$= |-s_{n-1} b_n + s_n(b_n - b_{n+1}) + \cdots + s_{m-1}(b_{m-1} - b_m) + s_m b_m|$$

$$\leq |-s_{n-1} b_n| + |s_n(b_n - b_{n+1})| + \cdots + |s_{m-1}(b_{m-1} - b_m)| + |s_m b_m|$$

$$\leq M(b_n + [b_n - b_m] + b_m) \leq 2M b_n < \varepsilon.$$

Notice that we have needed to use the fact that

$$b_{k-1} - b_k \geq 0$$

for each k. This is precisely the Cauchy criterion for the series $\sum_{k=1}^{\infty} a_k b_k$ and so we have proved convergence. ■

Example 3.45: The series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots$$

converges by the alternating series test. What other pattern of $+$ and $-$ signs could we insert and still have convergence? Let $a_k = \pm 1$. If the partial sums

$$\sum_{k=1}^{n} a_k$$

remain bounded, then, by Dirichlet's test, the series

$$\sum_{k=1}^{n} \frac{a_k}{k}$$

must converge. Thus, for example, the pattern

$$+ - + + - - + - + + - - + - + + - - \dots$$

would produce a convergent series (that is not alternating). ◀

3.6.14 Abel's Test

Adv.

The next test is another variant on the same theme as the Dirichlet test. There the series $\sum_{k=1}^{\infty} a_k b_k$ was proved to be convergent by assuming a fairly weak fact for the series $\sum_{k=1}^{\infty} a_k$ (i.e., bounded partial sums) and a strong fact for $\{b_k\}$ (i.e., monotone convergence to 0). Here we strengthen the first and weaken the second.

3.46 (Abel Test) *If $\{b_n\}$ is a convergent monotone sequence and the series $\sum_{k=1}^{\infty} a_k$ is convergent, then the series $\sum_{k=1}^{\infty} a_k b_k$ converges.*

Proof. Suppose first that b_k is decreasing to a limit B. Then $b_k - B$ decreases to zero. We can apply Dirichlet's test to the series

$$\sum_{k=1}^{\infty} a_k(b_k - B)$$

to obtain convergence, since if $\sum_{k=1}^{\infty} a_k$ is convergent, then it has a bounded sequence of partial sums.

But this allows us to express our series as the sum of two convergent series:

$$\sum_{k=1}^{\infty} a_k b_k = \sum_{k=1}^{\infty} a_k(b_k - B) + B\sum_{k=1}^{\infty} a_k.$$

If the sequence b_k is instead increasing to some limit then we can apply the first case proved to the series $-\sum_{k=1}^{\infty} a_k(-b_k)$. ■

Exercises

3.6.1 Let $\{a_n\}$ be a sequence of positive numbers. If $\lim_{n\to\infty} n^2 a_n = 0$, what (if anything) can be said about the series $\sum_{n=1}^{\infty} a_n$. If $\lim_{n\to\infty} n a_n = 0$, what (if anything) can be said about the series $\sum_{n=1}^{\infty} a_n$. (If we drop the assumption about the sequence $\{a_n\}$ being positive does anything change?)

3.6.2 Which of these series converge?

(a) $$\sum_{n=1}^{\infty} \frac{n(n+1)}{(n+2)^2}$$

(b) $$\sum_{n=1}^{\infty} \frac{3n(n+1)(n+2)}{n^3\sqrt{n}}$$

(c) $$\sum_{n=2}^{\infty} \frac{1}{n^s \log n}$$

(d) $$\sum_{n=1}^{\infty} a^{1/n} - 1$$

(e) $$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^t}$$

(f) $$\sum_{n=2}^{\infty} \frac{1}{n^s(\log n)^t}$$

(g) $$\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right)^{n^2}$$

3.6.3 For what values of x do the following series converge?

(a) $\sum_{n=2}^{\infty} \frac{x^n}{\log n}$

(b) $\sum_{n=2}^{\infty} (\log n) x^n$

(c) $\sum_{n=1}^{\infty} e^{-nx}$

(d) $1 + 2x + \frac{3^2 x^2}{2!} + \frac{4^3 x^3}{3!} + \dots.$

SEE NOTE 56

3.6.4 Let a_k be a sequence of positive numbers and suppose that

$$\lim_{k\to\infty} k a_k = L.$$

What can you say about the convergence of the series $\sum_{k=1}^{\infty} a_k$ if $L = 0$? What can you say if $L > 0$?

3.6.5 ✂ Let $\{a_k\}$ be a sequence of nonnegative numbers. Consider the following conditions:

(a) $\limsup_{k\to\infty} \sqrt{k} a_k > 0$

(b) $\limsup_{k\to\infty} \sqrt{k} a_k < \infty$

(c) $\liminf_{k\to\infty} \sqrt{k} a_k > 0$

(d) $\liminf_{k\to\infty} \sqrt{k} a_k < \infty$

Which condition(s) imply convergence or divergence of the series $\sum_{k=1}^{\infty} a_k$? Supply proofs. Which conditions are inconclusive as to convergence or divergence? Supply examples.

SEE NOTE 57

3.6.6 Suppose that $\sum_{n=1}^{\infty} a_n$ is a convergent series of positive terms. Must the series $\sum_{n=1}^{\infty} \sqrt{a_n}$ also be convergent?

3.6.7 Give examples of series both convergent and divergent that illustrate that the ratio test is inconclusive when the limit of the ratios L is equal to 1.

3.6.8 Give examples of series both convergent and divergent that illustrate that the root test is inconclusive when the limit of the roots L is equal to 1.

3.6.9 ✂ Apply both the root test and the ratio test to the series

$$\alpha + \alpha\beta + \alpha^2\beta + \alpha^2\beta^2 + \alpha^3\beta^2 + \alpha^3\beta^3 + \dots$$

where α, β are positive real numbers.

3.6.10 ✂ Show that the limit comparison test applied to series with positive terms can be replaced by the following version. If

$$\limsup_{k\to\infty} \frac{a_k}{b_k} < \infty$$

and if $\sum_{k=1}^{\infty} b_k$ converges, then so does $\sum_{k=1}^{\infty} a_k$. If

$$\liminf_{k\to\infty} \frac{a_k}{c_k} > 0$$

and if $\sum_{k=1}^{\infty} c_k$ diverges, then so does $\sum_{k=1}^{\infty} a_k$.

3.6.11 ✂ Show that the ratio test can be replaced by the following version. Compute

$$\liminf_{k\to\infty} \frac{a_{k+1}}{a_k} = L \text{ and} \limsup_{k\to\infty} \frac{a_{k+1}}{a_k} = M.$$

(a) If $M < 1$, then the series $\sum_{k=1}^{\infty} a_k$ is convergent.

(b) If $L > 1$, then the series $\sum_{k=1}^{\infty} a_k$ is divergent; moreover, the terms $a_k \to \infty$.

(c) If $L \le 1 \le M$, then the series $\sum_{k=1}^{\infty} a_k$ may diverge or converge, the test being inconclusive.

3.6.12 ✂ Show that the root test can be replaced by the following version. Compute

$$\limsup_{k\to\infty} \sqrt[k]{a_k} = L.$$

(a) If $L < 1$, then the series $\sum_{k=1}^{\infty} a_k$ is convergent.

(b) If $L > 1$, then the series $\sum_{k=1}^{\infty} a_k$ is divergent; moreover, some subsequence of the terms $a_{k_j} \to \infty$.

(c) If $L = 1$, then the series $\sum_{k=1}^{\infty} a_k$ may diverge or converge, the test being inconclusive.

3.6.13 ✂ Show that for any sequence of positive numbers $\{a_k\}$

$$\liminf_{k\to\infty} \frac{a_{k+1}}{a_k} \le \liminf_{k\to\infty} \sqrt[k]{a_k} \le \limsup_{k\to\infty} \sqrt[k]{a_k} \le \limsup_{k\to\infty} \frac{a_{k+1}}{a_k}.$$

What can you conclude about the relative effectiveness of the root and ratio tests?

3.6.14 ✂ Give examples of series for which one would clearly prefer to apply the root (ratio) test in preference to the ratio (root) test. How would you answer someone who claims that "Exercise 3.6.13 shows clearly that the ratio test is inferior and should be abandoned in favor of the root test?"

3.6.15 ✂ Let $\{a_n\}$ be a sequence of positive numbers and write

$$L_n = \frac{\log\left(\frac{1}{a_n}\right)}{\log n}.$$

Show that if $\liminf L_n > 1$, then $\sum a_n$ converges. Show that if $L_n \leq 1$ for all sufficiently large n, then $\sum a_n$ diverges.

3.6.16 Apply the test in Exercise 3.6.15 to obtain convergence or divergence of the following series (x is positive):

(a) $\sum_{n=2}^{\infty} x^{\log n}$

(b) $\sum_{n=2}^{\infty} x^{\log \log n}$

(c) $\sum_{n=2}^{\infty} (\log n)^{-\log n}$

3.6.17 Prove the alternating series test directly from the Cauchy criterion.

3.6.18 Determine for what values of p the series

$$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k^p} = 1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \dots$$

is absolutely convergent and for what values it is nonabsolutely convergent.

3.6.19 How many terms of the series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2}$$

must be taken to obtain a value differing from the sum of the series by less than 10^{-10}?

3.6.20 If the sequence $\{x_n\}$ is monotonically decreasing to zero then prove that the series

$$x_1 - \frac{1}{2}(x_1 + x_2) + \frac{1}{3}(x_1 + x_2 + x_3) - \frac{1}{4}(x_1 + x_2 + x_3 + x_4) + \dots$$

converges.

3.6.21 This exercise attempts to squeeze a little more information out of the integral test. In the notation of that test consider the sequence

$$e_n = \sum_{k=1}^{n} f(k) - \int_1^{n+1} f(x)\,dx$$

Show that the sequence $\{e_n\}$ is increasing and that $0 \leq e_n \leq f(1)$. What is the exact relation between $\sum_{k=1}^{\infty} f(k)$ and $\int_1^{\infty} f(x)\,dx$?

3.6.22 Show that

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \int_1^{n+1}\frac{1}{x}\,dx\right) = \gamma$$

for some number γ, $.5 < \gamma < 1$.

See Note 58

3.6.23 Show that

$$\lim_{n\to\infty}\sum_{k=n+1}^{2n}\frac{1}{k} = \log 2.$$

3.6.24 Let F be a positive function on $[1,\infty)$ with a positive, decreasing and continuous derivative F'.

(a) Show that $\sum_{k=1}^{\infty} F'(k)$ converges if and only if

$$\sum_{k=1}^{\infty}\frac{F'(k)}{F(k)}$$

converges.

(b) Suppose that $\sum_{k=1}^{\infty} F'(k)$ diverges. Show that

$$\sum_{k=1}^{\infty}\frac{F'(k)}{[F(k)]^p}$$

converges if and only if $p > 1$.

See Note 59

3.6.25 This collection of exercises develops some convergence properties of *power series*; that is, series of the form

$$\sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots .$$

A full treatment of power series appears in Chapter 16.

(a) Show that if a power series converges absolutely for some value $x = x_0$ then the series converges absolutely for all $|x| \leq |x_0|$.

(b) Show that if a power series converges for some value $x = x_0$ then the series converges absolutely for all $|x| < |x_0|$.

(c) Let

$$R = \sup \left\{ t : \sum_{k=0}^{\infty} a_k t^k \text{ converges } \right\}.$$

Show that the power series $\sum_{k=0}^{\infty} a_k x^k$ must converge absolutely for all $|x| < R$ and diverge for all $|x| > R$. [The number R is called the *radius of convergence* of the series. The explanation for the word "radius" (which conjures up images of circles) is that for complex series the set of convergence is a disk.]

(d) Give examples of power series with radius of convergence 0, ∞, 1, 2, and $\sqrt{2}$.

(e) Explain how the radius of convergence of a power series may be computed with the help of the ratio test.

(f) Explain how the radius of convergence of a power series may be computed with the help of the root test.

(g) ✂ ✂ Establish the formula

$$R = \frac{1}{\limsup_{k\to\infty} \sqrt[k]{|a_k|}}$$

for the radius of convergence of the power series $\sum_{k=0}^{\infty} a_k x^k$.

(h) Give examples of power series $\sum_{k=0}^{\infty} a_k x^k$ with radius of convergence R so that the series converges absolutely at both endpoints of the interval $[-R, R]$. Give another example so that the series converges at the right-hand endpoint but diverges at the left-hand endpoint of $[-R, R]$. What other possibilities are there?

3.6.26 The series

$$1 + mx + \frac{m(m-1)}{2!}x^2 +$$
$$\frac{m(m-1)(m-2)}{3!}x^3 + \frac{m(m-1)\dots(m-k+1)}{k!}x^k + \dots$$

is called the *binomial series*. Here m is any real number. (See Example 3.42.)

(a) Show that if m is a positive integer then this is precisely the expansion of $(1+x)^m$ by the binomial theorem.

(b) Show that this series converges absolutely for any m and for all $|x| < 1$.

(c) Obtain convergence for $x = 1$ if $m > -1$.

(d) Obtain convergence for $x = -1$ if $m > 0$.

3.7 Rearrangements

Any finite sum may be rearranged and summed in any order. This is because addition is commutative. We might expect the same to occur for series. We add up a series $\sum_{k=1}^{\infty} a_k$ by starting at the first term and adding in the order presented to us. If the terms are rearranged into a different order do we get the same result?

Example 3.47: The most famous example of a series that cannot be freely rearranged without changing the sum is the alternating harmonic series. We know that the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

is convergent (actually nonabsolutely convergent) with a sum somewhere between 1/2 and 1. If we rearrange this so that every positive term is followed by two negative terms, thus,

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \dots$$

we shall arrive at a different sum. Grouping these and adding, we obtain

$$\left(1-\frac{1}{2}\right)-\frac{1}{4}+\left(\frac{1}{3}-\frac{1}{6}\right)-\frac{1}{8}+\left(\frac{1}{5}-\frac{1}{10}\right)-\frac{1}{12}+\dots$$
$$=\frac{1}{2}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots\right)$$

whose sum is half the original series. Rearranging the series has changed the sum! ◀

For the theory of unordered sums there is no such problem. If an unordered sum $\sum_{j\in J} a_j$ converges to a number c, then so too does any rearrangement. Exercise 3.3.8 shows that if $\sigma : I \to I$ is one-to-one and onto, then

$$\sum_{i\in I} a_j = \sum_{i\in I} a_{\sigma(i)}.$$

We had hoped for the same situation for series. If $\sigma : \mathbb{N} \to \mathbb{N}$ is one-to-one and onto, then

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} a_{\sigma(k)}$$

may or may not hold. We call $\sum_{k=1}^{\infty} a_{\sigma(k)}$ a *rearrangement* of the series $\sum_{k=1}^{\infty} a_k$.

We propose now to characterize those series that allow unlimited rearrangements, and those that are more fragile (as is the alternating harmonic series) and cannot permit rearrangement.

3.7.1 Unconditional Convergence

A series is said to be *unconditionally convergent* if all rearrangements of that series converge and have the same sum. Those series that do not allow this but do converge are called *conditionally convergent*. Here the "conditional" means that the series converges in the arrangement given, but may diverge in another arrangement or may converge to a different sum in another arrangement. We shall see that conditionally convergent series are extremely fragile; there are rearrangements that exhibit any behavior desired. There are rearrangements that diverge and there are rearrangements that converge to any desired number.

Our first theorem asserts that any absolutely convergent series may be freely rearranged. All absolutely convergent series are unconditionally convergent. In fact, the two terms are equivalent

$$\textit{unconditionally convergent} \Leftrightarrow \textit{absolutely convergent}$$

although we must wait until the next section to prove that.

Theorem 3.48 (Dirichlet) *Every absolutely convergent series is unconditionally convergent.*

Proof. Let us prove this first for series $\sum_{k=1}^{\infty} a_k$ whose terms are all nonnegative. For such series convergence and absolute convergence mean the same thing.

Let $\sum_{k=1}^{\infty} a_{\sigma(k)}$ be any rearrangement. Then for any M

$$\sum_{k=1}^{M} a_{\sigma(k)} \leq \sum_{k=1}^{N} a_k \leq \sum_{k=1}^{\infty} a_k$$

by choosing an N large enough so that $\{1, 2, 3, \dots, N\}$ includes all the integers

$$\{\sigma(1), \sigma(2), \sigma(3), \dots, \sigma(M)\}.$$

By the bounded partial sums criterion this shows that $\sum_{k=1}^{\infty} a_{\sigma(k)}$ is convergent and to a sum smaller than $\sum_{k=1}^{\infty} a_k$. But this same argument would show that $\sum_{k=1}^{\infty} a_k$ is convergent and to a sum smaller than $\sum_{k=1}^{\infty} a_{\sigma(k)}$ and consequently all rearrangements converge to the same sum.

We now allow the series $\sum_{k=1}^{\infty} a_k$ to have positive and negative values. Write

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} [a_k]^+ - \sum_{k=1}^{\infty} [a_k]^-$$

(cf. Exercise 3.5.8) where we are using the notation

$$[X]^+ = \max\{X, 0\} \text{ and } [X]^- = \max\{-X, 0\}$$

and remembering that

$$X = [X]^+ - [X]^- \text{ and } |X| = [X]^+ + [X]^-.$$

Any rearrangement of the series on the left-hand side of this identity just results in a rearrangement in the two series of nonnegative terms on the right. We have just seen that this does nothing to alter the convergence or the sum. Consequently, any rearrangement of our series will have the same sum as required to prove the assertion of the theorem. ■

3.7.2 Conditional Convergence

A convergent series is said to be *conditionally convergent* if it is not unconditionally convergent. Thus such a series converges in the arrangement given, but either there is some rearrangement that diverges or else there is some rearrangement that has a different sum. In fact, both situations always occur.

We have already seen (Example 3.47) how the alternating harmonic series can be rearranged to have a different sum. We shall show that any nonabsolutely convergent series has this property. Our previous rearrangement took advantage of the special nature of the series; here our proof must be completely general and so the method is different.

The following theorem completes Theorem 3.48 and provides the connections:

$$\textit{conditionally convergent} \Leftrightarrow \textit{nonabsolutely convergent}$$

and

$$\textit{unconditionally convergent} \Leftrightarrow \textit{absolutely convergent}$$

Note. You may wonder why we have needed this extra terminology if these concepts are identical. One reason is to emphasize that this is part of the theory. Conditional convergence and nonabsolutely convergence may be equivalent, but they have different underlying meanings. Also, this terminology is used for series of other objects than real numbers and for series of this more general type the terms are not equivalent.

Theorem 3.49 (Riemann) *Every nonabsolutely convergent series is conditionally convergent. In fact, every nonabsolutely convergent series has a divergent rearrangement and can also be rearranged to sum to any preassigned value.*

Proof. Let $\sum_{k=1}^{\infty} a_k$ be an arbitrary nonabsolutely convergent series. To prove the first statement it is enough if we observe that both series

$$\sum_{k=1}^{\infty}[a_k]^+ \text{ and } \sum_{k=1}^{\infty}[a_k]^-$$

must diverge in order for $\sum_{k=1}^{\infty} a_k$ to be nonabsolutely convergent. We need to observe as well that $a_k \to 0$ since the series is assumed to be convergent.

Write p_1, p_2, p_3, for the sequence of positive numbers in the sequence $\{a_k\}$ (skipping any zero or negative ones) and write q_1, q_2, q_3, $\dots$ for the sequence of terms that we have skipped. We construct a new series

$$p_1 + p_2 + \cdots + p_{n_1} + q_1 + p_{n_1+1} + p_{n_1+2} + \cdots + p_{n_2} + q_2 + p_{n_2+1} + \dots$$

where we have chosen $0 = n_0 < n_1 < n_2 < n_3 < \dots$ so that

$$p_{n_k+1} + p_{n_k+2} + \cdots + p_{n_{k+1}} > 2^k$$

for each $k = 0, 1, 2, \dots$. Since $\sum_{k=1}^{\infty} p_k$ diverges, this is possible. The new series so constructed contains all the terms of our original series and so is a rearrangement. Since the terms $q_k \to 0$, they will not interfere with the goal of producing ever larger partial sums for the new series and so, evidently, this new series diverges to ∞.

The second requirement of the theorem is to produce a convergent rearrangement, convergent to a given number α. We proceed in much the same way but with rather more caution. We leave this to the exercises.
■

3.7.3 Comparison of $\sum_{i=1}^{\infty} a_i$ and $\sum_{i\in\mathbb{N}} a_i$

✂ Adv.

The unordered sum of a sequence of real numbers, written as,

$$\sum_{i\in\mathbb{N}} a_i,$$

has an apparent connection with the ordered sum

$$\sum_{i=1}^{\infty} a_i.$$

We should expect the two to be the same when both converge, but is it possible that one converges and not the other?

The answer is that the convergence of $\sum_{i\in\mathbb{N}} a_i$ is equivalent to the *absolute* convergence of $\sum_{i=1}^{\infty} a_i$.

Theorem 3.50: *A necessary and sufficient condition for $\sum_{i\in\mathbb{N}} a_i$ to converge is that the series $\sum_{i=1}^{\infty} a_i$ is absolutely convergent and in this case*

$$\sum_{i\in\mathbb{N}} a_i = \sum_{i=1}^{\infty} a_i.$$

Proof. We shall use a device we have seen before a few times: For any real number X write

$$[X]^+ = \max\{X, 0\} \text{ and } [X]^- = \max\{-X, 0\}$$

and note that

$$X = [X]^+ - [X]^- \text{ and } |X| = [X]^+ + [X]^-.$$

The absolute convergence of the series and the convergence of the sum in the statement in the theorem now reduce to considering the equality of the right-hand sides of

$$\sum_{i\in\mathbb{N}} a_i = \sum_{i\in\mathbb{N}} [a_i]^+ - \sum_{i\in\mathbb{N}} [a_i]^-$$

and

$$\sum_{i=1}^{\infty} a_i = \sum_{i=1}^{\infty} [a_i]^+ - \sum_{i=1}^{\infty} [a_i]^-.$$

This reduces our problem to considering just nonnegative series (sums).

Thus we may assume that each $a_i \geq 0$. For any finite set $I \subset \mathbb{N}$ it is clear that

$$\sum_{i \in I} a_i \leq \sum_{i=1}^{\infty} a_i.$$

It follows that if $\sum_{i=1}^{\infty} a_i$ converges, then (by Exercise 3.3.3) so too does $\sum_{i \in \mathbb{N}} a_i$ and

$$\sum_{i \in \mathbb{N}} a_i \leq \sum_{i=1}^{\infty} a_i. \tag{6}$$

Similarly, if N is finite,

$$\sum_{i=1}^{N} a_i \leq \sum_{i \in \mathbb{N}} a_i.$$

It follows that if $\sum_{i \in \mathbb{N}} a_i$ converges, then, by the boundedness criterion, so too does $\sum_{i=1}^{\infty} a_i$ and

$$\sum_{i=1}^{\infty} a_i \leq \sum_{i \in \mathbb{N}} a_i. \tag{7}$$

Together these two assertions and the equations (6) and (7) prove the theorem for the case of nonnegative series (sums). ■

Exercises

3.7.1 Let

$$s = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots.$$

Show that
$$\frac{3s}{2} = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \dots.$$

3.7.2 For what values of x does the following series converge and what is the sum?
$$1 + x^2 + x + x^4 + x^6 + x^3 + x^8 + x^{10} + x^5 + \dots$$

3.7.3 For what series is the computation
$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} a_{2k} + \sum_{k=1}^{\infty} a_{2k-1}$$
valid? Is this a rearrangement?

3.7.4 For what series is the computation
$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} (a_{2k} + a_{2k-1})$$
valid? Is this a rearrangement?

3.7.5 For what series is the computation
$$\sum_{k=1}^{\infty} a_k = a_2 + a_1 + a_4 + a_3 + a_6 + a_5 + \dots$$
valid? Is this a rearrangement?

3.7.6 Give an example of an absolutely convergent series for which is it much easier to compute the sum by rearrangement than otherwise.

3.7.7 For what values of α and β does the series
$$\frac{\alpha}{1} - \frac{\beta}{2} + \frac{\alpha}{3} - \frac{\beta}{4} + \dots$$
converge?

3.7.8 Let a series be altered by the insertion of zero terms in a completely arbitrary manner. Does this alter the convergence of the series?

3.7.9 Suppose that a convergent series contains only finitely many negative terms. Can it be safely rearranged?

3.7.10 Suppose that a nonabsolutely convergent series has been rearranged and that this rearrangement converges. Does this rearranged series converge absolutely or nonabsolutely?

3.7.11 Is there a divergent series that can be rearranged so as to converge? Can *every* divergent series be rearranged so as to converge? If $\sum_{k=1}^{\infty} a_k$ diverges, but does not diverge to ∞ or $-\infty$, can it be rearranged to diverge to ∞?

3.7.12 How many rearrangements of a nonabsolutely convergent series are there that do not alter the sum?

3.7.13 Complete the proof of Theorem 3.49 by showing that for any nonabsolutely convergent series series $\sum_{k=1}^{\infty} a_k$ and any α there is a rearrangement of the series so that

$$\sum_{k=1}^{\infty} a_{\sigma(k)} = \alpha.$$

See Note 60

3.7.14 Improve Theorem 3.49 by showing that for any nonabsolutely convergent series series $\sum_{k=1}^{\infty} a_k$ and any

$$-\infty \leq \alpha \leq \beta \leq \infty$$

there is a rearrangement of the series so that

$$\alpha = \liminf_{n\to\infty} \sum_{k=1}^{n} a_{\sigma(k)} \leq \limsup_{n\to\infty} \sum_{k=1}^{n} a_{\sigma(k)} = \beta.$$

3.8 Products of Series

ich.

The rule for the sum of two convergent series[2] in Theorem 3.8

$$\sum_{k=0}^{\infty}(a_k + b_k) = \sum_{k=0}^{\infty} a_k + \sum_{k=0}^{\infty} b_k$$

[2] In the formula for a product of series in this section we prefer to label the series starting with 0. This does not change the series in any way.

is entirely elementary to prove and comes directly from the rule for limits of sums of sequences. If A_n and B_n represent the sum of $n+1$ terms of the two series, then

$$\lim_{n\to\infty}\sum_{k=0}^{\infty}(a_k+b_k) = \lim_{n\to\infty}(A_n+B_n) = \lim_{n\to\infty}A_n + \lim_{n\to\infty}B_n$$

$$= \sum_{k=0}^{\infty}a_k + \sum_{k=0}^{\infty}b_k.$$

At first glance we might expect to have a similar rule for products of series, since

$$\lim_{n\to\infty}(A_n\times B_n) = \lim_{n\to\infty}A_n \times \lim_{n\to\infty}B_n$$

$$= \sum_{k=0}^{\infty}a_k \times \sum_{k=0}^{\infty}b_k.$$

But what is A_nB_n? If we write out this product we obtain

$$A_nB_n = (a_0+a_1+a_2+\cdots+a_n)(b_0+b_1+b_2+\cdots+b_n)$$

$$= \sum_{i=0}^{n}\sum_{j=1}^{n}a_ib_j.$$

From this all we can show is the curious observation that

$$\lim_{n\to\infty}\sum_{i=0}^{n}\sum_{j=1}^{n}a_ib_j = \sum_{k=0}^{\infty}a_k \times \sum_{k=0}^{\infty}b_k.$$

What we would rather see here is a result similar to the rule for sums:

“series + series = series.”

Can this result be interpreted as

$\times$	a_0	a_1	a_2	a_3	a_4	a_5	$\ldots$
b_0	a_0b_0	a_1b_0	a_2b_0	a_3b_0	a_4b_0	a_5b_0	$\ldots$
b_1	a_0b_1	a_1b_1	a_2b_1	a_3b_1	a_4b_1	a_5b_1	$\ldots$
b_2	a_0b_2	a_1b_2	a_2b_2	a_3b_2	a_4b_2	a_5b_2	$\ldots$
b_3	a_0b_3	a_1b_3	a_2b_3	a_3b_3	a_4b_3	a_5b_3	$\ldots$
b_4	a_0b_4	a_1b_4	a_2b_4	a_3b_4	a_4b_4	a_5b_4	$\ldots$
b_5	a_0b_5	a_1b_5	a_2b_5	a_3b_5	a_4b_5	a_5b_5	$\ldots$
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	

Figure 3.3. The product of the two series $\sum_0^\infty a_k$ and $\sum_0^\infty b_k$.

"series $\times$ series $=$ series?"

We need a systematic way of adding up the terms a_ib_j in the double sum so as to form a series. The terms are displayed in a rectangular array in Figure 3.3.

If we replace the series here by a power series, this systematic way will become much clearer. How should we add up

$$\left(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n\right)\left(b_0 + b_1x + b_2x^2 + \cdots + b_nx^n\right)$$

(which with $x = 1$ is the same question we just asked)? The now obvious answer is

$$a_0b_0 + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_1b_1 + a_2b_0)x^2$$

$$+(a_0b_3 + a_1b_2 + a_2b_1 + a_3b_0)x^3 + \ldots.$$

Notice that this method of grouping the terms corresponds to summing along diagonals of the rectangle in Figure 3.3.

This is the source of the following definition.

Definition 3.51: The series

$$\sum_{k=0}^{\infty} c_k$$

is called the *formal product* of the two series

$$\sum_{k=0}^{\infty} a_k \text{ and } \sum_{k=0}^{\infty} b_k$$

provided that

$$c_k = \sum_{i=0}^{k} a_i b_{k-i}.$$

Our main goal now is to determine if this “formal” product is in any way a genuine product; that is, if

$$\sum_{k=0}^{\infty} c_k = \sum_{k=0}^{\infty} a_k \times \sum_{k=0}^{\infty} b_k.$$

The reason we expect this might be the case is that the series $\sum_{k=0}^{\infty} c_k$ contains all the terms in the expansion of

$$(a_0 + a_1 + a_2 + a_3 + \dots)(b_0 + b_1 + b_2 + b_3 + \dots).$$

A good reason for caution, however, is that the series $\sum_{k=0}^{\infty} c_k$ contains these terms only in a particular arrangement and we know that series can be sensitive to rearrangement.

3.8.1 Products of Absolutely Convergent Series

It is a general rule in the study of series that absolutely convergent series permit the best theorems. We can rearrange such series freely as we have seen already in Section 3.7.1. Now we show that we can form products of such series. We shall have to be much more cautious about forming products of nonabsolutely convergent series.

Theorem 3.52 (Cauchy) *Suppose that $\sum_{k=0}^{\infty} c_k$ is the formal product of two absolutely convergent series*

$$\sum_{k=0}^{\infty} a_k \ \textit{and} \ \sum_{k=0}^{\infty} b_k.$$

Then $\sum_{k=0}^{\infty} c_k$ converges absolutely too and

$$\sum_{k=0}^{\infty} c_k = \sum_{k=0}^{\infty} a_k \times \sum_{k=0}^{\infty} b_k.$$

Proof. We write

$$A = \sum_{k=0}^{\infty} a_k,\ A' = \sum_{k=0}^{\infty} |a_k|,\ A_n = \sum_{k=0}^{n} a_k,$$

$$B = \sum_{k=0}^{\infty} b_k,\ B' = \sum_{k=0}^{\infty} |b_k|, \ \text{ and } \ B_n = \sum_{k=0}^{n} b_k.$$

By definition

$$c_k = \sum_{i=0}^{k} a_i b_{k-i}$$

and so

$$\sum_{k=0}^{N} |c_k| \le \sum_{k=0}^{N}\sum_{i=0}^{k} |a_i|\cdot|b_{k-i}| \le \left(\sum_{i=0}^{N} |a_i|\right)\left(\sum_{i=0}^{N} |b_i|\right) \le A'B'.$$

Since the latter two series converge, this provides an upper bound $A'B'$ for the sequence of partial sums $\sum_{k=1}^{N} |c_k|$ and hence the series $\sum_{k=0}^{\infty} c_k$ converges absolutely.

Let us recall that the formal product of the two series is just a particular rearrangement of the terms $a_i b_j$ taken over all $i \ge 0$, $j \ge 0$. Consider any arrangement of these terms. This must form an absolutely convergent series by the same argument as before since $A'B'$ will be an upper bound for the partial sums of the absolute values $|a_i b_j|$. Thus all rearrangements will converge to the same value by Theorem 3.48.

We can rearrange the terms $a_i b_j$ taken over all $i \geq 0$, $j \geq 0$ in the following convenient way "by squares." Arrange always so that the first $(m+1)^2$ $(m = 0, 1, 2, \dots)$ terms add up to $A_m B_m$. For example, one such arrangement starts off

$$a_0 b_0 + a_1 b_0 + a_0 b_1 + a_1 b_1 + a_2 b_0 + a_2 b_1 + a_0 b_2 + a_1 b_2 + a_2 b_2 + \dots.$$

(A picture helps considerably to see the pattern needed.) We know this arrangement converges and we know it must converge to

$$\lim_{m \to \infty} A_m B_m = AB.$$

In particular, the series $\sum_{k=0}^{\infty} c_k$ which is just another arrangement, converges to the same number AB as required. ■

It is possible to improve this theorem to allow one (but not both) of the series to converge nonabsolutely. The conclusion is that the product then converges (perhaps nonabsolutely), but different methods of proof will be needed. As usual, nonabsolutely convergent series are much more fragile, and the free and easy moving about of the terms in this proof is not allowed.

3.8.2 Products of Nonabsolutely Convergent Series

Let us give a famous example, due to Cauchy, of a pair of convergent series whose product diverges. We know that the alternating series

$$\sum_{k=0}^{\infty} (-1)^k \frac{1}{\sqrt{k+1}}$$

is convergent, but not absolutely convergent since the related absolute series is a p-harmonic series with $p = \frac{1}{2}$.

Let

$$\sum_{k=0}^{\infty} c_k$$

be the formal product of this series with itself. By definition the term c_k is given by

$$(-1)^k \left[\frac{1}{\sqrt{1\cdot(k+1)}} + \frac{1}{\sqrt{2\cdot(k)}} + \frac{1}{\sqrt{3\cdot(k-1)}} + \cdots + \frac{1}{\sqrt{(k+1)\cdot 1}} \right].$$

There are $k+1$ terms in the sum for c_k and each term is larger than $1/(k+1)$ so we see that $|c_k| \geq 1$. Since the terms of the product series $\sum_{k=0}^{\infty} c_k$ do not tend to zero, this is a divergent series.

This example supplies our observation: The formal product of two nonabsolutely convergent series need not converge. In particular, there may be no convergent series to represent the product

$$\sum_{k=0}^{\infty} a_k \times \sum_{k=0}^{\infty} b_k$$

for a pair of nonabsolutely convergent series. For absolutely convergent series the product always converges.

We should not be too surprised at this result. The theory begins to paint the following picture: Absolutely convergent series can be freely manipulated in most ways and nonabsolutely convergent series can hardly be manipulated in general in any serious manner. Interestingly, the following theorem can be proved that shows that even though, in general, the product might diverge, in cases where it does converge it converges to the "correct" value.

Theorem 3.53 (Abel) *Suppose that $\sum_{k=0}^{\infty} c_k$ is the formal product of two nonabsolutely convergent series $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$ and suppose that this product $\sum_{k=0}^{\infty} c_k$ is known to converge. Then*

$$\sum_{k=0}^{\infty} c_k = \sum_{k=0}^{\infty} a_k \times \sum_{k=0}^{\infty} b_k.$$

Proof. The proof requires more technical apparatus and will not be given until Section 3.9.2. ■

Exercises

3.8.1 Form the product of the series $\sum_{k=0}^{\infty} a_k x^k$ with the geometric series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

and obtain the formula

$$\frac{1}{1-x}\sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty}(a_0 + a_1 + a_2 + \dots + a_k)x^k.$$

For what values of x would this be valid?

3.8.2 Show that

$$(1-x)^2 = \sum_{k=0}^{\infty}(k+1)x^k$$

for appropriate values of x.

3.8.3 Using the fact that

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} = \log 2,$$

show that

$$\sum_{k=0}^{\infty} \frac{(-1)^k \sigma_k}{k+2} = \frac{(\log 2)^2}{2}$$

where $\sigma_k = 1 + 1/2 + 1/3 + \dots + 1/(k+1)$.

3.8.4 Verify that $e^{x+y} = e^x e^y$ by proving that

$$\sum_{k=0}^{\infty} \frac{(x+y)^k}{k!} = \sum_{k=0}^{\infty} \frac{x^k}{k!} \sum_{k=0}^{\infty} \frac{y^k}{k!}.$$

3.8.5 For what values of p and q are you able to establish the convergence of the product of the two series

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)^p} \quad \text{and} \quad \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)^q}?$$

3.9 Summability Methods

A first course in series methods often gives the impression of being obsessed with the issue of convergence or divergence of a series. The huge battery of tests in Section 3.6 devoted to determining the behavior of series might lead one to this conclusion. Accordingly, you may have decided that convergent series are useful and proper tools of analysis while divergent series are useless and without merit.

In fact divergent series are, in many instances, as important or more important than convergent ones. Many eighteenth century mathematicians achieved spectacular results with divergent series but without a proper understanding of what they were doing. The initial reaction of our founders of nineteenth-century analysis (Cauchy, Abel, and others) was that valid arguments could be based only on convergent series. Divergent series should be shunned. They were appalled at reasoning such as the following: The series

$$s = 1 - 1 + 1 - 1 + \dots$$

can be summed by noting that

$$s = 1 - (1 - 1 + 1 - \dots) = 1 - s$$

and so $2s = 1$ or $s = \frac{1}{2}$. But the sum $\frac{1}{2}$ proves to be a useful value for the "sum" of this series even though the series is clearly divergent.

There are many useful ways of doing rigorous work with divergent series. One way, which we now study, is the development of *summability methods.*

Suppose that a series $\sum_{k=0}^{\infty} a_k$ diverges and yet we wish to assign a "sum" to it by some method. Our standard method thus far is to take the limit of the sequence of partial sums. We write

$$s_n = \sum_{k=0}^{n} a_k$$

and the sum of the series is $\lim_{n\to\infty} s_n$. If the series diverges, this means precisely that this sequence does not have a limit. How can we use that sequence or that series nonetheless to assign a different meaning to the sum?

3.9.1 Cesàro's Method

Adv.

An infinite series $\sum_{k=0}^{\infty} a_k$ has a sum S if the sequence of partial sums

$$s_n = \sum_{k=0}^{n} a_k$$

converges to S. If the sequence of partial sums diverges, then we must assign a sum by a different method. We will still say that the series diverges but, nonetheless, we will be able to find a number that can be considered the sum.

We can replace $\lim_{n\to\infty} s_n$, which perhaps does not exist, by

$$\lim_{n\to\infty} \frac{s_0 + s_1 + s_2 + \cdots + s_n}{n+1} = C$$

if this exists and use this value for the sum of the series. This is an entirely natural method since it merely takes averages and settles for computing a kind of "average" limit where an actual limit might fail to exist.

For a series $\sum_{k=0}^{\infty} a_k$ often we can use this method to obtain a sum even when the series diverges.

Definition 3.54: If $\{s_n\}$ is the sequence of partial sums of the series $\sum_{k=0}^{\infty} a_k$ and

$$\lim_{n\to\infty} \frac{s_0 + s_1 + s_2 + \cdots + s_n}{n+1} = C$$

then the new sequence

$$\sigma_n = \frac{s_0 + s_1 + s_2 + \cdots + s_n}{n+1}$$

is called the sequence of *averages* or *Cesàro means* and we write

$$\sum_{k=0}^{\infty} a_k = C \quad \text{[Cesàro]}.$$

Thus the symbol [Cesàro] indicates that the value is obtained by this method rather than by the usual method of summation (taking limits of partial sums). The method is named after Ernesto Cesàro (1859–1906).

Our first concern in studying a summability method is to determine whether it assigns the "correct" value to a series that already converges. Does

$$\sum_{k=0}^{\infty} a_k = A \Rightarrow \sum_{k=0}^{\infty} a_k = A \text{ [Cesàro]?}$$

Any method of summing a series is said to be *regular* or a *regular summability method* if this is the case.

Theorem 3.55: *Suppose that a series $\sum_{k=0}^{\infty} a_k$ converges to a value A. Then $\sum_{k=0}^{\infty} a_k = A$ [Cesàro] is also true.*

Proof. This is an immediate consequence of Exercise 2.13.17. For any sequence $\{s_n\}$ write

$$\sigma_n = \frac{s_1 + s_2 + \cdots + s_n}{n}.$$

In that exercise we showed that

$$\liminf_{n\to\infty} s_n \leq \liminf_{n\to\infty} \sigma_n \leq \limsup_{n\to\infty} \sigma_n \leq \limsup_{n\to\infty} s_n.$$

If you skipped that exercise, here is how to prove it. Let

$$\beta > \limsup_{n\to\infty} s_n.$$

(If there is no such β, then $\limsup_{n\to\infty} s_n = \infty$ and there is nothing to prove.) Then $s_n < \beta$ for all $n \geq N$ for some N. Thus

$$\sigma_n \leq \frac{1}{n}\left(s_1 + s_2 + \cdots + s_{N-1}\right) + \frac{(n-N+1)\beta}{n}$$

for all $n \geq N$. Fix N, allow $n \to \infty$, and take limit superiors of each side to obtain

$$\limsup_{n\to\infty} \sigma_n \leq \beta.$$

It follows that

$$\limsup_{n\to\infty} \sigma_n \leq \limsup_{n\to\infty} s_n.$$

The other inequality is similar. In particular, if $\lim_{n\to\infty} s_n$ exists so too does $\lim_{n\to\infty} \sigma_n$ and they are equal, proving the theorem. ■

Example 3.56: As an example let us sum the series

$$1 - 1 + 1 - 1 + 1 - 1 + \dots.$$

The partial sums form the sequence 1, 0, 1, 0, ..., which evidently diverges. Indeed the series diverges merely by the trivial test: The terms do not tend to zero. Can we sum this series by the Cesàro summability method? The averages of the sequence of partial sums is clearly tending to $\frac{1}{2}$. Thus we can write

$$\sum_{k=0}^{\infty} (-1)^k = \frac{1}{2} \quad \text{[Cesàro]}$$

even though the series is divergent. ◀

3.9.2 Abel's Method

✂ *Adv.*

We require in this section that you recall some calculus limits. We shall need to compute a limit

$$\lim_{x\to 1-} F(x)$$

for a function F defined on $(0,1)$ where the expression $x \to 1-$ indicates a left-hand limit. In Chapter 5 we present a full account of such limits; here we need remember only what this means and how it is computed.

Suppose that a series $\sum_{k=0}^{\infty} a_k$ diverges and yet we wish to assign a "sum" to it by some other method. If the terms of the series do not get too large, then the series

$$F(x) = \sum_{k=0}^{\infty} a_k x^k$$

will converge (by the ratio test) for all $0 \le x < 1$. The value we wish for the sum of the series would appear to be $F(1)$, but for a divergent series inserting the value 1 for x gives us nothing we can use. Instead we compute

$$\lim_{x\to 1-} F(x) = \lim_{x\to 1-} \sum_{k=0}^{\infty} a_k x^k = A$$

and use this value for the sum of the series.

Definition 3.57: We write

$$\sum_{k=0}^{\infty} a_k x^k = A \ \ \text{[Abel]}$$

if

$$\lim_{x\to 1-} \sum_{k=0}^{\infty} a_k x^k = A.$$

Here the symbol [Abel] indicates that the value is obtained by this method rather than by the usual method of summation (taking limits of partial sums).

As before, our first concern in studying a summability method is to determine whether it assigns the "correct" value to a series that already converges. Does

$$\sum_{k=0}^{\infty} a_k = A \ \Rightarrow \ \sum_{k=0}^{\infty} a_k = A \text{ [Abel]?}$$

We are asking, in more correct language, whether Abel's method of summability of series is *regular*.

Theorem 3.58 (Abel) *Suppose that a series* $\sum_{k=0}^{\infty} a_k$ *converges to a value* A. *Then*

$$\lim_{x\to 1-} \sum_{k=0}^{\infty} a_k x^k = A.$$

Proof. Our first step is to note that the convergence of the series $\sum_{k=0}^{\infty} a_k$ requires that the terms $a_k \to 0$. In particular, the terms are bounded and so the root test will prove that the series $\sum_{k=0}^{\infty} a_k x^k$ converges absolutely for all $|x| < 1$ at least. Thus we can define

$$F(x) = \sum_{k=0}^{\infty} a_k x^k$$

for $0 \leq x < 1$.

Let us form the product of the series for $F(x)$ with the geometric series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

(cf. Exercise 3.8.1). Since both series are absolutely convergent for any $0 \leq x < 1$, we obtain

$$\frac{F(x)}{1-x} = \sum_{k=0}^{\infty} (a_0 + a_1 + a_2 + \cdots + a_k) x^k.$$

Writing

$$s_k = (a_0 + a_1 + a_2 + \cdots + a_k)$$

and using the fact that

$$s_k \to A = \sum_{k=0}^{\infty} a_k,$$

we obtain

$$F(x) = (1-x) \sum_{k=0}^{\infty} s_k x^k = A - (1-x) \sum_{k=0}^{\infty} (s_k - A) x^k.$$

Let $\varepsilon > 0$ and choose N so large that

$$|s_k - A| < \varepsilon/2$$

for $k > N$. Then the inequality

$$|F(x) - A| \le (1-x)\sum_{k=0}^{N} |s_k - A|x^k + \varepsilon/2$$

holds for all $0 \le x < 1$. The sum here is just a finite sum, and taking limits in finite sums is routine:

$$\lim_{x\to 1-} (1-x)\sum_{k=0}^{N}(s_k - A)x^k = 0.$$

Thus for $x < 1$ but sufficiently close to 1 we can make this smaller than $\varepsilon/2$ and conclude that

$$|F(x) - A| < \varepsilon.$$

We have proved that

$$\lim_{x\to 1-} F(x) = A$$

and the theorem is proved. ■

Example 3.59: Let us sum the series

$$\sum_{k=0}^{\infty}(-1)^k = 1 - 1 + 1 - 1 + 1 - 1 + \dots$$

by Abel's method. We form

$$F(x) = \sum_{k=0}^{\infty}(-1)^k x^k = \frac{1}{1+x}$$

obtaining the formula by recognizing this as a geometric series. Since

$$\lim_{x\to 1-} F(x) = \frac{1}{2}$$

we have proved that

$$\sum_{k=0}^{\infty}(-1)^k = \frac{1}{2} \quad \text{[Abel]}.$$

Recall that we have already obtained in Example 3.56 that

$$\sum_{k=0}^{\infty}(-1)^k = \frac{1}{2} \quad \text{[Cesàro]}$$

so these two different methods have assigned the same sum to this divergent series. You might wish to explore whether the same thing will happen with all series. ◀

As an interesting application we are now in a position to prove Theorem 3.53 on the product of series.

Theorem 3.60 (Abel) *Suppose that $\sum_{k=0}^{\infty} c_k$ is the formal product of two convergent series $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$ and suppose that $\sum_{k=0}^{\infty} c_k$ is known to converge. Then*

$$\sum_{k=0}^{\infty} c_k = \sum_{k=0}^{\infty} a_k \times \sum_{k=0}^{\infty} b_k.$$

Proof. The proof just follows on taking limits as $x \to 1-$ in the expression

$$\sum_{k=0}^{\infty} c_k x^k = \sum_{k=0}^{\infty} a_k x^k \times \sum_{k=0}^{\infty} b_k x^k.$$

Abel's theorem, Theorem 3.58, allows us to do this. How do we know, however, that this identity is true for all $0 \leq x < 1$? All three of these series are absolutely convergent for $|x| < 1$ and, by Theorem 3.52, absolutely convergent series can be multiplied in this way. ■

Exercises

3.9.1 Is the series

$$1 + 1 - 1 + 1 + 1 - 1 + 1 + 1 - 1 + \cdots$$

Cesàro summable?

3.9.2 Is the series

$$1 - 2 + 3 - 4 + 5 - 6 + 7 - 8 + \cdots$$

Cesàro summable?

3.9.3 Is the series

$$1 - 2 + 3 - 4 + 5 - 6 + 7 - 8 + \cdots$$

Abel summable?

3.9.4 Show that a divergent series of positive numbers cannot be Cesàro summable or Abel summable.

3.9.5 Find a proof from an appropriate source that demonstrates the exact relation between Cesàro summability and Abel summability.

3.9.6 In an appropriate source find out what is meant by a *Tauberian theorem* and present one such theorem appropriate to our studies in this section.

See Note 61

3.10 More on Infinite Sums

How should we form the sum of a double sequence $\{a_{jk}\}$ where both j and k can range over all natural numbers? In many applications of analysis such sums are needed. A variety of methods come to mind:

1. We might simply form the unordered sum

$$\sum_{(j,k)\in\mathbb{N}\times\mathbb{N}} a_{jk}.$$

2. We could construct "partial sums" in some systematic method and take limits just as we do for ordinary series:

$$\lim_{N\to\infty}\sum_{j=1}^{N}\sum_{k=1}^{N}a_{jk}.$$

These are called *square sums* and are quite popular. If you sketch a picture of the set of points

$$\{(j,k): 1\le j\le N,\ 1\le k\le N\}$$

in the plane the square will be plainly visible.

3. We could construct partial sums using *rectangular sums*:

$$\lim_{M,N\to\infty}\sum_{j=1}^{M}\sum_{k=1}^{N}a_{jk}.$$

Here the limit is a double limit, requiring both M and N to get large. If you sketch a picture of the set of points

$$\{(j,k): 1\le j\le M,\ 1\le k\le N\}$$

in the plane you will see the rectangle.

4. We could construct partial sums using *circular sums*:

$$\lim_{R\to\infty}\sum_{j^2+k^2\le R^2}a_{jk}.$$

Once again, a sketch would show the circles.

5. We could "iterate" the sums, by summing first over j and then over k:

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}a_{jk}$$

or, in the reverse order,

$$\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} a_{jk}.$$

Our experience in the study of ordinary series suggests that all these methods should produce the same sum if the numbers summed are all nonnegative, but that subtle differences are likely to emerge if we are required to add numbers both positive and negative.

In the exercises there are a number of problems that can be pursued to give a flavor for this kind of theory. At this stage in your studies it is important to grasp the fact that such questions arise. Later, when you have found a need to use these kinds of sums, you can develop the needed theory. The tools for developing that theory are just those that we have studied so far in this chapter.

Exercises

3.10.1 Decide on a meaning for the notion of a double series

$$\sum_{j,k=1}^{\infty} a_{jk} \tag{8}$$

and prove that if all the numbers a_{jk} are nonnegative then this converges if and only if

$$\sum_{(j,k)\in\mathbb{N}\times\mathbb{N}} a_{jk} \tag{9}$$

converges and that the values assigned to (8) and (9) are the same.

3.10.2 Decide on a meaning for the notion of an absolutely convergent double series

$$\sum_{j,k=1}^{\infty} a_{jk}$$

and prove that such a series is absolutely convergent if and only if

$$\sum_{(j,k)\in\mathbb{N}\times\mathbb{N}} a_{jk}$$

converges.

3.10.3 Show that the methods given in the text for forming a sum of a double sequence $\{a_{jk}\}$ are equivalent if all the numbers are nonnegative.

3.10.4 Show that the methods given in the text for forming a sum of a double sequence $\{a_{jk}\}$ are not equivalent in general.

3.10.5 What can you assert about the convergence or divergence of the double series

$$\sum_{j,k=1}^{\infty} \frac{1}{j\,k^4}?$$

3.10.6 What is the sum of the double series

$$\sum_{j,k=0}^{\infty} \frac{x^j y^k}{j!\,k!}?$$

3.11 Infinite Products

Enrich.

In this chapter we studied, quite extensively, infinite sums. There is a similar theory for infinite products, a theory that has much in common with the theory of infinite sums. In this section we shall briefly give an account of this theory, partly to give a contrast and partly to introduce this important topic.

Similar to the notion of an infinite sum

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \dots$$

is the notion of an infinite product

$$\prod_{n=1}^{\infty} p_n = p_1 \times p_2 \times p_3 \times p_4 \times \dots$$

with a nearly identical definition. Corresponding to the concept of "partial sums" for the former will be the notion of "partial products" for the latter.

The main application of infinite series is that of series representations of functions. The main application of infinite products is exactly the same. Thus, for example, in more advanced material we will find a representation of the sin function as an infinite series

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots$$

and also as an infinite product

$$\sin x = \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right)\left(1 - \frac{x^2}{16\pi^2}\right)\dots.$$

The most obvious starting point for our theory would be to define an infinite product as the limit of the sequence of partial products in exactly the same way that an infinite sum is defined as the limit of the sequence of partial sums. But products behave differently from sums in one important regard: The number zero plays a peculiar role. This is why the definition we now give is slightly different than a first guess might suggest. Our goal is to define an infinite product in such a way that a product can be zero only if one of the factors is zero (just like the situation for finite products).

Definition 3.61: Let $\{b_k\}$ be a sequence of real numbers. We say that the infinite product

$$\prod_{k=1}^{\infty} b_k$$

converges if there is an integer N so that all $b_k \neq 0$ for $k > N$ and if

$$\lim_{M\to\infty} \prod_{k=N+1}^{M} b_k$$

exists *and is not zero.* For the value of the infinite product we take

$$\prod_{k=1}^{\infty} b_k = b_1 \times b_2 \times \dots b_N \times \lim_{M\to\infty} \prod_{k=N+1}^{M} b_k.$$

This definition guarantees us that a product of factors can be zero if and only if one of the factors is zero. This is the case for finite products, and we are reluctant to lose this.

Theorem 3.62: *A convergent product*

$$\prod_{k=1}^{\infty} b_k = 0$$

if and only if one of the factors is zero.

Proof. This is built into the definition and is one of its features. ■

We expect the theory of infinite products to evolve much like the theory of infinite series. We recall that a series $\sum_{k=1}^{n} a_k$ could converge only if $a_k \to 0$. Naturally, the product analog requires the terms to tend to 1.

Theorem 3.63: *A product*

$$\prod_{k=1}^{\infty} b_k$$

that converges necessarily has $b_k \to 1$ as $k \to \infty$.

Proof. This again is a feature of the definition, which would not be possible if we had not handled the zeros in this way. Choose N so that none of the factors b_k is zero for $k > N$. Then

$$b_n = \lim_{n\to\infty} \frac{\prod_{k=N+1}^{n} b_k}{\prod_{k=N+1}^{n-1} b_k} = 1$$

as required. ■

As a result of this theorem it is conventional to write all infinite products in the special form

$$\prod_{k=1}^{\infty}(1+a_k)$$

and remember that the terms $a_k \to 0$ as $k \to \infty$ in a convergent product. Also, our assumption about the zeros allows for $a_k = -1$ only for finitely many values of k. The expressions $(1+a_k)$ are called the "factors" of the product and the a_k themselves are called the "terms."

A close linkage with series arises because the two objects

$$\sum_{k=1}^{\infty} a_k \quad \text{and} \quad \prod_{k=1}^{\infty}(1+a_k),$$

the series and the product, have much the same kind of behavior.

Theorem 3.64: *A product*

$$\prod_{k=1}^{\infty}(1+a_k)$$

where all the terms a_k are positive is convergent if and only if the series $\sum_{k=1}^{\infty} a_k$ converges.

Proof. Here we use our usual criterion that has served us through most of this chapter: A sequence that is monotonic is convergent if and only if it is bounded.

Note that

$$a_1 + a_2 + a_3 + \cdots + a_n \leq (1+a_1)(1+a_2)(1+a_3) \times \cdots \times (1+a_n)$$

so that the convergence of the product gives an upper bound for the partial sums of the series. It follows that if the product converges so must the series.

In the other direction we have

$$(1+a_1)(1+a_2)(1+a_3) \times \cdots \times (1+a_n) \leq e^{a_1+a_2+a_3+\cdots+a_n}$$

and so the convergence of the series gives an upper bound for the partial products of the infinite product. It follows that if the series converges, so must the product. ■

Exercises

3.11.1 Give an example of a sequence of positive numbers $\{b_k\}$ so that

$$\lim_{n\to\infty} b_1 b_2 b_3 \dots b_n$$

exists, but so that the infinite product

$$\prod_{n=1}^{\infty} b_k$$

nonetheless diverges.

3.11.2 Compute

$$\prod_{k=1}^{\infty}\left(1-\frac{1}{k^2}\right).$$

3.11.3 In Theorem 3.64 we gave no relation between the value of the product

$$\prod_{k=1}^{\infty}(1+a_k)$$

and the value of the series $\sum_{k=1}^{\infty} a_k$ where all the terms a_k are positive. What is the best you can state?

3.11.4 For what values of p does the product

$$\prod_{n=1}^{\infty}\left(1+\frac{1}{k^p}\right)$$

converge?

3.11.5 Show that

$$\prod_{k=1}^{\infty}(1+x^{2^k}) = (1+x^2)\times(1+x^4)\times(1+x^8)\times(1+x^{16})\times\dots$$

converges to $1/(1-x^2)$ for all $-1<x<1$ and diverges otherwise.

3.11.6 Find a Cauchy criterion for the convergence of infinite products.

3.11.7 A product

$$\prod_{k=1}^{\infty}(1+a_k)$$

is said to *converge absolutely* if the related product

$$\prod_{k=1}^{\infty}(1+|a_k|)$$

converges.

(a) Show that an absolutely convergent product is convergent.

(b) Show that an infinite product

$$\prod_{k=1}^{\infty}(1+a_k)$$

converges absolutely if and only if the series of its terms $\sum_{k=1}^{\infty} a_k$ converges absolutely.

(c) For what values of x does the product

$$\prod_{k=1}^{\infty}\left(1+\frac{x}{k}\right)$$

converge absolutely?

(d) For what values of x does the product

$$\prod_{k=1}^{\infty}\left(1+\frac{x}{k^2}\right)$$

converge absolutely?

(e) For what values of x does the product

$$\prod_{k=1}^{\infty}\left(1+x^k\right)$$

converge absolutely?

(f) Show that

$$\prod_{k=1}^{\infty}\left(1+\frac{(-1)^k}{k}\right)$$

converges but not absolutely.

3.11.8 Develop a theory that allows for the order of the factors in a product to be rearranged.

3.12 Challenging Problems for Chapter 3

3.12.1 If a_n is a sequence of positive numbers such that $\sum_{n=1}^{\infty} a_n$ diverges what (if anything) can you say about the following three series?

(a) $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$

(b) $\sum_{n=1}^{\infty} \frac{a_n}{1+na_n}$

(c) $\sum_{n=1}^{\infty} \frac{a_n}{1+n^2 a_n}$

3.12.2 Prove the following variant on the Dirichlet Test 3.44: If $\{b_n\}$ is a sequence of bounded variation (cf. Exercise 3.5.12) that converges to zero and the partial sums of the series $\sum_{k=1}^{\infty} a_k$ are bounded, then the series $\sum_{k=1}^{\infty} a_k b_k$ converges.

3.12.3 Prove this variant on the Cauchy condensation test: If the terms of a series $\sum_{k=1}^{\infty} a_k$ are nonnegative and decrease monotonically to zero, then that series converges if and only if the series

$$\sum_{j=1}^{\infty}(2j+1)a_{j^2}$$

converges.

3.12.4 Prove this more general version of the Cauchy condensation test: If the terms of a series $\sum_{k=1}^{\infty} a_k$ are nonnegative and decrease monotonically to zero, then that series converges if and only if the related series

$$\sum_{j=1}^{\infty}(m_{j+1}-m_j)a_{m_j}$$

converges. Here $m_1 < m_2 < m_3 < m_4 < \dots$ is assumed to be an increasing sequence of integers and

$$m_{j+1}-m_j \leq C\,(m_j - m_{j-1})$$

for some positive constant and all j.

3.12.5 For any two series of positive terms write

$$\sum_{k=1}^{\infty} a_k \preceq \sum_{k=1}^{\infty} b_k$$

if $a_k/b_k \to 0$ as $k \to \infty$.

(a) If both series converge, explain why this might be interpreted by saying that $\sum_{k=1}^{\infty} a_k$ is converging faster than $\sum_{k=1}^{\infty} b_k$.

(b) If both series diverge, explain why this might be interpreted by saying that $\sum_{k=1}^{\infty} a_k$ is diverging more slowly than $\sum_{k=1}^{\infty} b_k$.

(c) For convergent series is there any connection between

$$\sum_{k=1}^{\infty} a_k \preceq \sum_{k=1}^{\infty} b_k$$

and

$$\sum_{k=1}^{\infty} a_k \leq \sum_{k=1}^{\infty} b_k?$$

(d) For what values of p, q is

$$\sum_{k=1}^{\infty} \frac{1}{k^p} \preceq \sum_{k=1}^{\infty} \frac{1}{k^q}?$$

(e) For what values of r, s is

$$\sum_{k=1}^{\infty} r^k \preceq \sum_{k=1}^{\infty} s^k?$$

(f) Arrange the divergent series

$$\sum_{k=2}^{\infty} \frac{1}{k}, \sum_{k=2}^{\infty} \frac{1}{k\log k}, \sum_{k=2}^{\infty} \frac{1}{k\log(\log k)}, \sum_{k=2}^{\infty} \frac{1}{k\log(\log(\log k))} \dots$$

into the correct order.

(g) Arrange the convergent series

$$\sum_{k=2}^{\infty} \frac{1}{k^p}, \sum_{k=2}^{\infty} \frac{1}{k(\log k)^p}, \sum_{k=2}^{\infty} \frac{1}{k\log k(\log(\log k))^p},$$

$$\sum_{k=2}^{\infty} \frac{1}{k\log k(\log(\log k))(\log(\log(\log k)))^p} \dots$$

into the correct order. Here $p > 1$.

(h) Suppose that $\sum_{k=1}^{\infty} b_k$ is a divergent series of positive numbers. Show that there is a series

$$\sum_{k=1}^{\infty} a_k \preceq \sum_{k=1}^{\infty} b_k$$

that also diverges (but more slowly).

(i) Suppose that $\sum_{k=1}^{\infty} a_k$ is a convergent series of positive numbers. Show that there is a series

$$\sum_{k=1}^{\infty} a_k \preceq \sum_{k=1}^{\infty} b_k$$

that also converges (but more slowly).

(j) How would you answer this question? Is there a "mother" of all divergent series diverging so slowly that all other divergent series can be proved to be divergent by a comparison test with that series?

SEE NOTE 62

3.12.6 This collection of exercises develops some convergence properties of *trigonometric series*; that is, series of the form

$$a_0/2 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)\,. \tag{10}$$

(a) For what values of x does $\sum_{k=1}^{\infty} \frac{\sin kx}{k^2}$ converge?

(b) For what values of x does $\sum_{k=1}^{\infty} \frac{\sin kx}{k}$ converge?

(c) Show that the condition $\sum_{k=1}^{\infty} (|a_k| + |b_k|) < \infty$ ensures the absolute convergence of the trigonometric series (10) for all values of x.

SEE NOTE 63

3.12.7 Let $\{a_k\}$ be a decreasing sequence of positive real numbers with limit 0 such that

$$b_k = a_k - 2a_{k+1} + a_{k+2} \geq 0.$$

Prove that $\sum_{k=1}^{\infty} kb_k = a_1$.

SEE NOTE 64

3.12.8 Let $\{a_k\}$ be a monotonic sequence of real numbers such that $\sum_{k=1}^{\infty} a_k$ converges. Show that

$$\sum_{k=1}^{\infty} k(a_k - a_{k+1})$$

converges.

SEE NOTE 65

3.12.9 Show that every positive rational number can be obtained as the sum of a finite number of distinct terms of the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots.$$

SEE NOTE 66

3.12.10 Let $\sum_{k=1}^{\infty} x_k$ be a convergent series of positive numbers that is monotonically nonincreasing; that is, $x_1 \geq x_2 \geq x_3 \geq \dots$. Let P denote the set of all real numbers that are sums of finitely or infinitely many terms of the series. Show that P is an interval if and only if

$$x_n \leq \sum_{k=n+1}^{\infty} x_k$$

for every integer n.

SEE NOTE 67

3.12.11 Let p_1, p_2, p_3, be a sequence of distinct points that is dense in the interval $(0, 1)$. The points p_1, p_2, p_3, $\dots$, p_{n-1} decompose the interval $[0, 1]$ into n closed subintervals. The point p_n is an interior point of one of those intervals and decomposes that interval into two closed subintervals. Let a_n and b_n be the lengths of those two intervals. Prove that

$$\sum_{k=1}^{\infty} a_k b_k (a_k + b_k) = 3.$$

SEE NOTE 68

3.12.12 Let $\{a_n\}$ be a sequence of positive number such that the series $\sum_{k=1}^{\infty} a_k$ converges. Show that

$$\sum_{k=1}^{\infty} (a_k)^{n/(n+1)}$$

also converges.

SEE NOTE 69

3.12.13 Let $\{a_k\}$ be a sequence of positive numbers and suppose that

$$a_k \leq a_{2k} + a_{2k+1}$$

for all $k = 1, 2, 3, 4, \ldots$. Show that $\sum_{k=1}^{\infty} a_k$ diverges.

SEE NOTE 70

3.12.14 If $\{a_k\}$ is a sequence of positive numbers for which $\sum_{k=1}^{\infty} a_k$ diverges, determine all values of p for which

$$\sum_{k=1}^{\infty} \frac{a_k}{(a_1 + a_2 + \cdots + a_k)^p}$$

converges.

SEE NOTE 71

3.12.15 Let $\{a_n\}$ be a sequence of real numbers converging to zero. Show that there must exist a monotonic sequence $\{b_n\}$ such that the series $\sum_{k=1}^{\infty} b_k$ diverges and the series $\sum_{k=1}^{\infty} a_k b_k$ is absolutely convergent.

SEE NOTE 72

Notes

[36]Exercise 3.2.2. Define $\sum_{i \in I} a_i$ for I with zero or one elements. Suppose it is defined for I with n elements. Define it for I with $n+1$ elements and show well defined.

[37]Exercise 3.2.4. The answer is yes if I and J are disjoint. Otherwise the correct formula would be

$$\sum_{i\in I\cup J} a_i + \sum_{i\in I\cap J} a_i = \sum_{i\in I} a_i + \sum_{i\in J} a_i.$$

[38]Exercise 3.2.8. Try to interpret the "difference" $\Delta s_k = s_{k+1} - s_k = a_{k+1}$ as the analog of a derivative.

[39]Exercise 3.2.11. Use a telescoping sum method. Even if you cannot remember your trigonometric identities you can work backward to see which one is needed. Check the formula for values of θ with $\sin\theta/2 = 0$ and see that it can be interpreted by taking limits.

[40]Exercise 3.3.1. This is similar to the statement that convergent sequences have unique limits. Try to imitate that proof.

[41]Exercise 3.3.2. This is similar to the statement that convergent sequences are bounded. Try to imitate that proof.

[42]Exercise 3.3.3. This is similar to the statement that monotone, bounded sequences are convergent. Try to imitate that proof.

[43]Exercise 3.3.9. Compare with the sum

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2$$

given in the introduction to this chapter.

[44]Exercise 3.3.11. Here we are using, as elsewhere,

$$[X]^+ = \max\{X, 0\}$$

and

$$[X]^- = \max\{-X, 0\}$$

and note that

$$X = [X]^+ - [X]^- \text{ and } |X| = [X]^+ + [X]^-.$$

[45]Exercise 3.3.12. Note that the index set is

$$I = \mathbb{N} \times \mathbb{N}.$$

Thus we can study unordered sums of double sequences $\{a_{ij}\}$ in the form

$$\sum_{(i,j)\in\mathbb{N}\times\mathbb{N}} a_{ij}.$$

[46]Exercise 3.4.10. Handle the case where each $a_k \geq 0$ separately from the general case.

[47]Exercise 3.4.15. Using properties of the log function, you can view this series as a telescoping one.

[48]Exercise 3.4.16. Consider that

$$\frac{1}{r-1} - \frac{1}{r+1} = \frac{2}{r^2-1}.$$

[49]Exercise 3.4.24. Establish the inequalities

$$\sum_{k=1}^{2^n-1} \frac{1}{k^p} \leq \sum_{k=1}^{\infty} \frac{2^{k-1}}{(2^{k-1})^p}$$

$$= \sum_{j=0}^{\infty} (2^{1-p})^j = \frac{2^{p-1}}{2^{p-1}-1}.$$

Conclude that the partial sums of the p-harmonic series for $p > 1$ are increasing and bounded. Explain now why the series must converge.

[50]Exercise 3.4.26. As a first step show that

$$\int_{2k\pi+\pi/4}^{2k\pi+3\pi/4} \frac{|\sin x|}{x}\,dx$$

$$\geq \frac{1}{\sqrt{2}} \int_{2k\pi+\pi/4}^{2k\pi+3\pi/4} \frac{1}{x}\, dx.$$

(Remember that in calculus an integral $\int_0^\infty$ is interpreted as $\lim_{X\to\infty} \int_0^X$.)

[51]Exercise 3.4.28. Establish that

$$\left| x - \sum_{i=1}^{n} \frac{k_i}{p^i} \right| \leq \frac{1}{p^n}.$$

[52]Exercise 3.5.5. Add up the terms containing p digits in the denominator. Note that our deletions leave only $8 \times 9^{p-1}$ of them. The total sum is bounded by

$$8(1/1 + 9/10 + 9^2/100 + \dots) = 80.$$

[53]Exercise 3.5.8. Instead consider the series

$$\sum_{k=1}^{\infty} [a_k]^+ \ \text{ and } \ \sum_{k=1}^{\infty} [a_k]^-$$

where

$$[X]^+ = \max\{X, 0\}$$

and

$$[X]^- = \max\{-X, 0\}$$

and note that

$$X = [X]^+ - [X]^- \ \text{ and } \ |X| = [X]^+ + [X]^-.$$

[54]Exercise 3.5.15. Use the Cauchy-Schwarz inequality.

[55]Exercise 3.5.16. Use the Cauchy-Schwarz inequality.

[56]Exercise 3.6.3. The answer for (d) is $x < 1/e$.

[57]Exercise 3.6.5. Only one condition is sufficient to supply divergence. Give a proof for that one and counterexamples for the three others. Here is an idea that may help: Let $a_k = 0$ for all values of k except if $k = 2^m$ for some m in which case $a_k = 1/\sqrt{k}$. Note that $\limsup_{k\to\infty} \sqrt{k}a_k = 1$ in this case and that $\sum_{k=1}^{\infty} a_k$ will converge.

[58]Exercise 3.6.22. The exact value of γ, called *Euler's constant*, is not needed in the problem; it is approximately .5772156.

[59]Exercise 3.6.24. The integral test should occur to you while thinking of this problem. Start by checking that

$$\sum_{k=1}^{\infty} F'(k)$$

converges if and only if

$$\lim_{X\to\infty} F(X)$$

exists. Find similar statements for the other series.

[60]Exercise 3.7.13. Imitate the proof of the first part of Theorem 3.49 but arrange for the partial sums to go larger than α before inserting a term q_k. You must take the *first* opportunity to insert q_k when this occurs.

[61]Exercise 3.9.6. The name "Tauberian theorem" was coined by Hardy and Littlewood after a result of Alfred Tauber (1866–1942?). The date of his death is unknown; all that is certain is that he was sent by the Nazis to Theresienstadt concentration camp on June 28, 1942.

[62]Exercise 3.12.5. For (h) consider the series $\sum_{k=1}^{\infty}(s_{k+1} - s_k)/s_{k+1}$ where s_k is the sequence of partial sums of the series given.

[63]Exercise 3.12.6. For (b) use Abel's method and the computation in Exercise 3.2.11. Further treatment of some aspects of trigonometric series may be found in Section 16.8.

[64]Exercise 3.12.7. This is from the 1948 Putnam Mathematical Competition.

[65]Exercise 3.12.8. This is from the 1952 Putnam Mathematical Competition.

[66]Exercise 3.12.9. This is from the 1954 Putnam Mathematical Competition.

[67]Exercise 3.12.10. This is from the 1955 Putnam Mathematical Competition.

[68]Exercise 3.12.11. This is from the 1964 Putnam Mathematical Competition.

[69]Exercise 3.12.12. This is from the 1988 Putnam Mathematical Competition.

[70]Exercise 3.12.13. This is from the 1994 Putnam Mathematical Competition.

[71]Exercise 3.12.14. Problem posed by A. Torchinsky in *Amer. Math. Monthly*, **82** (1975), p. 936.

[72]Exercise 3.12.15. Problem posed by Jan Mycielski in *Amer. Math. Monthly*, **83** (1976), p. 284.

Chapter 4

SETS OF REAL NUMBERS

4.1 Introduction

Modern set theory and the world it has opened to mathematics has its origins in a problem in analysis. A young Georg Cantor in 1870 began to attack a problem given to him by his senior colleague Edward Heine, who worked at the same university. (We shall see Heine playing a key role in some ideas of this chapter too.)

The problem was to determine if the equation

$$\tfrac{1}{2}a_0 + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx) = 0 \tag{1}$$

must imply that all the coefficients of the series, the $\{a_k\}$ and the $\{b_k\}$ are zero. Cantor solved this using the methods of his time. It was a good achievement, but not the one that was to make him famous. What he did next was to ask, as any good mathematician would, whether his result could be generalized. Suppose that the series (1) converges to zero for all x except possibly for those in a given set E. If this set E is very small, then perhaps, the coefficients of the series should also have to be all zero.

The nature of these exceptional sets (nowadays called sets of uniqueness) required a language and techniques that were entirely new. Previously a number of authors had needed a language to describe sets that arose in various problems. What was used at the time was limited, and few interesting examples of sets were available. Cantor went beyond these, introducing a new collection of ideas that are now indispensable to analysis. We shall encounter in this chapter many of the notions that arose then: accumulation points, derived sets, countable sets, dense sets, nowhere dense sets.

Incidentally, Cantor never did finish his problem of describing the sets of uniqueness, as the development of the new set theory was more important and consumed his energies. In fact, the problem remains unsolved, although much interesting information about the nature of sets of uniqueness has been discovered.

The theory of sets that Cantor initiated has proved to be fundamental to all of mathematics. Very quickly the most talented analysts of that time began applying his ideas to the theory of functions, and by now this material is essential to an understanding of the subject. This chapter contains the most basic material. In Chapter 6 we will need some further concepts.

4.2 Points

In our studies of analysis we shall often need to have a language that describes sets of points and the points that belong to them. That language did not develop until late in the nineteenth century, which is why the early mathematicians had difficulty understanding some problems.

For example, consider the set of solutions to an equation

$$f(x) = 0$$

where f is some well-behaved function. In the simplest cases (e.g., if f is a polynomial function) the solution set could be empty or a finite number of points. There is no difficulty there. But in more general settings the solution set could be very complicated indeed. It may have points that are "isolated," points appearing in clusters, or it may contain intervals or merely fragments of intervals. You can see that we even lack the words to describe the possibilities.

The ideas in this section are all very geometric. Try to draw mental images that depict all of these ideas to get a feel for the definitions. The definitions themselves should be remembered but may prove hard to remember without some associated picture.

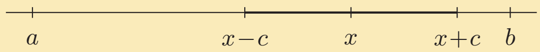

Figure 4.1. Every point in (a, b) is an interior point.

The simplest types of sets are intervals. We call

$$[a, b] = \{x : a \leq x \leq b\}$$

a closed interval, and

$$(a, b) = \{x : a < x < b\}$$

an open interval. The other sets that we often consider are the sets $\mathbb{N}$ of natural numbers, $\mathbb{Q}$ of rational numbers, and $\mathbb{R}$ of all real numbers. Use these in your pictures, as well as sets obtained by combining them in many ways.

4.2.1 Interior Points

Every point inside an open interval $I = (a, b)$ has the feature that there is a smaller open interval centered at that point that is also inside I. Thus if $x \in (a, b)$ then for any positive number c that is small enough

$$(x - c, x + c) \subset (a, b).$$

Indeed the arithmetic to show this is easy (and a picture makes it transparent). Let c be any positive number that is smaller than the shortest distance from x to either a or b. Then $(x - c, x + c) \subset (a, b)$. (See Figure 4.1.)

Note. Often we use the following suggestive language. An open interval that contains a point x is said to be a *neighborhood* of x. Thus each point in (a, b) possesses a neighborhood, indeed many neighborhoods, that lie entirely inside the set I. On occasion the point x itself is excluded from the neighborhood: We say that an interval (c,d) is a neighborhood of x if x belongs to the interval and we say that the set $(c,d)\setminus\{x\}$ is a *deleted neighborhood*. This is just the interval with the point x removed.

We can distinguish between points that are merely in a set and points that are more deeply inside the set. The word chosen to convey this image of "inside" is *interior*.

Definition 4.1: (Interior Point) Let E be a set of real numbers. Any point x that belongs to E is said to be *an interior point of* E provided that some interval

$$(x-c,x+c)\subset E.$$

Thus an interior point of the set E is not merely *in the set* E; it is, so to speak, deep inside the set, at a positive distance at least c away from every point that does not belong to E.

Example 4.2: The following examples are immediate if a picture is sketched.

1. Every point x of an open interval (a,b) is an interior point.
2. Every point x of a closed interval $[a,b]$, except the two endpoints a and b, is an interior point.
3. The set of natural numbers $\mathbb{N}$ has no interior points whatsoever.
4. Every point of $\mathbb{R}$ is an interior point.
5. No point of the set of rational numbers $\mathbb{Q}$ is an interior point. [This is because any interval $(x-c,x+c)$ must contain both rational numbers and irrational numbers and, hence, can never be a subset of $\mathbb{Q}$.]

In each case, we should try to find the interval $(x-c,x+c)$ inside the set or explain why there can be no such interval. ◀

4.2.2 Isolated Points

Most sets that we consider will have infinitely many points. Certainly any interval (a, b) or $[a, b]$ has infinitely many points. The set $\mathbb{N}$ of natural numbers also has infinitely many points, but as we look closely at any one of these points we see that each point is all alone, at a certain distance away from every other point in the set. We call these points *isolated points* of the set.

Definition 4.3: (Isolated Point) Let E be a set of real numbers. Any point x that belongs to E is said to be *an isolated point of* E provided that for some interval $(x - c, x + c)$

$$(x - c, x + c) \cap E = \{x\}.$$

Thus an isolated point of the set E is in the set E but has no close neighbors who are also in E. It is at some positive distance at least c away from every other point that belongs to E.

Example 4.4: As before, the examples are immediate if a picture is sketched.

1. No point x of an open interval (a, b) is an isolated point.
2. No point x of a closed interval $[a, b]$ is an isolated point.
3. Every point belonging to the set of natural numbers $\mathbb{N}$ is an isolated point.
4. No point of $\mathbb{R}$ is isolated.
5. No point of $\mathbb{Q}$ is isolated.

In each case, we should try to find the interval $(x - c, x + c)$ that meets the set at no other point or show that there is none. ◀

4.2.3 Points of Accumulation

Most sets that we consider will have infinitely many points. While the isolated points are of interest on occasion, more than likely we would be interested in points that are not isolated. These points have the property that every containing interval contains many points of the set. Indeed we are interested in any point x with the property that the intervals $(x-c, x+c)$ meet the set E at infinitely many points. This could happen even if x itself does not belong to E. We call these points *accumulation points* of the set. An accumulation point need not itself belong to the set.

Definition 4.5: (Accumulation Point) Let E be a set of real numbers. Any point x (not necessarily in E) is said to be *an accumulation point of* E provided that for every $c>0$ the intersection

$$(x-c, x+c) \cap E$$

contains infinitely many points.

Thus an accumulation point of E is a point that may or may not itself belong to E and that has many close neighbors who are in E.

Note. The definition requires that for all $c>0$ the intersection

$$(x-c, x+c) \cap E$$

contains infinitely many points of E. In checking for an accumulation point it may be preferable merely to check that there is at least one point in this intersection (other than possibly x itself). If there is always at least one point, then there must in fact be infinitely many (Exercise 4.2.18).

Example 4.6: Yet again, the examples are immediate if a picture is sketched.

1. Every point of an open interval (a, b) is a an accumulation point of (a, b). Moreover, the two endpoints a and b are also accumulation points of (a, b) [although they do not belong themselves to (a, b))].

2. Every point of a closed interval $[a, b]$ is an accumulation point of (a, b). No point outside can be.

3. No point at all is an accumulation point of the set of natural numbers $\mathbb{N}$.

4. Every point of $\mathbb{R}$ is an accumulation point.

5. Every point on the real line, both rational and irrational, is an accumulation point of the set $\mathbb{Q}$.

◀

4.2.4 Boundary Points

The intervals (a,b) and $[a,b]$ have what appears to be an "edge". The points a and b mark the boundaries between the inside of the set (i.e., the interior points) and the "outside" of the set. This inside/outside language with an idea of a boundary between them is most useful but needs a precise definition.

Definition 4.7: (Boundary Point) Let E be a set of real numbers. Any point x (not necessarily in E) is said to be *a boundary point of* E provided that every interval $(x-c, x+c)$ contains at least one point of E and also at least one point that does not belong to E.

This definition is easy to apply to the intervals (a,b) and $[a,b]$ but harder to imagine for general sets. For these intervals the only points that are immediately seen to satisfy the definition are the two endpoints that we would have naturally said to be at the boundary.

Example 4.8: The examples are not all transparent but require careful thinking about the definition.

1. The two endpoints a and b are the only boundary points of an open interval (a,b).

2. The two endpoints a and b are the only boundary points of a closed interval $[a,b]$.

3. Every point in the set $\mathbb{N}$ of natural numbers is a boundary point.

4. No point at all is boundary point of the set $\mathbb{R}$.

5. Every point on the real line, both rational and irrational, is a boundary point of the set $\mathbb{Q}$. (Think for a while about this one!)

◀

Exercises

4.2.1 Determine the set of interior points, accumulation points, isolated points, and boundary points for each of the following sets:

(a) $\{1, 1/2, 1/3, 1/4, 1/5, \dots\}$

(b) $\{0\} \cup \{1, 1/2, 1/3, 1/4, 1/5, \dots\}$

(c) $(0, 1) \cup (1, 2) \cup (2, 3) \cup (3, 4) \cup \cdots \cup (n, n+1) \cup \dots$

(d) $(1/2, 1) \cup (1/4, 1/2) \cup (1/8, 1/4) \cup (1/16, 1/8) \cup \dots$

(e) $\{x : |x - \pi| < 1\}$

(f) $\{x : x^2 < 2\}$

(g) $\mathbb{R} \setminus \mathbb{N}$

(h) $\mathbb{R} \setminus \mathbb{Q}$

4.2.2 Give an example of each of the following or explain why you think such a set could not exist.

(a) A nonempty set with no accumulation points and no isolated points

(b) A nonempty set with no interior points and no isolated points

(c) A nonempty set with no boundary points and no isolated points

4.2.3 Show that every interior point of a set must also be an accumulation point of that set, but not conversely.

4.2.4 Show that no interior point of a set can be a boundary point, that it is possible for an accumulation point to be a boundary point, and that every isolated point must be a boundary point.

4.2.5 Let E be a nonempty set of real numbers that is bounded above but has no maximum. Let $x = \sup E$. Show that x is a point of accumulation of E. Is it possible for x to also be an interior point of E? Is x a boundary point of E?

4.2.6 State and solve the version of Exercise 4.2.5 that would use the infimum in place of the supremum.

4.2.7 Let A be a set and $B = \mathbb{R} \setminus A$. Show that every boundary point of A is also a boundary point of B.

4.2.8 Let A be a set and $B = \mathbb{R} \setminus A$. Show that every boundary point of A is a point of accumulation of A or else a point of accumulation of B, perhaps both.

4.2.9 Must every boundary point of a set be also an accumulation point of that set?

4.2.10 Show that every accumulation point of a set that does not itself belong to the set must be a boundary point of that set.

4.2.11 Show that a point x is not an interior point of a set E if and only if there is a sequence of points $\{x_n\}$ converging to x and no point $x_n \in E$.

4.2.12 Let A be a set and $B = \mathbb{R} \setminus A$. Show that every interior point of A is not an accumulation point of B.

4.2.13 Let A be a set and $B = \mathbb{R} \setminus A$. Show that every accumulation point of A is not an interior point of of B.

4.2.14 Give an example of a set that has the set $\mathbb{N}$ as its set of accumulation points.

4.2.15 Show that there is no set which has the interval $(0, 1)$ as its set of accumulation points.

4.2.16 Show that there is no set which has the set $\mathbb{Q}$ as its set of accumulation points.

4.2.17 Give an example of a set that has the set

$$E = \{0\} \cup \{1, 1/2, 1/3, 1/4, 1/5, \dots\}$$

as its set of accumulation points.

4.2.18 Show that a point x is an accumulation point of a set E if and only if for every $\varepsilon > 0$ there are at least two points belonging to the set $E \cap (x - \varepsilon, x + \varepsilon)$.

4.2.19 Suppose that $\{x_n\}$ is a convergent sequence converging to a number L and that $x_n \neq L$ for all n. Show that the set

$$\{x : x = x_n \text{ for some } n\}$$

has exactly one point of accumulation, namely L. Of what importance was the assumption that $x_n \neq L$ for all n for this exercise?

4.2.20 Let E be a set and $\{x_n\}$ a sequence of distinct elements of E. Suppose that $\lim_{n\to\infty} x_n = x$. Show that x is a point of accumulation of E.

4.2.21 Let E be a set and $\{x_n\}$ a sequence of points, not necessarily elements of E. Suppose that $\lim_{n\to\infty} x_n = x$ and that x is an interior point of E. Show that there is an integer N so that $x_n \in E$ for all $n \geq N$.

4.2.22 Let E be a set and $\{x_n\}$ a sequence of elements of E. Suppose that

$$\lim_{n\to\infty} x_n = x$$

and that x is an isolated point of E. Show that there is an integer N so that $x_n = x$ for all $n \geq N$.

4.2.23 Let E be a set and $\{x_n\}$ a sequence of distinct points, not necessarily elements of E. Suppose that $\lim_{n\to\infty} x_n = x$ and that $x_{2n} \in E$ and $x_{2n+1} \notin E$ for all n. Show that x is a boundary point of E.

4.2.24 If E is a set of real numbers, then E', called the *derived set* of E, denotes the set of all points of accumulation of E. Give an example of each of the following or explain why you think such a set could not exist.

(a) A nonempty set E such that $E' = E$

(b) A nonempty set E such that $E' = \emptyset$

(c) A nonempty set E such that $E' \neq \emptyset$ but $E'' = \emptyset$

(d) A nonempty set E such that E', $E'' \neq \emptyset$ but $E''' = \emptyset$

(e) A nonempty set E such that E', E'', E''', ... are all different

(f) A nonempty set E such that $(E \cup E')' \neq (E \cup E')$

4.2.25 Show that there is no set with uncountably many isolated points.

SEE NOTE 73

4.3 Sets

We now begin a classification of sets of real numbers. Almost all of the concepts of analysis (limits, derivatives, integrals, etc.) can be better understood if a classification scheme for sets is in place. By far the most important notions are those of closed sets and open sets. This is the basis for much advanced mathematics and leads to the subject known as topology, which is fundamental to an understanding of many areas of mathematics. On the real line we can master open and closed sets and describe precisely what they are.

4.3.1 Closed Sets

In many parts of mathematics the word "closed" is used to indicate that some operation stays within a system. For example, the set of natural numbers $\mathbb{N}$ is closed under addition and multiplication (any sum or product of two of them is yet another) but not closed under subtraction or division (2 and 3 are natural numbers, but $2-3$ and $3/2$ are not). This same word was employed originally to indicate sets of real numbers that are "closed" under the operation of taking points of accumulation. If all points of accumulation turn out to be in the set, then the set is said to be closed. This terminology has survived and become, perhaps, the best known usage of the word "closed."

Definition 4.9: (Closed) Let E be a set of real numbers. The set E is said to be *closed* provided that every accumulation point of E belongs to the set E.

Thus a set E is not closed if there is some accumulation point of E that does not belong to E. In particular, a set with no accumulation points would have to be closed since there is no point that needs to be checked.

Example 4.10: The examples are immediate since we have previously described all of the accumulation points of these sets.

1. The empty set $\emptyset$ is closed since it contains all of its accumulation points (there are none).

2. The open interval (a, b) not closed because the two endpoints a and b are accumulation points of (a, b) and yet they do not belong to the set.

3. The closed interval $[a, b]$ is closed since only points that are already in the set are accumulation points.

4. The set of natural numbers $\mathbb{N}$ is closed because it has no points of accumulation.

5. The real line $\mathbb{R}$ is closed since it contains all of its accumulation points, namely every point.

6. The set of rational numbers $\mathbb{Q}$ is not closed. Every point on the real line, both rational and irrational, is an accumulation point of $\mathbb{Q}$, but the set fails to contain any irrationals.

◀

The Closure of a Set If a set is not closed it is because it neglects to contain points that "should" be there since they are accumulation points but not in the set. On occasions it is best to throw them in and consider a larger set composed of the original set together with the offending accumulation points that may not have belonged originally to the set.

Definition 4.11: (Closure) Let E be any set of real numbers and let E' denote the set of all accumulation points of E. Then the set

$$\overline{E} = E \cup E'$$

is called the *closure* of the set E.

For example, $\overline{(a,b)} = [a,b]$, $\overline{[a,b]} = [a,b]$, $\overline{\mathbb{N}} = \mathbb{N}$, and $\overline{\mathbb{Q}} = \mathbb{R}$. Each of these is an easy observation since we know what the points of accumulation of these sets are.

4.3.2 Open Sets

Originally, the word "open" was used to indicate a set that was not closed. In time it was realized that this is a waste of terminology, since the class of "not closed sets" is not of much general interest. Instead the word is now used to indicate a contrasting idea, an idea that is not quite an opposite—just at a different extreme. This may be a bit unfortunate since now a set that is not open need not be closed. Indeed some sets can be both open and closed, and some sets can be both not open and not closed.

Definition 4.12: (Open) Let E be a set of real numbers. Then E is said to be *open* if every point of E is also an interior point of E.

Thus every point of E is not merely a point *in the set* E; it is, so to speak, deep inside the set. For each point x_0 of E there is some positive number δ and all points outside E are at least a distance δ away from x_0. Note that this means that an open set cannot contain any of its boundary points.

Example 4.13: These examples are immediate since we have seen them before in the context of interior points in Section 4.2.1.

1. The empty set $\emptyset$ is open since it contains no points that are not interior points of the set. (This is the first example of a set that is both open and closed.)

2. The open interval (a, b) is open since every point x of an open interval (a, b) is an interior point.

3. The closed interval $[a, b]$ is not open since there are points in the set (namely the two endpoints a and b) that are in the set and yet are not interior points.

4. The set of natural numbers $\mathbb{N}$ has no interior points and so this set is not open; all of its points fail to be interior points.

5. Every point of $\mathbb{R}$ is an interior point and so $\mathbb{R}$ is open. (Remember, $\mathbb{R}$ is also closed so it is both open and closed. Note that $\mathbb{R}$ and $\emptyset$ are the only examples of sets that are both open and closed.)

6. No point of the set of rational numbers $\mathbb{Q}$ is an interior point and so $\mathbb{Q}$ definitely fails to be open.

◀

The Interior of a Set If a set is not open it is because it contains points that “shouldn’t” be there since they are not interior. On occasions it is best to throw them away and consider a smaller set composed entirely of the interior points.

Definition 4.14: (Interior) Let E be any set of real numbers. Then the set

$$\text{int}(E)$$

denotes the set of all interior points of E and is called the *interior* of the set E.

For example, $\text{int}((a,b)) = (a,b)$, $\text{int}([a,b]) = (a,b)$, $\text{int}(\mathbb{N}) = \emptyset$, and $\text{int}(\mathbb{Q}) = \emptyset$. Each of these is an easy observation since we know what the interior points of these sets are.

Component Intervals of Open Sets Think of the most general open set G that you can. A first feeble suggestion might be any open interval $G = (a,b)$. We can do a little better. How about the union of two of these

$$G = (a,b) \cup (c,d)?$$

If these are disjoint, then we would tend to think of G as having two "components." It is easy to see that every point is an interior point. We need not stop at two component intervals; any number would work:

$$G = (a_1,b_1) \cup (a_2,b_2) \cup (a_3,b_3) \cup \cdots \cup (a_n,b_n).$$

The argument is the same and elementary. If x is a point in this set, then x is an interior point. Indeed we can form the union of a sequence of such open intervals and it is clear that we shall obtain an open set. For a specific example consider

$$(-\infty,-3) \cup (1/2,1) \cup (1/8,1/4) \cup (1/32,1/16) \cup (1/128,1/64) \cup \ldots.$$

At this point our imagination stalls and it is hard to come up with any more examples that are not obtained by stringing together open intervals in exactly this way. This suggests that, perhaps, all open sets have this structure. They are either open intervals or else a union of a sequence of open intervals. This theorem characterizes all open sets of real numbers and reveals their exact structure.

Theorem 4.15: *Let G be a nonempty open set of real numbers. Then there is a unique sequence (finite or infinite) of disjoint, open intervals*

$$(a_1, b_1), (a_2, b_2), (a_3, b_3), \dots, (a_n, b_n), \dots$$

called the component intervals of G *such that*

$$G = (a_1, b_1) \cup (a_2, b_2) \cup (a_3, b_3) \cup \dots \cup (a_n, b_n) \cup \dots.$$

Proof. Take any point $x \in G$. We know that there must be some interval (a, b) containing the point x and contained in the set G. This is because G is open and so every point in G is an interior point. We need to take the largest such interval. The easiest way to describe this is to write

$$\alpha = \inf\{t : (t, x) \subset G\}$$

and

$$\beta = \sup\{t : (x, t) \subset G\}.$$

Note that $\alpha < x < \beta$. Then

$$I_x = (\alpha, \beta)$$

is called the *component* of G containing the point x. (It is possible here for $\alpha = -\infty$ or $\beta = \infty$.)

One feature of components that we require is this: If x and y belong to the same component, then

$$I_x = I_y$$

If x and y do not belong to the same component, then I_x and I_y have no points in common. This is easily checked (Exercise 4.3.21).

There remains the task of listing the components as the theorem requires. If the collection

$$\{I_x : x \in G\}$$

is finite, then this presents no difficulties. If it is infinite we need a clever strategy.

Let r_1, r_2, r_3, ... be a listing of all the rational numbers contained in the set G. We construct our list of components of G by writing for the first step

$$(a_1, b_1) = I_{r_1}.$$

The second component must be disjoint from this first component. We cannot simply choose I_{r_2} since if r_2 belongs to (a_1, b_1), then in fact

$$(a_1, b_1) = I_{r_1} = I_{r_2}.$$

Instead we travel along the sequence r_1, r_2, r_3, ... until we reach the first one, say r_{m_2}, that does not already belong to the interval (a_1, b_1). This then serves to define our next interval:

$$(a_2, b_2) = I_{r_{m_2}}.$$

If there is no such point, then the process stops. This process is continued inductively resulting in a sequence of open intervals:

$$(a_1, b_1), (a_2, b_2), (a_3, b_3), \dots, (a_n, b_n), \dots,$$

which may be infinite or finite. At the kth stage a point r_{m_k} is selected so that r_{m_k} does not belong to any component thus far selected. If this cannot be done, then the process stops and produces only a finite list of components.

The proof is completed by checking that (i) every point of G is in one of these intervals, (ii) every point in one of these intervals belongs to G, and (iii) the intervals in the sequence must be disjoint.

For (i) note that if $x \in G$, then there must be rational numbers in the component I_x. Indeed there is a first number r_k in the list that belongs to this component. But then $x \in I_{r_k}$ and so we must have chosen this interval I_{r_k} at some stage. Thus x does belong to one of these intervals.

For (ii) note that if x is in G, then $I_x \subset G$. Thus every point in one of the intervals belongs to G.

For (iii) consider some pair of intervals in the sequence we have constructed. The later one chosen was required to have a point r_{m_k} that did not belong to any of the preceding choices. But that means then that the new component chosen is disjoint from all the previous ones.

This completes the checking of the details and so the proof is done. ■

Exercises

4.3.1 Is it true that a set, all of whose points are isolated, must be closed?

SEE NOTE 74

4.3.2 If a set has no isolated points must it be closed? Must it be open?

4.3.3 A careless student, when asked, incorrectly remembers that a set is closed “if all its points are points of accumulation.” Must such a set be closed?

4.3.4 A careless student, when asked, incorrectly remembers that a set is open “if it contains all of its interior points.” Is there an example of a set that fails to have this property? Is there an example of a nonopen set that has this property?

4.3.5 Determine which of the following sets are open, which are closed, and which are neither open nor closed.

(a) $(-\infty, 0) \cup (0, \infty)$

(b) $\{1, 1/2, 1/3, 1/4, 1/5, \dots\}$

(c) $\{0\} \cup \{1, 1/2, 1/3, 1/4, 1/5, \dots\}$

(d) $(0, 1) \cup (1, 2) \cup (2, 3) \cup (3, 4) \cup \cdots \cup (n, n+1) \cup \dots$

(e) $(1/2, 1) \cup (1/4, 1/2) \cup (1/8, 1/4) \cup (1/16, 1/8) \cup \dots$

(f) $\{x : |x - \pi| < 1\}$

(g) $\{x : x^2 < 2\}$

(h) $\mathbb{R} \setminus \mathbb{N}$

(i) $\mathbb{R} \setminus \mathbb{Q}$

4.3.6 Show that the closure operation has the following properties:

(a) If $E_1 \subset E_2$, then $\overline{E_1} \subset \overline{E_2}$.

(b) $\overline{E_1 \cup E_2} = \overline{E_1} \cup \overline{E_2}$.

(c) $\overline{E_1 \cap E_2} \subset \overline{E_1} \cap \overline{E_2}$.

(d) Give an example of two sets E_1 and E_2 such that

$$\overline{E_1 \cap E_2} \neq \overline{E_1} \cap \overline{E_2}.$$

(e) $\overline{\overline{E}} = \overline{E}$.

4.3.7 Show that the interior operation has the following properties:

(a) If $E_1 \subset E_2$, then $\text{int}(E_1) \subset \text{int}(E_2)$.

(b) $\text{int}(E_1 \cap E_2) = \text{int}(E_1) \cap \text{int}(E_2)$.

(c) $\text{int}(E_1 \cup E_2) \supset \text{int}(E_1) \cup \text{int}(E_2)$.

(d) Give an example of two sets E_1 and E_2 such that

$$\text{int}(E_1 \cup E_2) \neq \text{int}(E_1) \cup \text{int}(E_2).$$

(e) $\text{int}(\text{int}(E)) = \text{int}(E)$.

4.3.8 Show that if the set E' of points of accumulation of E is empty, then the set E must be closed.

4.3.9 Show that the set E' of points of accumulation of any set E must be closed.

4.3.10 Show that the set $\text{int}(E)$ of interior points of any set E must be open.

4.3.11 Show that a set E is closed if and only if $\overline{E} = E$.

4.3.12 Show that a set E is open if and only if $\text{int}(E) = E$.

4.3.13 If A is open and B is closed, what can you say about the sets $A \setminus B$ and $B \setminus A$?

4.3.14 If A and B are both open or both closed, what can you say about the sets $A \setminus B$ and $B \setminus A$?

4.3.15 If E is a nonempty bounded, closed set, show that $\max\{E\}$ and $\min\{E\}$ both exist. If E is a bounded, open set, show that neither $\max\{E\}$ nor $\min\{E\}$ exist (although $\sup\{E\}$ and $\inf\{E\}$ do).

4.3.16 Show that if a set of real numbers E has at least one point of accumulation, then for every $\varepsilon > 0$ there exist points $x, y \in E$ so that $0 < |x - y| < \varepsilon$.

4.3.17 Construct an example of a set of real numbers E that has no points of accumulation and yet has the property that for every $\varepsilon > 0$ there exist points $x, y \in E$ so that $0 < |x - y| < \varepsilon$.

4.3.18 Let $\{x_n\}$ be a sequence of real numbers. Let E denote the set of all numbers z that have the property that there exists a subsequence $\{x_{n_k}\}$ convergent to z. Show that E is closed.

4.3.19 Determine the components of the open set $\mathbb{R} \setminus \mathbb{N}$.

4.3.20 Let $F = \{0\} \cup \{1, 1/2, 1/3, 1/4, 1/5, \dots\}$. Show that F is closed and determine the components of the open set $\mathbb{R} \setminus F$.

4.3.21 In the proof of Theorem 4.15 show that if x and y belong to the same component, then $I_x = I_y$, while if x and y do not belong to the same component, then I_x and I_y have no points in common.

4.3.22 In the proof of Theorem 4.15, after obtaining the collection of components $\{I_x : x \in G\}$, there remained the task of listing them. In classroom discussions the following suggestions were made as to how the components might be listed:

(a) List the components from largest to smallest.

(b) List the components from smallest to largest.

(c) List the components from left to right.

(d) List the components from right to left.

For each of these give an example of an open set with infinitely many components for which this strategy would work and also an example where it would fail.

4.3.23 In searching for interesting examples of open sets, you may have run out of ideas. Here is an example of a construction due to Cantor that has become the source for many important examples in analysis. We describe the component intervals of an open set G inside the interval $(0, 1)$. At each "stage" n we shall describe 2^{n-1} components.

At the first stage, stage 1, take $(1/3, 2/3)$ and at stage 2 take $(1/9, 2/9)$ and $(7/9, 8/9)$ and so on so that at each stage we take all the middle third intervals of the intervals remaining inside $(0, 1)$. The set G is the open subset of $(0, 1)$ having these intervals as components.

(a) Describe exactly the collection of intervals forming the components of G.

(b) What are the endpoints of the components. How do they relate to ternary expansions of numbers in $[0, 1]$?

(c) What is the sum of the lengths of all components?

(d) Sketch a picture of the set G by illustrating the components at the first three stages.

(e) Show that if $x,\ y \in G$, $x < y$, but x and y are not in the same component, then there are infinitely many components of G in the interval (x, y).

See Note 75

4.4 Elementary Topology

The study of open and closed sets in any space is called *topology*. Our goal now is to find relations between these ideas and examine the properties of these sets. Much of this is a useful introduction to topology in any space; some is very specific to the real line, where the topological ideas are easier to sort out.

The first theorem establishes the connection between the open sets and the closed sets. They are not quite opposites. They are better described as "complementary."

Theorem 4.16 (Open vs. Closed) *Let A be a set of real numbers and $B = \mathbb{R} \setminus A$ its complement. Then A is open if and only if B is closed.*

Proof. If A is open and B fails to be closed then there is a point z that is a point of accumulation of B and yet is not in B. Thus z must be in A. But if z is a point in an open set it must be an interior point. Hence there is an interval $(z - \delta, z + \delta)$ contained entirely in A; such an interval contains no points of B. Hence z cannot be a point of accumulation of B. This is a contradiction and so we have proved that B must be closed if A is open.

Conversely, if B is closed and A fails to be open, then there is a point $z \in A$ that is not an interior point of A. Hence every interval $(z - \delta, z + \delta)$ must contain points outside of A, namely points in B. By definition this means that z is a point of accumulation of B. But B is closed and so z, which is a point in A, should really belong to B. This is a contradiction and so we have proved that A must be open if B is closed. ■

Theorem 4.17 (Properties of Open Sets) *Open sets of real numbers have the following properties:*

1. *The sets $\emptyset$ and $\mathbb{R}$ are open.*
2. *Any intersection of a finite number of open sets is open.*
3. *Any union of an arbitrary collection of open sets is open.*
4. *The complement of an open set is closed.*

Proof. The first assertion is immediate and the last we have already proved. The third is easy. Thus it is enough for us to prove the second assertion. Let us suppose that E_1 and E_2 are open. To show that $E_1 \cap E_2$ is also open we need to show that every point is an interior point. Let $z \in E_1 \cap E_2$. Then, since z is in both of the sets E_1 and E_2 and both are open there are intervals

$$(z - \delta_1, z + \delta_1) \subset E_1$$

and

$$(z - \delta_2, z + \delta_2) \subset E_2.$$

Let $\delta = \min\{\delta_1, \delta_2\}$. We must then have

$$(z - \delta, z + \delta) \subset E_1 \cap E_2,$$

which shows that z is an interior point of $E_1 \cap E_2$. Since z is any point, this proves that $E_1 \cap E_2$ is open.

Having proved the theorem for two open sets, it now follows for three open sets since

$$E_1 \cap E_2 \cap E_3 = (E_1 \cap E_2) \cap E_3.$$

That any intersection of an arbitrary finite number of open sets is open now follows by induction. ■

Theorem 4.18 (Properties of Closed Sets) *Closed sets of real numbers have the following properties:*

1. *The sets $\emptyset$ and $\mathbb{R}$ are closed.*
2. *Any union of a finite number of closed sets is closed.*
3. *Any intersection of an arbitrary collection of closed sets is closed.*
4. *The complement of a closed set is open.*

Proof. Except for the second assertion these are easy or have already been proved. Let us prove the second one. Let us suppose that E_1 and E_2 are closed. To show that $E_1 \cup E_2$ is also closed we need to show that every accumulation point belongs to that set. Let z be an accumulation point of $E_1 \cup E_2$ that does not belong to the set. Since z is in neither of the closed sets E_1 and E_2, this point z cannot be an accumulation point of either. Thus some interval $(z-\delta, z+\delta)$ contains no points of either E_1 or E_2. Consequently, that interval contains no points of $E_1 \cup E_2$ and z is not an accumulation point after all, contradicting our assumption. Since z is any accumulation point, this proves that $E_1 \cup E_2$ is closed.

Having proved the theorem for two closed sets, it now follows for three closed sets since

$$E_1 \cup E_2 \cup E_3 = (E_1 \cup E_2) \cup E_3.$$

That any union of an arbitrary finite number of closed sets is closed now follows by induction. ■

Exercises

4.4.1 Explain why it is that the sets $\emptyset$ and $\mathbb{R}$ are open and also closed.

4.4.2 Show that a union of an arbitrary collection of open sets is open.

4.4.3 Show that an intersection of an arbitrary collection of closed sets is closed.

4.4.4 Give an example of a sequence of open sets G_1, G_2, G_3, ... whose intersection is neither open nor closed. Why does this not contradict Theorem 4.17?

4.4.5 Give an example of a sequence of closed sets F_1, F_2, F_3, ... whose union is neither open nor closed. Why does this not contradict Theorem 4.18?

4.4.6 Show that the set $\overline{E}$ can be described as the *smallest closed set that contains every point of* E.

SEE NOTE 76

4.4.7 Show that the set $\operatorname{int}(E)$ can be described as the *largest open set that is contained inside* E.

SEE NOTE 77

4.4.8 A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *bounded at a point* x_0 provided that there are positive numbers ε and M so that $|f(x)| < M$ for all $x \in (x_0 - \varepsilon, x_0 + \varepsilon)$. Show that the set of points at which a function is bounded is open. Let E be an arbitrary closed set. Is it possible to construct a function $f : \mathbb{R} \to \mathbb{R}$ so that the set of points at which f is not bounded is precisely the set E?

4.4.9 This exercise continues Exercise 4.3.23. Define the *Cantor ternary set* K to be the complement of the open set G of Exercise 4.3.23 in the interval $[0,1]$.

(a) If all the open intervals up to the nth stage in the construction of G are removed from the interval $[0,1]$, there remains a closed set K_n that is the union of a finite number of closed intervals. How many intervals?

(b) What is the sum of the lengths of these closed intervals that make up K_n?

(c) Show that $K = \bigcap_{n=1}^{\infty} K_n$.

(d) Sketch a picture of the set K by illustrating the sets K_1, K_2, and K_3.

(e) Show that if x, $y \in K$, $x < y$, then there is an open subinterval $I \subset (x,y)$ containing no points of K.

(f) Give an example of a number $z \in K \cap (0,1)$ that is not an endpoint of a component of G.

4.4.10 Express the closed interval $[0,1]$ as an intersection of a sequence of open sets. Can it also be expressed as a union of a sequence of open sets?

4.4.11 Express the open interval $(0,1)$ as a union of a sequence of closed sets. Can it also be expressed as an intersection of a sequence of closed sets?

4.5 Compactness Arguments

✂ Parts of this section could be cut in a short course. For a minimal approach to compactness arguments, you may wish to skip over all but the Bolzano-Weierstrass property. For all purposes of elementary real analysis this is sufficient. Proofs in the sequel that require a compactness argument will be supplied with one that uses the Bolzano-Weierstrass property and, perhaps, another that can be omitted.

In analysis we frequently encounter the problem of arguing from a set of "local" assumptions to a "global" conclusion. Let us focus on just one problem of this type and see the kind of arguments that can be used.

Local Boundedness of a Function Suppose that a function f is *locally bounded* at each point of a set E. By this we mean that for every point $x \in E$ there is an interval $(x - \delta, x + \delta)$ and f is bounded on the points in E that belong to that interval. Can we conclude that f is bounded on the whole of the set E?

Thus we have been given a local condition at each point x in the set E. There must be numbers δ_x and M_x so that

$$|f(t)| \leq M_x \text{ for all } t \in E \text{ in the interval } (x - \delta_x, x + \delta_x).$$

The global condition we want, if possible, is to have some single number M that works for all $t \in E$; that is,

$$|f(t)| \leq M \text{ for all } t \in E.$$

Two examples show that this depends on the nature of the set E.

Example 4.19: The function $f(x) = 1/x$ is locally bounded at each point x in the set $(0, 1)$ but is not bounded on the set $(0, 1)$. It is clear that f cannot be bounded on $(0, 1)$ since the statement

$$\frac{1}{t} \leq M \text{ for all } t \in (0, 1)$$

cannot be true for any M. But this function is locally bounded at each point x here. Let $x \in (0, 1)$. Take $\delta_x = x/2$ and $M_x = 2/x$. Then

$$f(t) = \frac{1}{t} \leq \frac{2}{x} = M_x$$

if

$$x/2 = x - \delta_x < t < x + \delta_x.$$

What is wrong here? What is there about this set $E = (0, 1)$ that does not allow the conclusion? The point 0 is a point of accumulation of $(0, 1)$ that does not belong to $(0, 1)$, and so there is no assumption that f is bounded at that point. We avoid this difficulty if we assume that E is closed. ◀

Example 4.20: The function $f(x) = x$ is locally bounded at each point x in the set $[0, \infty)$ but is not bounded on the set $[0, \infty)$. It is clear that f cannot be bounded on $[0, \infty)$ since the statement

$$f(t) = t \leq M \text{ for all } t \in [0, \infty)$$

cannot be true for any M. But this function is locally bounded at each point x here. Let $x \in [0, \infty)$. Take $\delta_x = 1$ and $M_x = x + 1$. Then

$$f(t) = t \leq x + 1 = M_x$$

if $x - 1 < t < x + 1$.

What is wrong here? What is there about this set $E = [0, \infty)$ that does not allow the conclusion? This set is closed and so contains all of its accumulation points so that the difficulty we saw in the preceding example does not arise. The difficulty is that the set is too big, allowing larger and larger bounds as we move to the right. We could avoid this difficulty if we assume that E is bounded. ◄

Indeed, as we shall see, we have reached the correct hypotheses now for solving our problem. The version of the theorem we were searching for is this:

> **Theorem** *Suppose that a function f is locally bounded at each point of a closed and bounded set E. Then f is bounded on the whole of the set E.*

Arguments that exploit the special features of closed and bounded sets of real numbers are called *compactness arguments*. Most often they are used to prove that some local property has global implications, which is precisely the nature of our boundedness theorem. We now solve our problem using various different compactness arguments. Each of these arguments will become a formidable tool in proving theorems in analysis. Many situations will arise in which some local property must be proved to hold globally, and compactness will play a huge role in these.

4.5.1 Bolzano-Weierstrass Property

A closed and bounded set has a special feature that can be used to design compactness arguments. This property is essentially a repeat of a property about convergent subsequences that we saw in Section 2.11.

Theorem 4.21 (Bolzano-Weierstrass Property) *A set of real numbers E is closed and bounded if and only if every sequence of points chosen from the set has a subsequence that converges to a point that belongs to E.*

Proof. Suppose that E is both closed and bounded and let $\{x_n\}$ be a sequence of points chosen from E. Since E is bounded this sequence $\{x_n\}$ must be bounded too. We apply the Bolzano-Weierstrass theorem for sequences (Theorem 2.40) to obtain a subsequence $\{x_{n_k}\}$ that converges. If $x_{n_k} \to z$ then since all the points of the subsequence belong to E either the sequence is constant after some term or else z is a point of accumulation of E. In either case we see that $z \in E$. This proves the theorem in one direction.

In the opposite direction we suppose that a set E, which we do not know in advance to be either closed or bounded, has the Bolzano-Weierstrass property. Then E cannot be unbounded. For example, if E is unbounded then there is a sequence of points $\{x_n\}$ of E with $x_n \to \infty$ or $-\infty$ and no subsequence of that sequence converges, contradicting the assumption.

Also, E must be closed. If not, there is a point of accumulation z that is not in E. This means that there is a sequence of points $\{x_n\}$ in E converging to z. But any subsequence of $\{x_n\}$ would also converge to z and, since $z \notin E$, we again have a contradiction. ■

This theorem can also be interpreted as a statement about accumulation points.

Corollary 4.22: *A set of real numbers E is closed and bounded if and only if every infinite subset of E has a point of accumulation that belongs to E.*

Let us use the Bolzano-Weierstrass property to prove our theorem about local boundedness.

> **Theorem** *Suppose that a function f is locally bounded at each point of a closed and bounded set E. Then f is bounded on the whole of the set E.*

Proof. **(Bolzano-Weierstrass compactness argument)** To use this argument we will need to construct a sequence of points in E that we can use. Our proof is a proof by contradiction. If f is not bounded on E there must be a sequence of points $\{x_n\}$ chosen from E so that

$$|f(x_n)| > n$$

for all n. If such a sequence could not be chosen, then at some stage, N say, there are no more points with $|f(x_N)| > N$ and N is an upper bound.

By compactness (i.e., by Theorem 4.21) there is a convergent subsequence $\{x_{n_k}\}$ converging to a point $z \in E$. By the local boundedness assumption there is an open interval $(z - \delta, z + \delta)$ and a number M_z so that

$$|f(t)| \leq M_z$$

whenever t is in E and inside that interval. But for all sufficiently large values of k, the point x_{n_k} must belong to the interval $(z - \delta, z + \delta)$. The two statements

$$|f(x_{n_k})| > n_k \text{ and } |f(x_{n_k})| \leq M_z$$

cannot both be true for all large k and so we have reached a contradiction, proving the theorem. ■

4.5.2 Cantor's Intersection Property

A famous compactness argument, one that is used often in analysis, involves the intersection of a *descending* sequence of sets; that is, a sequence with

$$E_1 \supset E_2 \supset E_3 \supset E_4 \supset \dots.$$

What conditions on the sequence will imply that

$$\bigcap_{n=1}^{\infty} E_n \neq \emptyset?$$

Example 4.23: An example shows that some conditions are needed. Suppose that for each $n \in \mathbb{N}$ we let $E_n = (0, 1/n)$. Then

$$E_1 \supset E_2 \supset E_3 \supset \dots,$$

so $\{E_n\}$ is a descending sequence of sets with empty intersection. The same is true of the sequence $F_n = [n, \infty)$. Observe that the sets in the sequence $\{E_n\}$ are bounded (but not closed) while the sets in the sequence $\{F_n\}$ are closed (but not bounded). ◀

In a paper in 1879 Cantor described the following theorem and the role it plays in analysis. He pointed out that variants on this idea had been already used throughout most of that century, notably by Lagrange, Legendre, Dirichlet, Cauchy, Bolzano, and Weierstrass.

Theorem 4.24: *Let $\{E_n\}$ be a sequence of nonempty closed and bounded subsets of real numbers such that $E_1 \supset E_2 \supset E_3 \supset \dots$. Let $E = \bigcap_{n=1}^\infty E_n$. Then E is not empty.*

Proof. For each $i \in \mathbb{N}$ choose $x_i \in E_i$. The sequence $\{x_i\}$ is bounded since every point lies inside the bounded set E_1. Therefore, because of Theorem 4.21, $\{x_i\}$ has a convergent subsequence $\{x_{i_k}\}$. Let z denote that limit. Fix an integer m. Because the sets are descending, $x_{i_k} \in E_m$ for all sufficiently large $k \in \mathbb{N}$. But E_m is closed, from which it follows that $z \in E_m$. This is true for all $m \in \mathbb{N}$, so $z \in E$. ■

Corollary 4.25 (Cantor Intersection Theorem) *Suppose that $\{E_n\}$ is a sequence of nonempty closed subsets of real numbers such that*

$$E_1 \supset E_2 \supset E_3 \supset \dots.$$

If

$$\text{diameter } E_n \to 0,$$

then the intersection

$$E = \bigcap_{n=1}^\infty E_n$$

consists of a single point.

Proof. Here the diameter of a nonempty, closed bounded set E would just be $\max E - \min E$, which exists and is finite for such a set (see Exercise 4.3.15). Since we are assuming that the diameters shrink to zero it follows that, at least for all sufficiently large n, E_n must be bounded.

That $E \neq \emptyset$ follows from Theorem 4.24. It remains to show that E contains only one point. Let $x \in E$ and $y \in \mathbb{R}$, $y \neq x$. Since diameter $E_n \to 0$, there exists $i \in \mathbb{N}$ such that diameter $E_i < |x - y|$. Since $x \in E_i$, y cannot be in E_i. Thus $y \notin E$ and $E = \{x\}$ as required. ■

Now we prove our theorem about local boundedness by using the Cantor intersection property to frame an argument.

> **Theorem** *Suppose that a function f is locally bounded at each point of a closed and bounded set E. Then f is bounded on the whole of the set E.*

Proof. **(Cantor intersection compactness argument).** To use this argument we will need to construct a sequence of closed and bounded sets shrinking to a point. Our proof is again a proof by contradiction. Suppose that f is not bounded on E.

Since E is bounded we may assume that E is contained in some interval $[a, b]$. Divide that interval in half, forming two subintervals of the same length, namely $(b-a)/2$. At least one of these intervals contains points of E and f is unbounded on that interval. Call it $[a_1, b_1]$.

Now do the same to the interval $[a_1, b_1]$. Divide that interval in half, forming two subintervals of the same length, namely $(b-a)/4$. At least one of these intervals contains points of E and f is unbounded on that interval. Call it $[a_2, b_2]$. Continue this process inductively, producing a descending sequence of intervals $\{[a_n, b_n]\}$ so that the nth interval $[a_n, b_n]$ has length $(b-a)/2^n$, contains points of E, and f is unbounded on $E \cap [a_n, b_n]$.

By the Cantor intersection property there is a single point $z \in E$ contained in all of these intervals. But by our local boundedness assumption there is an interval $(z-c, z+c)$ so that f is bounded on the points of E in that interval. For any large enough value of n, though, the interval $[a_n, b_n]$ would be contained inside the interval $(z-c, z+c)$. This would be impossible and so we have reached a contradiction, proving the theorem. ■

4.5.3 Cousin's Property

ich.

Another compactness argument dates back to Pierre Cousin in the last years of the nineteenth century. This exploits the order of the real line and considers how small intervals may be pieced together to give larger intervals. The larger interval $[a, b]$ is subdivided

$$a = x_0 < x_1 < \cdots < x_n = b$$

and then expressed as a finite union of nonoverlapping subintervals said to form a *partition*:

$$[a,b] = \bigcup_{i=1}^{n} [x_{i-1}, x_i].$$

This again provides us with a compactness argument since it allows a way to argue from the local to the global.

Lemma 4.26 (Cousin) *Let $\mathcal{C}$ be a collection of closed subintervals of $[a,b]$ with the property that for each $x \in [a,b]$ there exists $\delta = \delta(x) > 0$ such that $\mathcal{C}$ contains all intervals $[c,d] \subset [a,b]$ that contain x and have length smaller than δ. Then there exists a partition*

$$a = x_0 < x_1 < \cdots < x_n = b$$

of $[a,b]$ such that $[x_{i-1}, x_i] \in \mathcal{C}$ for $i = 1, \ldots, n$.

This lemma makes precise the statement that if a collection of closed intervals contains all "sufficiently small" ones for $[a,b]$, then it contains a partition of $[a,b]$. We shall frequently see the usefulness of such a partition. This is the most elementary of a collection of tools called *covering theorems*. Roughly, a *cover* of a set is a family of intervals covering the set in the sense that each point in the set is contained in one or more of the intervals.

We formalize the assumption in Cousin's lemma in this language:

Definition 4.27: (Cousin Cover) A collection $\mathcal{C}$ of closed intervals satisfying the hypothesis of Cousin's lemma is called a *Cousin cover* of $[a,b]$.

Proof. **(Proof of Cousin's lemma)** Let us, in order to obtain a contradiction, suppose that $\mathcal{C}$ does not contain a partition of the interval $[a,b]$. Let c be the midpoint of that interval and consider the two subintervals $[a,c]$ and $[c,b]$. If $\mathcal{C}$ contains a partition of both intervals $[a,c]$ and $[c,b]$, then by putting those partitions together we can obtain a partition of $[a,b]$, which we have supposed is impossible.

Let $I_1 = [a,b]$ and let I_2 be either $[a,c]$ or $[c,b]$ chosen so that $\mathcal{C}$ contains no partition of I_2. Inductively we can continue in this fashion, obtaining a shrinking sequence of intervals $I_1 \supset I_2 \supset I_3 \supset \dots$ so that the length of I_n is $(b-a)/2^{n-1}$ and $\mathcal{C}$ contains no partition of I_n.

By the Cantor intersection theorem (Theorem 4.25) there is a single point z in all of these intervals.

For sufficiently large n, the interval I_n contains z and has length smaller than $\delta(z)$. Thus, by definition, $I_n \in \mathcal{C}$. In particular, $\mathcal{C}$ does indeed contain a partition of that interval I_n since the single interval $\{I_n\}$ is itself a partition. But this contradicts the way in which the sequence was chosen and this contradiction completes our proof. ■

Now we reprove our theorem about local boundedness by using Cousin's property to frame an argument.

> **Theorem** *Suppose that a function f is locally bounded at each point of a closed and bounded set E. Then f is bounded on the whole of the set E.*

Proof. **(Cousin compactness argument)** The set E is bounded and so is contained in some interval $[a,b]$. Let us say that an interval $[c,d] \subset [a,b]$ is "black" if the following statement is true:

> There is a number M (which may depend on $[c,d]$) so that $|f(t)| \le M$ for all $t \in E$ that are in the interval $[c,d]$.

The collection of all black intervals is a Cousin cover of $[a,b]$. This is because of the local boundedness assumption on f. Consequently, by Cousin's lemma, there is a partition of the interval $[a,b]$ consisting of black intervals. The function f is bounded in E on each of these finitely many black intervals and so, since there are only finitely many of them, f must be bounded on E in $[a,b]$. But $[a,b]$ includes all of E and so the proof is complete. ■

4.5.4 Heine-Borel Property

Another famous compactness property involves covers too, as in the Cousin lemma, but this time covers consisting of open intervals. This theorem has wide applications, including again extensions of local properties to global ones. You may find this compactness argument more difficult to work with than the others. On the real line all of the arguments here are equivalent and, in most cases, any one will do the job. Why not use the simpler ones then? The answer is that in more general spaces than the real line these other versions may be more useful. Time spent learning them now will pay off in later courses.

The property we investigate is named after two mathematicians, Émile Borel (1871–1956) and Heinrich Eduard Heine (1821–1881), whose names have become closely attached to these ideas.

We begin with some definitions.

Definition 4.28: (Open Cover) Let $A \subset \mathbb{R}$ and let $\mathcal{U}$ be a family of open intervals. If for every $x \in A$ there exists at least one interval $U \in \mathcal{U}$ such that $x \in U$, then $\mathcal{U}$ is called an *open cover* of A.

Definition 4.29: (Heine-Borel Property) A set $A \subset \mathbb{R}$ is said to have the *Heine-Borel property* if every open cover of A can be reduced to a finite subcover. That is, if $\mathcal{U}$ is an open cover of A, then there exists a finite subset of $\mathcal{U}$, $\{U_1, U_2, \dots, U_n\}$ such that

$$A \subset U_1 \cup U_2 \cup \cdots \cup U_n.$$

Example 4.30: Any finite set has the Heine-Borel property. Just take one interval from the cover for each element in the finite set. ◀

Example 4.31: The set $\mathbb{N}$ does not have the Heine-Borel property. Take, for example, the collection of open intervals

$$\{(0, n) : n = 1, 2, 3, \dots\}.$$

While this forms an open cover of $\mathbb{N}$, no finite subcollection could also be an open cover. ◀

Example 4.32: The set $A = \{1/n : n \in \mathbb{N}\}$ does not have the Heine-Borel property. Take, for example, the collection of open intervals

$$\{(1/n, 2) : n = 1, 2, 3, \dots\}.$$

While this forms an open cover of A, no finite subcollection could also be an open cover. ◀

Observe in these examples that $\mathbb{N}$ is closed (but not bounded) while A is bounded (but not closed). We shall prove, in Theorem 4.33, that a set A has the Heine-Borel property if and only if that set is both closed and bounded.

Theorem 4.33 (Heine-Borel) *A set $A \subset \mathbb{R}$ has the Heine-Borel property if and only if A is both closed and bounded.*

Proof. Suppose $A \subset \mathbb{R}$ is both closed and bounded, and $\mathcal{U}$ is an open cover for A. We may assume $A \neq \emptyset$, otherwise there is nothing to prove. Let $[a, b]$ be the smallest closed interval containing A; that is,

$$a = \inf\{x : x \in A\} \text{ and } \ b = \sup\{x : x \in A\}.$$

Observe that $a \in A$ and $b \in A$. We shall apply Cousin's lemma to the interval $[a, b]$, so we need to first define an appropriate Cousin cover of $[a, b]$.

For each $x \in A$, since $\mathcal{U}$ is an open cover of A, there exists an open interval $U_x \in \mathcal{U}$ such that $x \in U_x$. Since U_x is open, there exists $\delta(x) > 0$ for which $(x-t, x+t) \subset U_x$ for all $t \in (0, \delta(x))$. This defines $\delta(x)$ for points in A. Now consider points in $V = [a, b] \setminus A$. We must define $\delta(x)$ for points of V. Since A is closed and $\{a, b\} \subset A$, V is open (why?); thus for each $x \in V$ there exists $\delta(x) > 0$ such that $(x-t, x+t) \subset V$ for all $t \in (0, \delta(x))$. We can therefore obtain a Cousin cover $\mathcal{C}$ of $[a, b]$ as follows: An interval $[c, d]$ is a member of $\mathcal{C}$ if there exists $x \in [a, b]$ such that either (i) $x \in A$ and $x \in [c, d] \subset U_x$ or (ii) $x \in V$ and $x \in [c, d] \subset V$.

Observe that an interval of type (i) can contain points of V, but an interval of type (ii) cannot contain points of A. Figure 4.2 illustrates examples of both types of intervals. In that figure $[c, d] \subset U_x$ is an interval of type (i) in $\mathcal{C}$; $[c', d'] \subset V$ is an interval of type (ii) in $\mathcal{C}$.

It is clear that $\mathcal{C}$ forms a Cousin cover of $[a, b]$. From Cousin's lemma we infer the existence of a partition $a = x_0 < x_1 < \cdots < x_n = b$ with $[x_{i-1}, x_i] \in \mathcal{C}$ for $i = 1, \ldots, n$. Each of the intervals $[x_{i-1}, x_i]$ is either contained in V (in which case it is disjoint from A) or is contained in some member $U_i \in \mathcal{U}$. We now "throw away" from the partition those intervals that contain only points of V, and the union of the remaining closed intervals covers all of A. Each interval of this finite collection is contained in some open interval U from the cover $\mathcal{U}$. More precisely, let

$$S = \{i : 1 \leq i \leq n \text{ and } [x_{i-1}, x_i] \subset U_i\}.$$

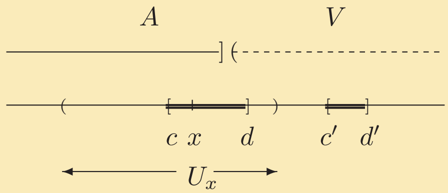

Figure 4.2. The two types of intervals in the proof of Theorem 4.33.

Then

$$A \subset \bigcup_{i \in S} [x_{i-1}, x_i] \subset \bigcup_{i \in S} U_i,$$

so

$$\{U_i : i \in S\}$$

is the required subcover of A.

To prove the converse, we must show that if A is not bounded or if A is not closed, then there exists an open cover of A with no finite subcover. Suppose first that A is not bounded. Consider the family of open intervals

$$\mathcal{U} = \{(-n, n) : n \in \mathbb{N}\}.$$

Clearly $\mathcal{U}$ is an open cover of A. (Indeed it is an open cover of all of $\mathbb{R}$.) But it is also clear that $\mathcal{U}$ contains no finite subcover of A since a finite subcover will cover only a bounded set and we have assumed that A is unbounded.

Now suppose A is not closed. Then there is a point of accumulation z of A that does not belong to A. Consider the family of open intervals

$$\mathcal{U} = \left\{ \left(-\infty, z - \frac{1}{n} \right) : n \in \mathbb{N} \right\} \cup \left\{ \left(z + \frac{1}{n}, \infty \right) : n \in \mathbb{N} \right\}.$$

Clearly $\mathcal{U}$ is an open cover of A. (Indeed it is an open cover of all of $\mathbb{R} \setminus \{z\}$.) But it is also clear that $\mathcal{U}$ contains no finite subcover of A since a finite subcover contains no points of the interval $(z-c, z+c)$ for some small positive c and yet, since z is an accumulation point of A, this interval must contain infinitely many points of A. ■

Once again, we return to our sample theorem, which shows how a local property can be used to prove a global condition, this time using a Heine-Borel compactness argument.

> **Theorem** *Suppose that a function f is locally bounded at each point of a closed and bounded set E. Then f is bounded on the whole of the set E.*

Proof. **(Heine-Borel compactness argument).** As f is locally bounded at each point of E, for every $x \in E$ there exists an open interval U_x containing x and a positive number M_x such that $|f(t)| < M_x$ for all $t \in U_x \cap E$. Let

$$\mathcal{U} = \{U_x : x \in E\}.$$

Then $\mathcal{U}$ is an open cover of E. By the Heine-Borel theorem there exists

$$\{U_{x_1}, U_{x_2}, \dots, U_{x_n}\}$$

such that

$$E \subset U_{x_1} \cup U_{x_2} \cup \cdots \cup U_{x_n}.$$

Let

$$M = \max\{M_{x_1}, M_{x_2}, \dots, M_{x_n}\}.$$

Let $x \in E$. Then there exists i, $1 \leq i \leq n$, for which $x \in U_{x_i}$. Since

$$|f(x)| \leq M_{x_i} \leq M$$

we conclude that f is bounded on E. ■

Our ability to reduce $\mathcal{U}$ to a *finite* subcover in the proof of this theorem was crucial. You may wish to use the function $f(x) = 1/x$ on $(0, 1]$ to appreciate this statement.

4.5.5 Compact Sets

We have seen now a wide range of techniques called compactness arguments that can be applied to a set that is closed and bounded. We now introduce the modern terminology for such sets.

Definition 4.34: A set of real numbers E is said to be *compact* if it has any of the following equivalent properties:

1. E is closed and bounded.
2. E has the Bolzano-Weierstrass property.
3. E has the Heine-Borel property.

In spaces more general than the real line there may be analogues of the notions of closed, bounded, convergent sequences, and open covers. Thus there can also be analogues of closed and bounded sets, the Bolzano-Weierstrass property, and the Heine-Borel property. In these more general spaces the three properties are not always equivalent and it is the Heine-Borel property that is normally chosen as the definition of compact sets there. Even so, a thorough understanding of compactness arguments on the real line is an excellent introduction to these advanced and important ideas in other settings.

If we return to our sample theorem we see that now, perhaps, it should best be described in the language of compact sets:

> **Theorem** *Suppose that E is compact. Then every function $f : E \to \mathbb{R}$ that is locally bounded on E is bounded on the whole of the set E. Conversely, if every function $f : E \to \mathbb{R}$ that is locally bounded on E is bounded on the whole of the set E, then E must be compact.*

In real analysis there are many theorems of this type. The concept of compact set captures exactly when many local conditions can have global implications.

Exercises

4.5.1 Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ that is not locally bounded at any point.

See Note 78

4.5.2 Show directly that the interval $[0, \infty)$ does not have the Bolzano-Weierstrass property.

4.5.3 ✂ Show directly that the interval $[0, \infty)$ does not have the Heine-Borel property.

4.5.4 ✂ Show directly that the set $[0, 1] \cap \mathbb{Q}$ does not have the Heine-Borel property.

4.5.5 Develop the properties of compact sets: (i) is the union of a pair of compact sets compact? (ii) is the union of a finite sequence of compact sets compact? (iii) is the union of a sequence of compact sets compact? Do the same for intersections.

See Note 79

4.5.6 Show directly that the union of two sets with the Bolzano-Weierstrass property must have the Bolzano-Weierstrass property.

4.5.7 ✂ Show directly that the union of two sets with the Heine-Borel property must have the Heine-Borel property.

4.5.8 We defined an open cover of a set E to consist of open *intervals* covering E. Let us change that definition to allow an open cover to consist of any family of open *sets* covering E. What changes are needed in the proof of Theorem 4.33 so that it remains valid in this greater generality?

See Note 80

4.5.9 A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *locally increasing* at a point x_0 if there is a $\delta > 0$ so that

$$f(x) < f(x_0) < f(y)$$

whenever

$$x_0 - \delta < x < x_0 < y < x_0 + \delta.$$

Show that a function that is locally increasing at every point in $\mathbb{R}$ must be increasing; that is, that $f(x) < f(y)$ for all $x < y$.

See Note 81

4.5.10 Let $f : E \to \mathbb{R}$ have this property: For every $e \in E$ there is an $\varepsilon > 0$ so that

$$f(x) > \varepsilon \text{ if } x \in E \cap (e - \varepsilon, e + \varepsilon).$$

Show that if the set E is compact then there is some positive number c so that

$$f(e) > c$$

for all $e \in E$. Show that if E is not closed or is not bounded, then this conclusion may not be valid.

4.5.11 Prove the following variant of Lemma 4.26:

> Let $\mathcal{C}$ be a collection of closed subintervals of $[a, b]$ with the property that for each $x \in [a, b]$ there exists $\delta = \delta(x) > 0$ such that $\mathcal{C}$ contains all intervals $[c, d] \subset [a, b]$ that contain x and have length smaller than δ. Suppose that $\mathcal{C}$ has the property that if $[\alpha, \beta]$ and $[\beta, \gamma]$ both belong to $\mathcal{C}$ then so too does $[\alpha, \gamma]$. Then $[a, b]$ belongs to $\mathcal{C}$.

4.5.12 Use the version of Cousin's lemma given in Exercise 4.5.11 to give a simpler proof of the sample theorem on local boundedness.

4.5.13 ✂ Give an example of an open covering of the set $\mathbb{Q}$ of rational numbers that does not reduce to a finite subcover.

4.5.14 Suppose that E is closed and K is compact. Show that $E \cap K$ is compact. Do this in two ways (using the "closed + bounded" definition and also using the Bolzano-Weierstrass property).

4.5.15 Prove that every function $f : E \to \mathbb{R}$ that is locally bounded on E is bounded on the whole of the set E only if the set E is compact, by supplying the following two constructions:

(a) Show that if the set E is not bounded, then there is an unbounded function $f : E \to \mathbb{R}$ so that f is locally bounded on E.

(b) Show that if the set E is not closed, then there is an unbounded function $f : E \to \mathbb{R}$ so that f is locally bounded on E.

4.5.16 ✂ Suppose that E is closed and K is compact. Show that $E \cap K$ is compact using the Heine-Borel property.

4.5.17 Suppose that E is compact. Is the set of boundary points of E also compact?

4.5.18 ✂ Prove Lindelöff's covering theorem:

> *Let $\mathcal{C}$ be a collection of open intervals such that every point of a set E belongs to at least one of the intervals. Then there is a sequence of intervals I_1, I_2, I_3, ... chosen from $\mathcal{C}$ that also covers E.*

SEE NOTE 82

4.5.19 Describe briefly the distinction between the covering theorem of Lindelöff (expressed in Exercise 4.5.18) and that of Heine-Borel.
See Note 83

4.5.20 ✂ We have seen that the following four conditions on a set $A \subset \mathbb{R}$ are equivalent:

(a) A is closed and bounded.
(b) Every infinite subset of A has a limit point in A.
(c) Every sequence of points from A has a subsequence converging to a point in A.
(d) Every open cover of A has a finite subcover.

Prove directly that (b)⇒(c), (b)⇒(d) and (c)⇒(d).
See Note 84

4.5.21 Let f be a function that is locally bounded on a compact interval $[a, b]$. Let

$$S = \{a < x \leq b : \ f \text{ is bounded on } [a, x]\}.$$

(a) Show that $S \neq \emptyset$.
(b) Show that if $z = \sup S$, then $a < z \leq b$.
(c) Show that $z \in S$.
(d) Show that $z = b$ by showing that $z < b$ is impossible.

Using these steps, construct a proof of the sample theorem on local boundedness.

4.6 Countable Sets

As part of our discussion of properties of sets in this chapter let us review a special property of sets that relates, not to their topological properties, but to their size. We can divide sets into finite sets and infinite sets. How do we divide infinite sets into "large" and "larger" infinite sets?

We did this in our discussion of sequences in Section 2.3. (If you skipped over that section now is a good time to go back.) If an infinite set E has the property that the elements of E can be written as a list (i.e., as a sequence)

$$e_1, e_2, e_3, \dots, e_n, \dots,$$

then that set is said to be *countable*. Note that this property has nothing particularly to do with the other properties of sets encountered in this chapter. It is yet another and different way of classifying sets.

The following properties review our understanding of countable sets. Remember that the empty set, any finite set, and any infinite set that can be listed are all said to be countable. An infinite set that cannot be listed is said to be *uncountable*.

Theorem 4.35: *Countable sets have the following properties:*

1. *Any subset of a countable set is countable.*
2. *Any union of a sequence of countable sets is countable.*
3. *No interval is countable.*

Exercises

4.6.1 Give examples of closed sets that are countable and closed sets that are uncountable.

4.6.2 Is there a nonempty open set that is countable?

4.6.3 If a set is countable, what can you say about its complement?

4.6.4 Is the intersection of two uncountable sets uncountable?

4.6.5 Show that the Cantor set of Exercise 4.3.23 is infinite and uncountable.
SEE NOTE 85

4.6.6 Give (if possible) an example of a set with

(a) Countably many points of accumulation

(b) Uncountably many points of accumulation

(c) Countably many boundary points

(d) Uncountably many boundary points

(e) Countably many interior points

(f) Uncountably many interior points

4.6.7 A set is said to be *co-countable* if it has a countable complement. Show that the intersection of finitely many co-countable sets is itself co-countable.

4.6.8 Let E be a set and $f : \mathbb{R} \to \mathbb{R}$ be an *increasing* function [i.e., if $x < y$, then $f(x) < f(y)$]. Show that E is countable if and only if the image set $f(E)$ is countable. (What property other than "increasing" would work here?)

4.6.9 Show that every uncountable set of real numbers has a point of accumulation.

SEE NOTE 86

4.6.10 Let $\mathcal{F}$ be a family of (nondegenerate) intervals; that is, each member of $\mathcal{F}$ is an interval (open, closed or neither) but is not a single point. Suppose that any two intervals I and J in the family have no point in common. Show that the family $\mathcal{F}$ can be arranged in a sequence $I_1, I_2, \ldots$.

SEE NOTE 87

4.7 Challenging Problems for Chapter 4

4.7.1 Cantor, in 1885, defined a set E to be *dense-in-itself* if $E \subset E'$. Develop some facts about such sets. Include illustrative examples.

4.7.2 One of Cantor's early results in set theory is that for every closed set E there is a set S with $E = S'$. Attempt a proof.

4.7.3 Can the closed interval $[0, 1]$ be expressed as the union of a sequence of disjoint closed subintervals each of length smaller than 1?

4.7.4 In many applications of open sets and closed sets we wish to work just inside some other set A. It is convenient to have a language for this. A set $E \subset A$ is said to be *open relative to* A if $E = A \cap G$ for some set $G \subset \mathbb{R}$ that is open. A set $E \subset A$ is said to be *closed relative to* A if $E = A \cap F$ for some set $F \subset \mathbb{R}$ that is closed. Answer the following questions.

(a) Let $A = [0, 1]$ describe, if possible, sets that are open relative to A but not open as subsets of $\mathbb{R}$.

(b) Let $A = [0, 1]$ describe, if possible, sets that are closed relative to A but not closed as subsets of $\mathbb{R}$.

(c) Let $A = (0, 1)$ describe, if possible, sets that are open relative to A but not open as subsets of $\mathbb{R}$.

(d) Let $A = (0, 1)$ describe, if possible, sets that are closed relative to A but not closed as subsets of $\mathbb{R}$.

4.7.5 Let $A = \mathbb{Q}$. Give examples of sets that are neither open nor closed but are both relative to $\mathbb{Q}$.

4.7.6 Show that all the subsets of $\mathbb{N}$ are both open and closed relative to $\mathbb{N}$.

4.7.7 Introduce for any set $E \subset \mathbb{R}$ the notation

$$\partial E = \{x : \ x \text{ is a boundary point of } E\}.$$

(a) Show for any set E that $\partial E = \overline{E} \cap \overline{(\mathbb{R} \setminus E)}$.

(b) Show that for any set E the set ∂E is closed.

(c) For what sets E is it true that $\partial E = \emptyset$?

(d) Show that $\partial E \subset E$ for any closed set E.

(e) If E is closed, show that $\partial E = E$ if and only if E has no interior points.

(f) If E is open, show that ∂E can contain no interval.

4.7.8 Let E be a nonempty set of real numbers and define the function

$$f(x) = \inf\{|x - e| : e \in E\}.$$

(a) Show that $f(x) = 0$ for all $x \in E$.

(b) Show that $f(x) = 0$ if and only if $x \in \overline{E}$.

(c) Show for any nonempty closed set E that

$$\{x \in \mathbb{R} : f(x) > 0\} = (\mathbb{R} \setminus E).$$

4.7.9 Let $f : \mathbb{R} \to \mathbb{R}$ have this property: For every $x_0 \in \mathbb{R}$ there is a $\delta > 0$ so that

$$|f(x) - f(x_0)| < |x - x_0|$$

whenever $0 < |x - x_0| < \delta$. Show that for all x, $y \in \mathbb{R}$, $x \neq y$,

$$|f(x) - f(y)| < |x - y|.$$

4.7.10 Let $f : E \to \mathbb{R}$ have this property: For every $e \in E$ there is an $\varepsilon > 0$ so that

$$f(x) > \varepsilon \text{ if } x \in E \cap (e - \varepsilon, e + \varepsilon).$$

Show that if the set E is compact, then there is some positive number c so that

$$f(e) > c$$

for all $e \in E$. Show that if E is not closed or is not bounded, then this conclusion may not be valid.

4.7.11 **(Separation of Compact Sets)** Let A and B be nonempty sets of real numbers and let

$$\delta(A, B) = \inf\{|a - b| : a \in A, b \in B\}.$$

$\delta(A, B)$ is often called the "distance" between the sets A and B.

(a) Prove $\delta(A, B) = 0$ if $A \cap B \neq \emptyset$.

(b) Give an example of two closed, disjoint sets in $\mathbb{R}$ for which $\delta(A, B) = 0$.

(c) Prove that if A is compact, B is closed, and $A \cap B = \emptyset$, then $\delta(A, B) > 0$.

SEE NOTE 88

4.7.12 Show that every closed set can be expressed as the intersection of a sequence of open sets.

4.7.13 Show that every open set can be expressed as the union of a sequence of closed sets.

4.7.14 A collection of sets $\{S_\alpha : \alpha \in A\}$ is said to have the *finite intersection property* if every finite subfamily has a nonempty intersection.

(a) Show that if $\{S_\alpha : \alpha \in A\}$ is a family of compact sets that has the finite intersection property, then

$$\bigcap_{\alpha \in A} S_\alpha \neq \emptyset.$$

(b) Give an example of a collection of closed sets $\{S_\alpha : \alpha \in A\}$ that has the finite intersection property and yet

$$\bigcap_{\alpha \in A} S_\alpha = \emptyset.$$

4.7.15 A set $S \subset \mathbb{R}$ is said to be *disconnected* if there exist two disjoint open sets U and V each containing a point of S so that $S \subset U \cup V$. A set that is not disconnected is said to be *connected*.

(a) Give an example of a disconnected set.

(b) Show that every compact interval $[a, b]$ is connected.

(c) Show that $\mathbb{R}$ is connected.

(d) Show that every nonempty connected set is an interval.

4.7.16 Show that the only subsets of $\mathbb{R}$ that are both open and closed are $\emptyset$ and $\mathbb{R}$.

4.7.17 Given any uncountable set of real numbers E show that it is possible to extract a sequence $\{a_k\}$ of distinct terms of E so that the series $\sum_{k=1}^{\infty} a_k/k$ diverges.

Notes

[73]Exercise 4.2.25. Let $\{q_n\}$ be an enumeration of the rationals. If x is isolated, then there is an open interval I_x containing x and containing no other point of the set. Pick the least integer n so that $q_n \in I_x$. This associates integers with the isolated points in a set.

[74]Exercise 4.3.1. Consider the set $\{1/n : n \in \mathbb{N}\}$.

[75]Exercise 4.3.23. The ternary expansion of a number $x \in [0, 1]$ is given as

$$x = 0.a_1a_2a_3a_4 \cdots = \sum_{i=1}^{\infty} a_i/3^i$$

where the $a_i \in \{0,1,2\}$. (Thus this is merely the "base 3" version of a decimal expansion.) Observe that $1/3$ and $2/3$ can be expressed as $0.0222222\ldots$ and $0.200000\ldots$ in ternary. Observe that each number in the interval $(1/3, 2/3)$, that is the first stage component of G, must be written as $0.1a_2a_3a_4\ldots$ in ternary. How might this lead to a description of the points in G?

[76]Exercise 4.4.6. Consider the intersection of the family of *all* closed sets that contain the set E.

[77]Exercise 4.4.7. Consider the union of the family of *all* open sets that are contained in the set E.

[78]Exercise 4.5.1. Try this one: Define $f(x) = 0$ for x irrational and $f(x) = q$ if $x = p/q$ where p/q is a rational with p, q integers and with no common factors.

[79]Exercise 4.5.5. Take compact to mean closed and bounded. Show that a finite union or arbitrary intersection of compact sets is again compact. Check that an arbitrary union of compact sets need not be compact. Show that any closed subset of a compact set is compact. Show that any finite set is compact.

[80]Exercise 4.5.8. For a course in functions of one variable open covers can consist of intervals. In more general settings there may be nothing that corresponds to an "interval;" thus the more general covering by open sets is needed. Your task is just to look through the proof and spot where an "open interval" needs to be changed to an "open set."

[81]Exercise 4.5.9. Cousin's lemma offers the easiest proof, although any other compactness argument would work. Take the family of all intervals $[c, d]$ for which $f(c) < f(d)$ and check that the hypotheses of that lemma hold on any interval $[x, y]$.

[82]Exercise 4.5.18. Let $\mathcal{C} = \{V_\alpha : \alpha \in A\}$ be the open cover. Let N_1, N_2, ... be a listing of all open intervals with rational endpoints. For each $x \in E$ there is a V_α and a k so that the interval N_k satisfies $x \in N_k \subset V_\alpha$. Call this choice $k(x)$. Thus

$$\mathcal{N} = \{N_{k(x)} : x \in E\}$$

is a countable open cover of E (but not the countable open cover that we want). But corresponding to each member of $\mathcal{N}$ is a member of $\mathcal{C}$ that contains it. Using that correspondence we construct the countable subcollection of $\mathcal{C}$ that forms a cover of E.

[83]Exercise 4.5.19. Lindelöff's theorem asserts that an open cover of any set of reals can be reduced to a countable

subcover. The Heine-Borel theorem asserts that an open cover of any compact set of reals can be reduced to a finite subcover.

[84]Exercise 4.5.20. For (b)⇒(d) and for (c)⇒(d). Suppose that there is an open cover of A but no finite subcover. Step 1: You may assume that the open cover is just a sequence of open sets. (This is because of Exercise 4.5.18.) Step 2: You may assume that the open cover is an increasing sequence of open sets $G_1 \subset G_2 \subset G_3 \subset \dots$ (just take the union of the first terms in the sequence you were given). Step 3: Now choose points x_i to be in $G_i \cap A$ but not in any previous G_j for $j < i$. Step 4: Now apply (b) [or (c)] to get a point $z \in A$ that is an accumulation point of the points x_i. This would have to be a point in some set G_N (since these cover A) but for $n > N$ none of the points x_n can belong to G_N.

[85]Exercise 4.6.5. This result may seem surprising at first since the Cantor set, at first sight, seems to contain only the endpoints of the open intervals that are removed at each stage, and that set of endpoints would be countable. (That view is mistaken; there are many more points.) Show that a point x in $[0, 1]$ belongs to the Cantor set if and only if it can be written as a ternary expansion $x = 0.c_1c_2c_3\dots$ (base 3) in such a way that only 0's and 2's occur. This is now a simple characterization of the Cantor set (in terms of string of 0's and 2's) and you should be able to come up with some argument as to why it is now uncountable.

[86]Exercise 4.6.9. You will need the Bolzano-Weierstrass theorem (Theorem 4.21). But this uncountable set E might be unbounded. How could we prove that an uncountable set would have to contain an infinite bounded subset? Consider

$$E = \bigcup_{n=1}^{\infty} E \cap [-n, n].$$

[87]Exercise 4.6.10. Select a rational number from each member of the family and use that to place them in an order.

[88]Exercise 4.7.11. For part (b) look ahead to part (c): Any such example must have A and B unbounded. For part (c) assume $\delta(A, B) = 0$. Then there must be points $x_n \in A$ and $y_n \in B$ with $|x_n - y_n| < 1/n$. As A is compact there is a convergent subsequence x_{n_k} converging to a point z in A. What is happening to y_{n_k}? (Be sure to use here the fact that B is closed.)

Chapter 5

CONTINUOUS FUNCTIONS

5.1 Introduction to Limits

The definition of the limit of a function

$$\lim_{x\to x_0} f(x)$$

is given in calculus courses, but in many classes it is not explored to any great depth. Computation of limits is interesting and offers its challenges, but for a course in real analysis we must master the definition itself and derive its consequences.

Our viewpoint is larger than that in most calculus treatments. There it is common to insist, in order for a limit to be defined, that the function f must be defined at least in some interval $(x_0 - \delta, x_0 + \delta)$ that contains the point x_0 (with the possible exception of x_0 itself). Here we must allow a function f that is defined only on some set E and study limits for points x_0 that are not too remote from E. We do not insist that x_0 be in the domain of f but we do require that it be "close." This requirement is expressed using our language from Chapter 4. We must have x_0 a point of accumulation of E.

Except for this detail about the domain of the function the definition we use is the usual ε-δ definition from calculus. Readers familiar with the sequence limit definitions of Chapter 2 will have no trouble handling this definition. It is nearly the same in general form as the ε-N definition for sequences, and many of the proofs use similar ideas.

5.1.1 Limits (ε-δ Definition)

The definition of a sequence limit, $\lim_{n\to\infty} s_n$, made precise the statement that s_n is arbitrarily close to L if n is sufficiently large. The definition of a function limit

$$\lim_{x\to x_0} f(x)$$

is intended, in much the same way, to make precise the statement that $f(x)$ is arbitrarily close to L if x is sufficiently close to x_0. One feature of the definition must be to exclude the value at the point x_0 from consideration; it should be irrelevant to the value of the limit. It is possible (likely even) that $f(x_0) = L$, but whether this is true or false should not be any influence on the existence of the limit.

Thus the definition assumes the following form. The requirement that x_0 be a point of accumulation of E may seem strange at first sight, but we will see that it is needed in order for the definition to have some meaning. Without it any number would be the limit and the theory of limits would be useless.

Definition 5.1: (Limit) Let $f : E \to \mathbb{R}$ be a function with domain E and suppose that x_0 is a point of accumulation of E. Then we write

$$\lim_{x\to x_0} f(x) = L$$

if for every $\varepsilon > 0$ there is a $\delta > 0$ so that

$$|f(x) - L| < \varepsilon$$

whenever x is a point of E differing from x_0 and satisfying $|x - x_0| < \delta$.

Note. The condition on x can be written as

$$0 < |x - x_0| < \delta$$

or as

$$x \in (x_0 - \delta, x_0 + \delta)\,, \quad x \neq x_0$$

or, yet again, as

$$x_0 - \delta < x < x_0 + \delta\,, \quad x \neq x_0.$$

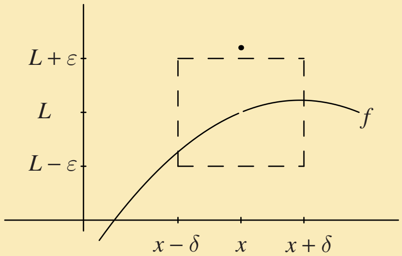

Figure 5.1. Graphical interpretation of the ε-δ limit definition.

The exclusion of $x = x_0$ should be seen as an advantage here. An inequality is required to be true for all x satisfying some condition, and we are allowed *not* to have to check $x = x_0$. It may happen to be true that $|f(x) - L| < \varepsilon$ when $x = x_0$ but it is irrelevant to the definition. For example, you will recall that the limit used to define a derivative

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

must require that the value for $x = x_0$ be excluded; the expression is not defined when $x = x_0$.

See Figure 5.1 for a graphical interpretation of the definition. In the picture a particular value of ε is illustrated and for that value the figure shows a choice of δ that works. Every smaller value of δ would have worked, too. The definition requires doing this, however, for *every* positive ε, and the figure cannot convey that.

We now present some examples illustrating how to prove the existence of a limit directly from the definition. These are to be considered as exercises in understanding the definition. We would rarely use the definition to compute a limit, and we hope seldom to use the definition to verify one; we will use the definition to develop a theory that will verify limits for us.

Example 5.2: Any function $f(x) = ax + b$ will have the easily predicted limit

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} (ax + b) = ax_0 + b.$$

If you sketch a picture similar to that of Figure 5.1 you see easily that the choice of δ is monitored by the slope of the line $y = ax + b$. The steeper the slope, the smaller the δ has to be taken in comparison with ε.

Let us do this for the linear function $f(x) = 10x - 11$. We expect that

$$\lim_{x \to 5} (10x - 11) = 10(5) - 11 = 39.$$

Let us prove this. We need a condition ensuring that the expression

$$|(10x - 11) - 39|$$

is smaller than ε. Some arithmetic converts this to

$$|(10x - 11) - 39| = |10x - 50| = |10|\ |x - 5|\,.$$

Now it is clear that, if we insist that $|x - 5| < \varepsilon/10$, we will have

$$|(10x - 11) - 39| < \varepsilon.$$

That completes the proof. Better, though, would be to write it in a more straightforward manner that obscures how we did it but gets to the point of the proof more simply:

> Let $\varepsilon > 0$. Let $\delta = \varepsilon/10$. Then for all x with $|x - 5| < \delta$ we have
>
> $$|(10x - 11) - 39| = |10|\ |x - 5| < 10\delta = \varepsilon.$$
>
> By definition, $\lim_{x \to 5}(10x - 11) = 39$ as required.

An alert reader of our short proof will know that the choice of δ as $\varepsilon/10$ took some time to compute and is not just an inspired second sentence of the proof. ◀

Example 5.3: Let us use the definition to verify the existence of

$$\lim_{x \to x_0} x^2.$$

Again the definition gives no hints as how to compute the limit; it can be used only to verify the correctness of a limit statement. To keep it simple let us show that $\lim_{x\to 3} x^2 = 9$. We need a condition ensuring that the expression

$$\left|x^2 - 9\right|$$

is smaller than ε. Some arithmetic converts this to

$$\left|x^2 - 9\right| = |x-3|\,|x+3|\,.$$

If we insist that

$$|x-3| < \varepsilon/M,$$

where M is bigger than any value of $|x+3|$, then we will have $\left|x^2-9\right| < \varepsilon$ exactly as we need. But just how big might $|x+3|$ be? If we remember that we are interested only in values of x close to 3 (not huge values of x), then this is not too big. For example, if x stays inside $(2,4)$, then $|x+3| < 7$. These are enough computations to allow us to write up a proof.

> Let $\varepsilon > 0$. Let $\delta = \varepsilon/7$ or $\delta = 1$, whichever is smaller (i.e., $\delta = \min\{\varepsilon/7, 1\}$). Then if $|x-3| < \delta$ it follows that
>
> $$|x+3| = |x-3+6| \le |x-3| + 6 < 7$$
>
> and hence that
>
> $$\left|x^2-9\right| = |x-3|\,|x+3| < 7\,|x-3| < 7(\varepsilon/7) = \varepsilon.$$
>
> By definition, $\lim_{x\to 3} x^2 = 9$ as required.

The finished proof is shorter and lacks all the motivating steps that we just went through. ◀

In spite of these examples and the necessity in elementary courses such as this to work through similar examples, the main goal of our definition is to build up a theory of limits that can then be used to justify other methods of computation and lead to new discoveries. On occasions we must, however, return to the definition to handle an unusual case.

Exercises

5.1.1 Prove the existence of the limit $\lim_{x\to x_0}(4-12x)$.

SEE NOTE 89

5.1.2 Prove the validity of the limit $\lim_{x\to x_0}(ax+b)=ax_0+b$.

SEE NOTE 90

5.1.3 Prove the existence of the limit $\lim_{x\to -4}x^2$.

SEE NOTE 91

5.1.4 Prove the validity of the limit $\lim_{x\to x_0}x^2=x_0^2$.

SEE NOTE 92

5.1.5 Suppose in the definition of the limit that the phrase "x_0 be a point of accumulation of the domain of f" is deleted. Show that then the limit statement $\lim_{x\to -2}\sqrt{x}=L$ would be true for every number L.

5.1.6 Recall that in the definition of $\lim_{x\to x_0}f(x)$ there is a requirement that x_0 be a point of accumulation of the domain of f. Which values of x_0 would be excluded from consideration in the limit

$$\lim_{x\to x_0}\sqrt{x^2-2}?$$

5.1.7 Which values of x_0 would be excluded from consideration in the limit

$$\lim_{x\to x_0}\arcsin|x+2|?$$

5.1.8 Prove the validity of the limit $\lim_{x\to x_0}\sqrt{x}=\sqrt{x_0}$.

SEE NOTE 93

5.1.9 Prove that the limit $\lim_{x\to 0}\frac{1}{x}$ fails to exist.

5.1.10 Prove that the limit $\lim_{x\to 0}\sin(1/x)$ fails to exist.

5.1.11 Using the definition, show that if $\lim_{x\to x_0}f(x)=L$, then

$$\lim_{x\to x_0}|f(x)|=|L|.$$

5.1.12 Suppose that x_0 is a point of accumulation of both A and B and that $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$. We insist that f and g must agree in the sense that $f(x) = g(x)$ if x is in both A and B.

(a) What conditions on A and B ensure that if the limit $\lim_{x\to x_0} f(x)$ exists so too must the limit $\lim_{x\to x_0} g(x)$?

(b) What conditions on A and B ensure that if

$$\lim_{x\to x_0} f(x) \text{ and } \lim_{x\to x_0} g(x)$$

both exist then they must be equal.

See Note 94

5.1.2 Limits (Sequential Definition)

The theory of function limits can be reduced to the theory of sequence limits. This is a popular device in mathematics. Some new theory turns out to be contained in an old theory. This allows easy proofs of results since the old theory has all the tools needed for constructing proofs in the new subject. If our goal were merely to prove all the properties of limits, this would allow us to skip over ε-δ proofs. But since we are trying in this elementary course to learn many methods of analysis, we shall not escape from learning to use ε-δ arguments. Even so, this is an interesting tool for us to use. We can call upon our sequence experience to discover new facts about function limits.

Definition 5.4: (Limit) Let $f : E \to \mathbb{R}$ be a function with domain E and suppose that x_0 is a point of accumulation of E. Then we write

$$\lim_{x\to x_0} f(x) = L$$

if for every sequence $\{e_n\}$ of points of E with $e_n \neq x_0$ and $e_n \to x_0$ as $n \to \infty$,

$$\lim_{n\to\infty} f(e_n) = L.$$

Note. If x_0 is not a point of accumulation of E, then there would be no sequence $\{e_n\}$ of points of E with $e_n \neq x_0$ for all n and $e_n \to x_0$ as $n \to \infty$. Thus once again this is an essential ingredient of our limit definition.

Before we can use this definition we need to establish that it is equivalent to the ε-δ definition. We prove that now.

Proof. (**Definitions 5.1 and 5.4 are equivalent**) Suppose first that

$$\lim_{x \to x_0} f(x) = L$$

according to Definition 5.1 and that $\{e_n\}$, $e_n \neq x_0$, is a sequence of points in the domain of f converging to x_0. Let $\varepsilon > 0$. There must be a positive number δ so that

$$|f(x) - L| < \varepsilon$$

if $0 < |x - x_0| < \delta$. But $e_n \to x_0$ and $e_n \neq x_0$ so there is number N such that $0 < |e_n - x_0| < \delta$ for all $n \geq N$. Putting these together, we find that

$$|f(e_n) - L| < \varepsilon$$

if $n \geq N$. This proves that $\{f(e_n)\}$ converges to L. This verifies that Definition 5.1 implies Definition 5.4.

Conversely, suppose that L is not the limit of $f(x)$ as $x \to x_0$ according to Definition 5.1. We must find a sequence of points $\{e_n\}$ in the domain of f and converging to x_0 such that $f(e_n)$ does not converge to L. Because L is not the limit, there must be some $\varepsilon_0 > 0$ so that for any $\delta > 0$ there will be points x in the domain of f with $0 < |x - x_0| < \delta$ and yet the inequality

$$|f(x) - L| < \varepsilon_0$$

fails. Applying this to $\delta = 1$, $1/2$, $1/3$, $1/4$ ... we obtain a sequence of points x_n with x_n in the domain of f and

$$0 < |x_n - x_0| < 1/n$$

and yet

$$|f(x_n) - L| \geq \varepsilon_0.$$

This is precisely the sequence we wanted since $\{f(x_n)\}$ cannot converge to L. Thus we have shown that Definition 5.4 implies Definition 5.1. ■

Since the two definitions are equivalent, we can use either a sequential argument or an ε-δ argument in our discussions of limits.

Example 5.5: Suppose we wish to prove that $\lim_{x\to x_0} f(x) = L$ implies that

$$\lim_{x\to x_0} \sqrt{f(x)} = \sqrt{L}.$$

We could convert this into an ε-δ statement, which will involve us in some unpleasant inequality work. Or we can see that, alternatively, we need to prove that if we know $f(x_n) \to L$, then we can conclude $\sqrt{f(x_n)} \to \sqrt{L}$. But we did study just such problems in our investigation of sequence limits (Exercise 2.4.16). ◀

Exercises

5.1.13 Prove the existence of the limit $\lim_{x\to x_0}(4-12x)$ by converting to a statement about sequences.

5.1.14 Prove the validity of the limit

$$\lim_{x\to x_0} (ax+b) = ax_0 + b$$

by converting to a statement about sequences.

5.1.15 Prove the validity of the limit $\lim_{x\to x_0} x^2 = x_0^2$ by converting to a statement about sequences.

5.1.16 Show that $\lim_{x\to 0} |x|/x$ does not exist by using the sequential definition of limit.

SEE NOTE 95

5.1.17 Prove that the limit $\lim_{x\to 0} \frac{1}{x}$ fails to exist by converting to a statement about sequences.

5.1.18 Prove that the limit $\lim_{x\to 0} \sin(1/x)$ fails to exist by converting to a statement about sequences.

5.1.19 Let x_0 be an accumulation point of the domain E of a function f. Prove that the limit $\lim_{x\to x_0} f(x)$ fails to exist if and only if there is a sequence of distinct points $\{e_n\}$ of E converging to x_0 but with $\{f(e_n)\}$ divergent.

5.1.20 Let f be the *characteristic function* of the rational numbers; that is, f is defined for all real numbers by setting $f(x) = 1$ if x is a rational number and $f(x) = 0$ if x is not a rational number. Determine where, if possible, the limit $\lim_{x\to x_0} f(x)$ exists.

5.1.21 Using the sequential definition, show that if $\lim_{x\to x_0} f(x) = L$, then

$$\lim_{x\to x_0} |f(x)| = |L|.$$

5.1.22 Find hypotheses under which you can prove that if $\lim_{x\to x_0} f(x) = L$, then

$$\lim_{x\to x_0} \sqrt{f(x)} = \sqrt{L}.$$

SEE NOTE 96

5.1.3 Limits (Mapping Definition)

Enrich.

The essential idea behind a limit

$$\lim_{x\to x_0} f(x) = L$$

is that values of x close to x_0 get mapped by f into values close to L. We have been able to express this idea by using inequalities that express this closeness: δ-close for the x values and ε-close for the $f(x)$ values. This is essentially a mapping property that can be expressed by arbitrary open sets.

The following definition is equivalent to both Definitions 5.1 and 5.4.

Definition 5.6: (Limit) Let $f : E \to \mathbb{R}$ be a function with domain E and suppose that x_0 is a point of accumulation of E. Then we write

$$\lim_{x\to x_0} f(x) = L$$

if for every open set V containing the point L there is an open set U containing the point x_0 and every point $x \neq x_0$ of U that is in the domain of f is mapped into a point in V; that is,

$$f : E \cap U \setminus \{x_0\} \to V.$$

Once again, we must show that this definition is equivalent to the ε-δ definition. We prove that now.

Proof. (**Definitions 5.1 and 5.6 are equivalent**) Suppose first that

$$\lim_{x\to x_0} f(x) = L$$

according to Definition 5.1. Let V be an open set containing the point L. Then, since L is an interior point of V there is a positive number ε with

$$(L-\varepsilon, L+\varepsilon) \subset V.$$

Choose $\delta > 0$ so that

$$|f(x) - L| < \varepsilon$$

if $0 < |x - x_0| < \delta$ whenever x is a point in E (the domain of f). Let U be the open set $(x_0 - \delta, x_0 + \delta)$. Then the inequality we have shows that every point $x \neq x_0$ of U that is in the domain of f is mapped into a point in V. This is precisely Definition 5.6.

Conversely, suppose that $\lim_{x\to x_0} f(x) = L$ according to Definition 5.6. Let $\varepsilon > 0$. Choose $V = (L-\varepsilon, L+\varepsilon)$. By our definition there must be an open set U containing the point x_0 and every point $x \neq x_0$ of U that is in the domain of f is mapped into a point in V. Since x_0 is an interior point of U there must be a positive number δ so that

$$(x_0 - \delta, x_0 + \delta) \subset U.$$

This mapping property implies that

$$|f(x) - L| < \varepsilon$$

if $0 < |x - x_0| < \delta$. This is exactly our ε-δ definition of Definition 5.1. ■

Since all three of our definitions are equivalent we can use either a sequential argument, a mapping argument, or an ε-δ argument in our discussions of limits.

Exercises

5.1.23 Show that $\lim_{x\to 0} |x|/x$ does not exist using the mapping definition of limit.

5.1.24 Prove directly that the sequential definition of limit is equivalent to the mapping definition.

5.1.4 One-Sided Limits

It is possible for a function to fail to have a limit at a point and yet appear to have limits on one side. If we ignore what is happening on the right for a function, perhaps it will have a "left-hand limit." This is easy to achieve. Let f be defined everywhere near a point x_0 and define a new function

$$g(x) = f(x) \text{ for all } x < x_0.$$

This new function g is defined on a set to the left of x_0 and knows nothing of the values of f on the right. Thus the limit

$$\lim_{x \to x_0} g(x)$$

can be thought of as a left-hand limit for f. It would be written as

$$\lim_{x \to x_0-} f(x)$$

where the "x_0-" is the indication that a left-hand limit is used, not an ordinary limit. Similarly, the notation

$$\lim_{x \to x_0+} f(x)$$

denotes a right-hand limit with the "x_0+" indicating the limit on the positive or right side of x_0.

Since these one-sided limits are really just ordinary limits for a different function, they must satisfy all the theory of ordinary limits with no further fuss. We can use them quite freely without worrying that they need a different definition or a different theory. Even so, it is convenient to translate our usual definitions into one-sided limits just to have an expression for them. We give the right-hand version. You can supply a left-hand version.

Definition 5.7: (Right-Hand Limit) Let $f : E \to \mathbb{R}$ be a function with domain E and suppose that x_0 is a point of accumulation of $E \cap (x_0, \infty)$. Then we write

$$\lim_{x\to x_0+} f(x) = L$$

if for every $\varepsilon > 0$ there is a $\delta > 0$ so that

$$|f(x) - L| < \varepsilon$$

whenever $x_0 < x < x_0 + \delta$ and $x \in E$.

An equivalent sequential version can be established.

Definition 5.8: (Right-Hand Limit) Let $f : E \to \mathbb{R}$ be a function with domain E and suppose that x_0 is a point of accumulation of $E \cap (x_0, \infty)$. Then we write

$$\lim_{x\to x_0+} f(x) = L$$

if for every decreasing sequence $\{e_n\}$ of points of E with $e_n > x_0$ and $e_n \to x_0$ as $n \to \infty$,

$$\lim_{n\to\infty} f(e_n) = L.$$

Exercises

5.1.25 Show directly that Definitions 5.7 and 5.8 are equivalent.

5.1.26 Under appropriate additional assumptions about the domain of the function f show that $\lim_{x\to x_0} f(x) = L$ if and only if both

$$\lim_{x\to x_0+} f(x) = L \text{ and } \lim_{x\to x_0-} f(x) = L$$

are valid.

5.1.27 If the two limits

$$\lim_{x\to x_0+} f(x) = L_1 \text{ and } \lim_{x\to x_0-} f(x) = L_2$$

exist and are different, then the function is said to have a *jump discontinuity* at that point. The value $L_1 - L_2$ is called the *magnitude of the jump*. Give an example of a function with a jump of magnitude 3 at the value $x_0 = 2$. Give an example with a jump of magnitude -3.

5.1.28 Compute the one-sided limits of the function

$$f(x) = \frac{x}{|x|}$$

at any point x_0.

SEE NOTE 97

5.1.29 Compute, if possible, the one-sided limits of the function

$$f(x) = e^{1/x}$$

at 0.

SEE NOTE 98

5.1.30 According to our definitions, is there any distinction between the assertions

$$\lim_{x \to 0} \sqrt{x} = 0 \text{ and } \lim_{x \to 0+} \sqrt{x} = 0?$$

What is the meaning of $\lim_{x \to 0-} \sqrt{x} = 0$?

SEE NOTE 99

5.1.5 Infinite Limits

We can easily check that the limits

$$\lim_{x \to 0+} \frac{1}{x} \quad \text{and} \quad \lim_{x \to 0-} \frac{1}{x}$$

fail to exist. A glance at the graph of the function $f(x) = 1/x$ suggests that we should write instead

$$\lim_{x \to 0+} \frac{1}{x} = \infty \quad \text{and} \quad \lim_{x \to 0-} \frac{1}{x} = -\infty$$

as a way of conveying more information about what is happening rather than saying merely that the limits do not exist.

In this we are following our custom in the study of divergent sequences. Some sequences merely diverge, some diverge to ∞ or to $-\infty$. If we look back at the definition for sequences and compare it with our function limit definition, we should arrive at the following definition.

Definition 5.9: (Infinite Limit) Let $f : E \to \mathbb{R}$ be a function with domain E and suppose that x_0 is a point of accumulation of $E \cap (x_0, \infty)$. Then we write

$$\lim_{x \to x_0+} f(x) = \infty$$

if for every $M > 0$ there is a $\delta > 0$ so that $f(x) \geq M$ whenever

$$x_0 < x < x_0 + \delta \ \text{ and } \ x \in E.$$

Similarly, we can define

$$\lim_{x \to x_0+} f(x) = -\infty$$

if for every $m < 0$ there is a $\delta > 0$ so that $f(x) \leq m$ whenever $x_0 < x < x_0 + \delta$ and $x \in E$. The infinite limits on the left are similarly defined and denoted $\lim_{x \to x_0-} f(x) = \infty$ and $\lim_{x \to x_0-} f(x) = -\infty$. Also, two-sided limits are defined in the same manner, but with a two-sided condition.

Note. Just as for sequences, we do not say that the limit of a function *exists* unless that limit is finite. Thus, for example, we would say that the limit $\lim_{x \to 0+} 1/x$ does not exist, and that in fact $\lim_{x \to 0+} 1/x = \infty$. A limit is a real number. The symbols ∞ and $-\infty$ are used to describe certain situations, but they are not interpreted as numbers themselves.

Exercises

5.1.31 Give an equivalent formulation for infinite limits using a sequential version.

5.1.32 Formulate a definition for the statement that $\lim_{x \to x_0-} f(x) = \infty$. Using your definition, show that

$$\lim_{x \to x_0-} f(x) = \infty$$

if and only if

$$\lim_{x\to(-x_0)+} f(-x) = \infty.$$

5.1.33 Where does the function

$$f(x) = \frac{1}{\sqrt{x^2-1}}$$

have infinite limits? Give proofs using the definition.

5.1.34 Formulate a definition for the statements

$$\lim_{x\to\infty} f(x) = L \text{ and } \lim_{x\to-\infty} f(x) = L.$$

See Note 100

5.1.35 Formulate a definition for the statements

$$\lim_{x\to\infty} f(x) = \infty \text{ and } \lim_{x\to-\infty} f(x) = \infty.$$

5.1.36 Let $f:(0,\infty)\to\mathbb{R}$. Show that

$$\lim_{x\to\infty} f(x) = L \text{ if and only if } \lim_{x\to 0+} f(1/x) = L.$$

5.1.37 What are the limits $\lim_{x\to\infty} x^p$ for various real numbers p?

5.1.38 Show that one of the limits $\lim_{x\to 0+} f(x)$ and $\lim_{x\to 0-} f(x)$ of the function

$$f(x) = e^{1/x}$$

at 0 is infinite and one is finite. What can you say about the limits

$$\lim_{x\to\infty} f(x) \quad\text{and}\quad \lim_{x\to-\infty} f(x)?$$

See Note 101

5.2 Properties of Limits

The computation of limits in calculus courses depended on a theory of limits. For most simple computations it was enough to know how to handle functions that were put together by adding, subtracting, multiplying, or dividing other functions. Later, more subtle problems required advanced techniques (e.g., L'Hôpital's rule). Here we develop the rudiments of a theory of function limits.

We start with the uniqueness property, the boundedness property and continue to the algebraic properties. In this we are following much the same path we did when we began our study of sequential limits. Indeed the definitions of sequential limits and function limits are so similar that the theories are necessarily themselves quite similar.

5.2.1 Uniqueness of Limits

When we write the statement

$$\lim_{x\to x_0} f(x) = L$$

we wish to be assured that it is not also true for some other numbers different from L.

Theorem 5.10 (Uniqueness of Limits) *Suppose that*

$$\lim_{x\to x_0} f(x) = L.$$

Then the number L is unique: No other number has this same property.

Proof. We suppose that

$$\lim_{x\to x_0} f(x) = L$$

and

$$\lim_{x\to x_0} f(x) = L_1$$

are both true. To prove the theorem we must show that $L = L_1$. If we convert this to a statement about sequences this asserts that any sequence $x_n \to x_0$ with $x_n \neq x_0$ and all points in the domain of f must have

$$f(x_n) \to L$$

and also must have

$$f(x_n) \to L_1.$$

For these limits to exist the point x_0 must be a point of accumulation for the domain of f and so there exists at least one such sequence. But we have already established for sequence limits that this is impossible (Theorem 2.8) unless $L = L_1$. ■

Exercises

5.2.1 Give an ε-δ proof of Theorem 5.10.

SEE NOTE 102

5.2.2 Explain why the proof fails if the part of the limit definition that asserts x_0 is to be a point of accumulation of the domain of f were omitted.

5.2.2 Boundedness of Limits

We recall that convergent sequences are bounded. There is a similar statement for functions. If a function limit exists the function cannot be too large; the statement must be made precise, however, since it is really only valid close to the point where the limit is taken.

For example, you will recall from our discussion of local boundedness in Section 4.5 that the function $f(x) = 1/x$ is unbounded and yet locally bounded at each point other than at 0. In the same way we will see that the existence of the limit

$$\lim_{x \to x_0} \frac{1}{x} = \frac{1}{x_0}$$

for every value of $x_0 \neq 0$ also requires that local boundedness property.

Theorem 5.11 (Boundedness of Limits) *Suppose that the limit*

$$\lim_{x\to x_0} f(x) = L$$

exists. Then there is an interval $(x_0 - c, x_0 + c)$ *and a number* M *such that*

$$|f(x)| \leq M$$

for every value of x *in that interval that is in the domain of* f.

Proof. There is a $\delta > 0$ so that

$$|f(x) - L| < 1$$

whenever x is a point in the domain of f differing from x_0 and satisfying $|x - x_0| < \delta$. If x_0 is not in the domain of f, then this means that

$$|f(x)| = |f(x) - L + L| \leq |f(x) - L| + |L| < |L| + 1$$

for all x in $(x_0 - \delta, x_0 + \delta)$ that are in the domain of f. This would complete the proof since we can take $M = |L| + 1$.

If x_0 is in the domain of f, then take instead

$$M = |L| + 1 + |f(x_0)|.$$

Then

$$|f(x)| \leq M$$

for all x in $(x_0 - \delta, x_0 + \delta)$ that are in the domain of f. ■

A similar statement can be made about boundedness away from zero. This shows that if a function has a nonzero limit, then close by to the point the function stays away from zero. The proof uses similar ideas and is left for the exercises.

Theorem 5.12 (Boundedness Away from Zero) *If the limit*

$$\lim_{x\to x_0} f(x)$$

exists and is not zero, then there is an interval $(x_0 - c, x_0 + c)$ *and a positive number* m *such that*

$$|f(x)| \geq m > 0$$

for every value of $x \neq x_0$ *in that interval and that belongs to the domain of* f.

Exercises

5.2.3 Prove Theorem 5.11 using the sequential definition of limit instead.

See Note 103

5.2.4 Use Theorem 5.11 to show that $\lim_{x\to 0} \frac{1}{x}$ cannot exist.

5.2.5 Prove Theorem 5.12 using an ε-δ argument.

5.2.6 Prove Theorem 5.12 using a sequential argument.

5.2.7 Prove Theorem 5.12 by deriving it from Theorem 5.11 and the fact (proved later) that if

$$\lim_{x\to x_0} f(x) = L \neq 0$$

then

$$\lim_{x\to x_0} \frac{1}{f(x)} = \frac{1}{L}.$$

5.2.3 Algebra of Limits

Functions can be combined by the usual arithmetic operations (addition, subtraction, multiplication and division). Indeed most functions we are likely to have encountered in a calculus course can be seen to be composed of simpler functions combined together in this way.

Example 5.13: The computations

$$\lim_{x\to 3}\frac{2x^3+4}{3x^2+1}=\frac{\lim_{x\to 3}(2x^3+4)}{\lim_{x\to 3}(3x^2+1)}$$

$$=\frac{2(\lim_{x\to 3}x^3)+4}{3(\lim_{x\to 3}x^2)+1}=\frac{2\times 3^3+4}{3\times 3^2+1}$$

should return fond memories of calculus homework assignments. But how are these computations properly justified? ◀

Because of our experience with sequence limits, we can anticipate that there should be an "algebra of function limits" just as there was an algebra of sequence limits. The proofs can be obtained either by imitating the proofs we constructed earlier for sequences or by using the fact that function limits can be reduced to sequential limits.

There is an extra caution here. An example illustrates.

Example 5.14: We know that $\lim_{x\to 0}\sqrt{-x}=0$ and $\lim_{x\to 0}\sqrt{x}=0$. Does it follow that

$$\lim_{x\to 0}\left(\sqrt{x}+\sqrt{-x}\right)=0?$$

There is only one point in the domain of the function

$$f(x)=\sqrt{x}+\sqrt{-x}$$

and so no limit statement is possible. ◀

The extra hypothesis throughout the following theorems appears in order to avoid examples like this. We must assume that the domain of f, call it $\text{dom}(f)$, and the domain of g, call it $\text{dom}(g)$, must have enough points in common to define the limit at the point x_0 being considered. In most simple applications the domains of the functions do not cause any troubles.

For proofs we have a number of strategies available. We can reduce these limit theorems to statements about sequences and then appeal to the theory of sequential limits that we developed in Chapter 2.

Alternatively, we can construct ε-δ proofs by modeling them after the similar statements that we proved for sequences. We do not need any really new ideas. The proofs have, accordingly, been left to the exercises.

Theorem 5.15 (Multiples of Limits) *Suppose that the limit*

$$\lim_{x\to x_0} f(x)$$

exists and that C is a real number. Then

$$\lim_{x\to x_0} Cf(x) = C\left(\lim_{x\to x_0} f(x)\right).$$

Theorem 5.16 (Sums and Differences) *Suppose that the limits*

$$\lim_{x\to x_0} f(x) \quad \textit{and} \quad \lim_{x\to x_0} g(x)$$

exist and that x_0 is a point of accumulation of $dom(f) \cap dom(g)$. Then

$$\lim_{x\to x_0} (f(x)+g(x)) = \lim_{x\to x_0} f(x) + \lim_{x\to x_0} g(x)$$

and

$$\lim_{x\to x_0} (f(x)-g(x)) = \lim_{x\to x_0} f(x) - \lim_{x\to x_0} g(x).$$

Theorem 5.17 (Products of Limits) *Suppose that the limits*

$$\lim_{x\to x_0} f(x) \quad \textit{and} \quad \lim_{x\to x_0} g(x)$$

exist and that x_0 is a point of accumulation of $dom(f) \cap dom(g)$. Then

$$\lim_{x\to x_0} f(x)g(x) = \left(\lim_{x\to x_0} f(x)\right)\left(\lim_{x\to x_0} g(x)\right).$$

Theorem 5.18 (Quotients of Limits) *Suppose that the limits*

$$\lim_{x\to x_0} f(x) \quad and \quad \lim_{x\to x_0} g(x)$$

exist and that the latter is not zero and that x_0 is a point of accumulation of $dom(f)\cap dom(g)$. Then

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{\lim_{x\to x_0} f(x)}{\lim_{x\to x_0} g(x)}.$$

Exercises

5.2.8 Let f and g be functions with domains $\mathrm{dom}(f)$ and $\mathrm{dom}(g)$. What are the domains of the functions listed below obtained by combining these functions algebraically or by a composition?

(a) $f+g$

(b) $f-g$

(c) fg

(d) f/g

(e) $f\circ g$

(f) $\sqrt{f+g}$

(g) $\sqrt{fg}$

5.2.9 What exactly is the trouble that arises in the theorems of this section that required us to assume "that x_0 is a point of accumulation of $\mathrm{dom}(f)\cap\mathrm{dom}(g)$?"

SEE NOTE 104

5.2.10 Is it true that if both $\lim_{x\to x_0} f(x)$ and $\lim_{x\to x_0} g(x)$ fail to exist, then

$$\lim_{x\to x_0} (f(x)+g(x))$$

must also fail to exist?

5.2.11 In the statement of Theorem 5.18 don't we also have to assume that $g(x)$ is never zero?

SEE NOTE 105

5.2.12 A careless student gives the following as a proof of Theorem 5.17. Find the flaw: "Suppose that $\varepsilon > 0$. Choose δ_1 so that

$$|f(x) - L| < \frac{\varepsilon}{2|M| + 1}$$

if $0 < |x - x_0| < \delta_1$ and also choose δ_2 so that

$$|g(x) - M| < \frac{\varepsilon}{2|f(x)| + 1}$$

if $0 < |x - x_0| < \delta_2$ Define $\delta = \min\{\delta_1, \delta_2\}$. If $0 < |x - x_0| < \delta$, then we have

$$|f(x)g(x) - LM| \leq |f(x)|\,|g(x) - M| + |M|\,|f(x) - L|$$

$$\leq |f(x)| \left(\frac{\varepsilon}{2|f(x)| + 1}\right) + |M| \left(\frac{\varepsilon}{2|M| + 1}\right) < \varepsilon.$$

Well, that shows $f(x)g(x) \to LM$ if $f(x) \to L$ and $g(x) \to M$."

5.2.13 Prove Theorem 5.15 by using an ε-δ proof and by using the sequential definition of limit.

5.2.14 Prove Theorem 5.16 by using an ε-δ proof and by using the sequential definition of limit.

5.2.15 Prove Theorem 5.18 by using the sequential definition of limit.

5.2.16 Prove Theorem 5.17 by correcting the flawed ε-δ proof in Exercise 5.2.12 and by using the sequential definition of limit. Which method is easier?

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5.2.4 Order Properties

Just as we saw that sequence limits preserve both the algebraic structure and the order structure, so we will find that function limits have the same properties. We have just completed the algebraic properties. We turn now to the order properties.

If $f(x) \leq g(x)$ for all x, then we expect to conclude that

$$\lim_{x \to x_0} f(x) \leq \lim_{x \to x_0} g(x).$$

We now prove this and several other properties that relate directly to the order structure of the real numbers.

Theorem 5.19: *Suppose that the limits*

$$\lim_{x\to x_0} f(x) \quad \textit{and} \quad \lim_{x\to x_0} g(x)$$

exist and that x_0 *is a point of accumulation of* $dom(f)\cap dom(g)$*. If*

$$f(x) \le g(x)$$

for all $x \in dom(f)\cap dom(g)$*, then*

$$\lim_{x\to x_0} f(x) \le \lim_{x\to x_0} g(x).$$

Proof. Let us give an indirect proof. Let

$$L = \lim_{x\to x_0} f(x) \text{ and } \ M = \lim_{x\to x_0} g(x)$$

and suppose, contrary to the theorem, that $L > M$. Choose ε so small that $M+\varepsilon < L-\varepsilon$; that is, choose

$$\varepsilon < (L-M)/2.$$

By the definition of limits there are numbers δ_1 and δ_2 so that

$$f(x) > L-\varepsilon$$

if $x \ne x_0$ is within δ_1 of x_0 and in the domain of f and

$$g(x) < M+\varepsilon$$

if $x \ne x_0$ is within δ_2 of x_0 and is in the domain of g. But the conditions in the theorem assure us that there must be at least one point, $x = z$ say, that satisfies both conditions. That would mean

$$g(z) < M+\varepsilon < L-\varepsilon < f(z).$$

This is impossible as it contradicts the fact that all the values of $f(x)$ are less than the values $g(x)$. This contradiction completes the proof. ■

Note. There is a trap here that we encountered in our discussions of sequence limits. We remember that the condition $s_n < t_n$ does not imply that

$$\lim_{n\to\infty} s_n < \lim_{n\to\infty} t_n.$$

In the same way the condition $f(x) < g(x)$ does not imply

$$\lim_{x\to x_0} f(x) < \lim_{x\to x_0} g(x).$$

Be careful with this, too.

Corollary 5.20: *Suppose that the limit*

$$\lim_{x\to x_0} f(x)$$

exists and that $\alpha \leq f(x) \leq \beta$ for all x in the domain of f. Then

$$\alpha \leq \lim_{x\to x_0} f(x) \leq \beta.$$

Note. Again, don't forget the trap. The condition $\alpha < f(x) < \beta$ for all x implies at best that

$$\alpha \leq \lim_{x\to x_0} f(x) \leq \beta.$$

It would not imply that

$$\alpha < \lim_{x\to x_0} f(x) < \beta.$$

The next theorem is another useful variant on these themes. Here an unknown function is sandwiched between two functions whose limit behavior is known, allowing us to conclude that a limit exists. This theorem is often taught as "the squeeze theorem" just as the version for sequences in Theorem 2.20 was labeled. Here we need the functions to have the same domain.

Theorem 5.21 (Squeeze Theorem) *Suppose that f, g, $h : E \to \mathbb{R}$ and that x_0 is a point of accumulation of the common domain E. Suppose that the limits*

$$\lim_{x\to x_0} f(x) = L \quad \textit{and} \quad \lim_{x\to x_0} g(x) = L$$

exist and that

$$f(x) \leq h(x) \leq g(x)$$

for all $x \in E$ except perhaps at $x = x_0$. Then $\lim_{x\to x_0} h(x) = L$.

Proof. The easiest proof is to use a sequential argument. This is left as Exercise 5.2.19. ■

Example 5.22: Let us prove that the limit

$$\lim_{x\to 0} x\sin(1/x) = 0$$

is valid. Certainly the expression $\sin(1/x)$ seems troublesome at first. But we notice that the inequalities

$$-|x| \leq x\sin(1/x) \leq |x|$$

are valid for all x (except $x = 0$ where the function is undefined). Since

$$\lim_{x\to 0} |x| = \lim_{x\to 0} -|x| = 0$$

Theorem 5.21 supplies our result. ◄

A final theorem on the theme of order structure is often needed. The absolute value, we recall, is defined directly in terms of the order structure. Is absolute value preserved by the limit operation? As the proof does not require any new ideas, it is left as Exercise 5.2.21.

Theorem 5.23 (Limits of Absolute Values) *Suppose that the limit*

$$\lim_{x \to x_0} f(x) = L$$

exists. Then

$$\lim_{x \to x_0} |f(x)| = |L|.$$

Since maxima and minima can be expressed in terms of absolute values, there is a corollary that is sometimes useful.

Corollary 5.24 (Max/Min of Limits) *Suppose that the limits*

$$\lim_{x \to x_0} f(x) = L \quad \text{and} \quad \lim_{x \to x_0} g(x) = M$$

exist and that x_0 is a point of accumulation of $dom(f) \cap dom(g)$. Then

$$\lim_{x \to x_0} \max\{f(x), g(x)\} = \max\{L, M\}$$

and

$$\lim_{x \to x_0} \min\{f(x), g(x)\} = \min\{L, M\}.$$

Proof. The first of the these follows from the identity

$$\max\{f(x), g(x)\} = \frac{f(x) + g(x)}{2} + \frac{|f(x) - g(x)|}{2}$$

and the theorem on limits of sums and the theorem on limits of absolute values. In the same way the second assertion follows from

$$\min\{f(x), g(x)\} = \frac{f(x) + g(x)}{2} - \frac{|f(x) - g(x)|}{2}.$$

■

Exercises

5.2.17 Show that the condition $f(x) < g(x)$ does not imply that

$$\lim_{x\to x_0} f(x) < \lim_{x\to x_0} g(x).$$

5.2.18 Give a sequential type proof for Theorem 5.19.

5.2.19 Give a sequential type proof for Theorem 5.21.

5.2.20 Give an ε-δ proof of Theorem 5.23.

5.2.21 Give a proof of Theorem 5.23 by converting it to a statement about sequences.

5.2.22 Extend Corollary 5.24 to the case of more than two functions; that is, determine

$$\lim_{x\to x_0} \max\{f_1(x), f_2(x), \dots, f_n(x)\}.$$

5.2.5 Composition of Functions

You will have observed a pattern that is attractive in the study of limits. These examples suggest the pattern:

$$\lim_{x\to x_0} [f(x)]^2 = \left(\lim_{x\to x_0} f(x)\right)^2,$$

$$\lim_{x\to x_0} \sqrt{f(x)} = \sqrt{\lim_{x\to x_0} f(x)},$$

$$\lim_{x\to x_0} e^{f(x)} = e^{\lim_{x\to x_0} f(x)}.$$

The first is easy to prove since $[f(x)]^2 = f(x)f(x)$ and we can use the product rule. The square root example is harder but could be proved using an ε-δ argument and requires only the assumption that $\lim_{x\to x_0} f(x)$ is positive. It could be false if $\lim_{x\to x_0} f(x) = 0$ and definitely is false if $\lim_{x\to x_0} f(x) < 0$.

The third will require some familiarity with the exponential function and is harder still, though always true.

The general pattern is the following. Some function F is composed with f, and the limit computation we wish to use is

$$\lim_{x\to x_0} F(f(x)) = F\left(\lim_{x\to x_0} f(x)\right).$$

Can this be justified? More correctly, what are the conditions under which it can be justified?

Let us analyze this using a sequence argument since that often simplifies function limits. We suppose $x_n \to x_0$. We have then our supposition that $f(x_n) \to L$. Can we conclude

$$F(f(x_n)) \to F(L)?$$

This is exactly what we are doing when we try to use

$$\lim_{x\to x_0} F(f(x)) = F\left(\lim_{x\to x_0} f(x)\right).$$

The property of the function F that we desire is simple:

$$\textit{If } z_n \to z_0 \textit{ then } F(z_n) \to F(z_0).$$

Think of $z_n = f(x_n)$; then $z_n \to L$ and the required property is

$$F(z_n) \to F(L) \text{ whenever } z_n \to L.$$

This is the same as requiring that

$$\lim_{z\to L} F(z) = F(L).$$

Thus we have proved the following theorem, which completely answers our question about justifying the preceding operations.

Theorem 5.25: *Let F be a function defined in a neighborhood of the point L and such that*

$$\lim_{z\to L} F(z) = F(L).$$

If

$$\lim_{x\to x_0} f(x) = L$$

then

$$\lim_{x\to x_0} F(f(x)) = F\left(\lim_{x\to x_0} f(x)\right) = F(L).$$

The condition on the function F that

$$\lim_{z\to L} F(z) = F(L)$$

is called *continuity at the point L* and is the subject of Section 5.4.

Exercises

5.2.23 Show that

$$\lim_{x\to x_0} \sqrt{f(x)} = \sqrt{\lim_{x\to x_0} f(x)}$$

could be false if $\lim_{x\to x_0} f(x) = 0$ and definitely is false if $\lim_{x\to x_0} f(x) < 0$.

5.2.24 Give a formal proof of Theorem 5.25 using the sequential method sketched in the text.

5.2.25 Give a formal proof of Theorem 5.25 using an ε-δ method.

5.2.26 Give a formal proof of Theorem 5.25 using the mapping idea.

5.2.27 Give an example of a limit for which

$$\lim_{x\to x_0} F(f(x)) \neq F\left(\lim_{x\to x_0} f(x)\right)$$

even though both of the limits in the statement do exist.

5.2.28 Show that

$$\lim_{x\to x_0} |f(x)| = \left|\lim_{x\to x_0} f(x)\right|$$

under some appropriate assumption by applying Theorem 5.25.

See Note 106

5.2.29 Show that

$$\lim_{x\to x_0} \sqrt{|f(x)|} = \sqrt{\left|\lim_{x\to x_0} f(x)\right|}$$

under some appropriate assumption by applying Theorem 5.25.

See Note 107

5.2.30 Obtain Corollary 5.24 as an application of Theorem 5.25.

5.2.6 Examples

There are a number of well-known examples of limits that every student should know. Partly this is because there will be an expectation in later courses that these should have been seen. But, more important, an abundance of examples is needed to gain some insight into when limits exist and when they do not and how they behave.

For any function f defined near a point x_0 there are several possibilities we should look for.

1. Does the limit $\lim_{x\to x_0} f(x)$ exist?

2. If the limit does exist, is the limit the most likely value, namely

$$\lim_{x\to x_0} f(x) = f(x_0)?$$

(Such functions are said to be *continuous* at the point x_0.)

3. If the limit fails to exist, then could it be that the one-sided limits do exist but happen to be unequal; that is,
$$\lim_{x\to x_0+} f(x) \neq \lim_{x\to x_0-} f(x)?$$
(Such a function is said to have a *jump discontinuity* at the point x_0.)

The case that is most familiar, namely where
$$\lim_{x\to x_0} f(x) = f(x_0),$$
is described by the language of continuity. Our study of continuity comes in the next section. But let us be aware now of when a function has this property.

Polynomials All polynomial functions have entirely predictable limits. If
$$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n,$$
then
$$\lim_{x\to x_0} p(x) = p(x_0)$$
at every value. (In the language we shall use, these functions are continuous.) To prove this we can use the fact that $\lim_{x\to x_0} a_0 = a_0$ and the fact that $\lim_{x\to x_0} x = x_0$. These are trivial to prove. Then the polynomial is built up from this by additions and multiplications. The theorems of Section 5.2.3 can be used to complete the verification [e.g., $\lim_{x\to x_0} x^2 = x_0^2$ by the product rule, $\lim_{x\to x_0} x^3 = \lim_{x\to x_0}(x)(x^2) = x_0^3$ by the product rule applied again].

Rational Functions A rational function is a function of the form
$$R(x) = \frac{p(x)}{q(x)}$$
where p and q are polynomials (i.e., a ratio of polynomials and hence the name). Since we can take limits
$$\lim_{x\to x_0} R(x) = \frac{\lim_{x\to x_0} p(x)}{\lim_{x\to x_0} q(x)}$$

freely, excepting only the case where the denominator is zero, we have found that

$$\lim_{x\to x_0} R(x) = R(x_0)$$

except at those points where $q(x_0) = 0$. At those points, it is possible that the limit exists. Note, however, that it cannot equal $R(x_0)$ since $R(x_0)$ is not defined. It is also possible that the right-hand and left-hand limits are infinite. There are some examples in the exercises to illustrate these possibilities.

Exponential Functions The exponential function e^x can be proved to have the limiting value that we would expect, namely

$$\lim_{x\to x_0} e^x = e^{x_0}.$$

To prove this depends on how we have defined the exponential function in the first place. There are many ways in which we can develop such a theory. It is usual to wait for more theoretical apparatus and then define the exponential function in an appropriate way that allows that to be exploited. Recall that we mentioned in Example 2.36 that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \ldots.$$

Sums like this are called power series. As part of the theory of power series we will discover precisely when they are continuous. Then it is possible to define the exponential function as a power series and claim continuity immediately.

Most of the elementary functions of the calculus (trigonometric functions, inverse trigonometric functions, etc.) can be handled in this way. We do not pause here to worry about limits of such functions.

Characteristic Function of the Rationals The *characteristic function* of a set E of real numbers is the function that assigns value 1 at points in E and value 0 at points outside E. Some authors call it an *indicator function* since it does, indeed, indicate when points are or are not in the set. For an interesting example of a function that would have been considered bizarre in the early days of calculus, consider the characteristic function of the rationals:

$$\chi_{\mathbb{Q}}(x) = 1 \text{ if } x \in \mathbb{Q}$$

and

$$\chi_{\mathbb{Q}}(x) = 0 \text{ if } x \notin \mathbb{Q}.$$

It is an easy exercise to check that

$$\lim_{x \to x_0+} \chi_{\mathbb{Q}}(x) \text{ and } \lim_{x \to x_0-} \chi_{\mathbb{Q}}(x)$$

both fail to exist.

Dirichlet Function The Dirichlet function is defined on $[0, 1]$ by

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is irrational or } x = 0 \\ 1/q, & \text{if } x = p/q, \ \ p, q \in \mathbb{N}, \ \ p/q \text{ in lowest terms.} \end{cases}$$

To examine the limiting behavior of this function, we need to observe that while there are many points where this function is positive (all rationals) there are not many points where it assumes a value greater than some positive number ε. Indeed if we count them we will see that for any positive integer q the set of points

$$S_q = \{x \in [0, 1] : f(x) \geq 1/q\}$$

contains at most $q(q-1)/2$ points. The exact number is not important; all we need to observe is that there are only finitely many such points.

Thus let $\varepsilon > 0$ and choose any integer q large enough so that $1/q < \varepsilon$. For any point x_0 we can choose $\delta > 0$ in such a way that both intervals $(x_0 - \delta, x_0)$ and $(x_0, x_0 + \delta)$ contain no points of the finite set S_q. That must mean that every point x in $(x_0 - \delta, x_0)$ or $(x_0, x_0 + \delta)$ satisfies

$$0 \leq f(x) < 1/q < \varepsilon.$$

Thus it follows that

$$\lim_{x \to x_0} f(x) = 0$$

at every point x_0. In particular, the equation

$$\lim_{x \to x_0} f(x) = f(x_0)$$

will hold at every irrational point x_0 but must fail at every rational point. In the language of continuity we have proved that this function is continuous at every irrational point but discontinuous at every rational point. A curious function: It appears to be continuous at nearly every point and discontinuous at nearly every point. Nineteenth-century mathematicians were quite intrigued by such functions and called them pointwise discontinuous, a term that seems not to have survived.

(We shall return to this example occasionally. For example, Exercise 7.5.4 asks for an account of the local extrema of this function.)

Nondecreasing Functions with Jumps The simplest example of a function with a discontinuity is perhaps

$$H(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

This function fails to have a limit at $x = 0$ since $\lim_{x\to 0+} H(x) = 1$ and $\lim_{x\to 0-} H(x) = 0$. In the language introduced earlier in this section we would say that H has a jump (or a jump discontinuity) at the point 0.

The discontinuity can be placed at any point. The function $H(x - c)$ has a jump at $x = c$. Moreover, if $c_1 < c_2 < c_3 < \cdots < c_k$ is a finite sequence of distinct points, then the function

$$F(x) = \sum_{i=1}^{k} H(x - c_i)$$

is a nondecreasing function with jumps at each of the points c_1, c_2, c_3, …, c_k. At every other point x_0 it is the case that $\lim_{x\to x_0} F(x) = F(x_0)$.

An interesting question now occurs. We have succeeded in constructing a function that is nondecreasing and has jumps at a prescribed finite set of points. Can we construct such a function if we wish to have jumps at a given infinite set of points? This is a question to which we will return.

Step Functions A function f is a *step function* if it assumes finitely many values, say b_1, b_2, …, b_N and for each $1 \leq i \leq N$ the set

$$f^{-1}(b_i) = \{x : f(x) = b_i\},$$

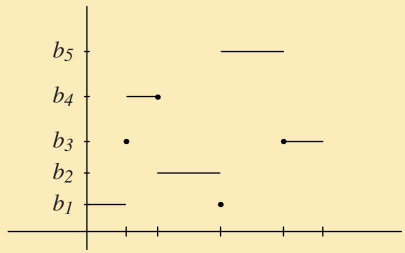

Figure 5.2. Graph of a step function.

which represents the set of points at which f assumes the value b_i, is a finite union of intervals and singleton point sets. Another way to think about this is that a function of the form

$$f(x) = \sum_{i=1}^{M} a_i \chi_{A_i}(x)$$

is a step function if all the A_i are intervals or singleton sets. (See Figure 5.2 for an illustration.)

Step functions play an important role in integration theory. They offer a crude way of approximating functions. The function

$$H(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

that we have just seen is a simple step function since it assumes just two values, 0 and 1, where 0 is assumed on the interval $(-\infty, 0)$ and 1 is assumed on $[0, \infty)$.

The discontinuities of a step function are easy to detect.

Distance of a Closed Set to a Point Let C be a closed set and define a function by writing

$$d(x, C) = \inf\{|x - y| : y \in C\}.$$

This function gives a meaning to the distance between a set C and a point x. If $x_0 \in C$, then $d(x_0, C) = 0$, and if $x_0 \notin C$, then $d(x_0, C) > 0$.

This function is continuous at every point; that is, this function has the property that

$$\lim_{x \to x_0} d(x, C) = d(x_0, C). \tag{1}$$

We can interpret (1) geometrically: If two points x_1 and x_2 are close together, then they are at roughly the same distance from the closed set C.

The Characteristic Function of the Cantor Set Let K be the Cantor set and let χ_K be its characteristic function; that is, let $\chi_K = 1$ if $x \in K$ and $\chi_K(x) = 0$ otherwise. This function has the property that

$$\lim_{x \to x_0} \chi_K(x) = 0$$

if x_0 is not in the Cantor set and the limit exists at no point in the Cantor set. For an easy proof of this you will have to review the properties of the Cantor set and its complement in Exercises 4.3.23 and 4.4.9.

Exercises

5.2.31 Give a proof that includes all necessary details that the limit

$$\lim_{x \to x_0} p(x) = p(x_0)$$

for all polynomials p.

5.2.32 Suppose that you know that

$$\lim_{x \to 2} e^x = e^2.$$

Prove that $\lim_{x \to x_0} e^x = e^{x_0}$ for all x_0.

See Note 108

5.2.33 Suppose that you know that

$$\lim_{x \to 0} \cos x = 1 \text{ and } \lim_{x \to 0} \sin x = 0.$$

Prove that $\lim_{x \to x_0} \sin x = \sin x_0$ for all x_0.

See Note 109

5.2.34 In the text we constructed a nondecreasing function with jumps at each of the points c_1, c_2, c_3, ..., c_k and continuous everywhere else. Construct an *increasing* function with this property.

See Note 110

5.2.35 Let $f : [a, b] \to \mathbb{R}$ be a step function. Show that there is a partition

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

so that f is constant on each interval (x_{i-1}, x_i), $i = 1, 2, \ldots, n$.

5.2.36 Suppose that

$$f(x) = \sum_{i=1}^{M} a_i \chi_{A_i}$$

where the A_i are intervals. Show that f is a step function; that is, that f assumes finitely many values, and for each b in the range of f the set $f^{-1}(b)$ is a finite union of intervals or singleton sets. Where are the discontinuities of such a function?

See Note 111

5.2.37 Show that the characteristic function of the rationals can also be defined by the formula

$$\chi_{\mathbb{Q}}(x) = \lim_{m\to\infty} \lim_{n\to\infty} |\cos(m!\pi x)|^n.$$

5.2.38 Show that

$$\lim_{x\to x_0+} \chi_{\mathbb{Q}}(x) \text{ and } \lim_{x\to x_0-} \chi_{\mathbb{Q}}(x)$$

both fail to exist, where $\chi_{\mathbb{Q}}$ is the characteristic function of the rationals. What would be the answer to the corresponding question for the characteristic function of the irrationals?

5.2.39 Describe the graph of the function $\chi_{\mathbb{Q}}$. What kind of a sketch would convey this set?

5.2.40 Give an example of a set E such that the characteristic function χ_E of E has limits at every point. Can you describe the most general set E with this property?

5.2.41 Give an example of a set E such that the characteristic function χ_E of E has one-sided limits at every point. Can you describe the most general set E with this property?

5.2.42 Show that

$$\lim_{x \to x_0} d(x, C) = d(x_0, C)$$

at every point x_0 where $d(x, C)$ is the distance from x to the closed set C as defined in this section.

5.2.43 Sketch the graph of the function $d(x, C)$ for several closed sets C (e.g., $\{0\}$, $\mathbb{N}$, $[0, 1]$, $\{0\} \cup \{1, 1/2, 1/3, 1/4, \dots\}$, and $[0, 1] \cup [2, 3]$).

5.2.44 Sketch the graph of the characteristic function χ_K of the Cantor set (Exercises 4.3.23 and 4.4.9) and show that

$$\lim_{x \to x_0} \chi_K(x) = 0$$

at all points x not in the Cantor set and that this limit fails to exist at all points in Cantor set.

SEE NOTE 112

5.3 Limits Superior and Inferior

✂ *Adv.*

If limits fail to exist we need not abandon all hope of discussing the limiting behavior. We saw this situation in our study of sequence limits in Section 2.13. Even if $\{s_n\}$ diverges so that $\lim_{n\to\infty} s_n$ fails to exist, it is possible that the two extreme limits

$$\liminf_{n\to\infty} s_n \text{ and } \limsup_{n\to\infty} s_n$$

provide some meaningful information. These two concepts always exist (possibly as ∞ or $-\infty$). A similar situation occurs for functions. The theory is nearly identical in many respects.

Definition 5.26: (Lim Sup and Lim Inf) Let $f : E \to \mathbb{R}$ be a function with domain E and suppose that x_0 is a point of accumulation of E. Then we write

$$\limsup_{x\to x_0} f(x) = \inf_{\delta>0} \sup\{f(x) : x \in (x_0 - \delta, x_0 + \delta) \cap E,\ x \neq x_0\}$$

and

$$\liminf_{x\to x_0} f(x) = \sup_{\delta>0} \inf\{f(x) : x \in (x_0 - \delta, x_0 + \delta) \cap E,\ x \neq x_0\}$$

As this section is for more advanced readers we have left the development of this concept to the exercises.

Exercises

5.3.1 Show from the definition that

$$\limsup_{x\to x_0} f(x) \geq \liminf_{x\to x_0} f(x).$$

5.3.2 Compute each of the following.

(a) $\limsup_{x\to 0} \sin x^{-1}$

(b) $\limsup_{x\to 0} x \sin x^{-1}$

(c) $\limsup_{x\to 0} x^{-1} \sin x^{-1}$

5.3.3 Formulate an equivalent definition for $\limsup_{x\to 0} f(x)$ expressed in terms of sequential limits; that is, in terms of limits of $f(x_n)$ for $x_n \to x_0$. Show that your definition is equivalent to that in the text.

5.3.4 Give an example of a function f so that

$$\liminf_{x\to 0} f(x) = 0 \text{ and } \limsup_{x\to 0} f(x) = 1.$$

5.3.5 What changes, if any, are there if the definition of lim sup had been written as

$$\limsup_{x\to x_0} f(x) = \inf_{\delta>0} \sup\{f(x) : x \in (x_0-\delta, x_0+\delta) \cap E\}?$$

SEE NOTE 113

5.3.6 Formulate a definition for the one-sided concepts

$$\limsup_{x\to x_0+} f(x) \text{ and } \limsup_{x\to x_0-} f(x).$$

5.3.7 Give an example of a function f with the properties

$$\liminf_{x\to 0+} f(x) = 0,$$

$$\limsup_{x\to 0+} f(x) = \infty,$$

$$\liminf_{x\to 0-} f(x) = -\infty,$$

and

$$\limsup_{x\to 0-} f(x) = 1.$$

5.3.8 Show that $\lim_{x\to 0} f(x)$ exists if and only if all four of

$$\liminf_{x\to 0+} f(x),$$

$$\limsup_{x\to 0+} f(x),$$

$$\liminf_{x\to 0-} f(x),$$

and

$$\limsup_{x\to 0-} f(x)$$

are equal and finite.

5.3.9 Show that $\lim_{x\to 0} f(x) = \infty$ if and only if

$$\liminf_{x\to 0+} f(x) = \infty,$$

$$\limsup_{x\to 0+} f(x) = \infty,$$

$$\liminf_{x\to 0-} f(x) = \infty,$$

and

$$\limsup_{x\to 0-} f(x) = \infty.$$

5.3.10 For the function $\chi_{\mathbb{Q}}$, the characteristic function of the rationals, determine the values of each of the limits

$$\liminf_{x\to x_0+} \chi_{\mathbb{Q}}(x),\ \limsup_{x\to x_0+} \chi_{\mathbb{Q}}(x),\ \liminf_{x\to x_0-} \chi_{\mathbb{Q}}(x),\ \text{and} \limsup_{x\to x_0-} \chi_{\mathbb{Q}}(x)$$

at any point x_0.

5.3.11 Give an example of a function f such that
$$\{x_0 : \limsup_{x\to x_0-} f(x) > \limsup_{x\to x_0+} f(x)\}$$
is infinite.

5.4 Continuity

The earliest use of the term "continuity" is somewhat clouded by misconceptions of the nature of a function. If a function was given by a single formula then it was considered in the eighteenth century to be "continuous." If, however, the function had a "break" in the formula—defined differently in one interval than in another—it was considered as "discontinuous." As the subject developed these notions continued to obscure the really important ideas. Augustin Cauchy (1789–1857) was the first to give the modern definition and to focus attention on the concept that has now assumed such an important role in analysis.

5.4.1 How to Define Continuity

Before we proceed to the present day definition, let us consider another notion. Even as late as the middle of the nineteenth century, some mathematicians believed this notion should form the basis for a definition of continuity. This concept is suggested by the phrase "the graph has no jumps." While some instructors of calculus courses might use such phrases to convey a sense of continuity to students, the phrase is not a precise one, nor does it fully convey all we wish a continuous function to be.

This notion *is* related to continuity, however, and has some importance in its own right. We'll begin with a brief discussion of it. Here is one attempt at making our phrase precise. (See Figure 5.3.)

Definition 5.27: (Intermediate Value Property) Let f be defined on an interval I. Suppose that for each $a, b \in I$ with $f(a) \neq f(b)$, and for each d between $f(a)$ and $f(b)$, there exists c between a and b for which $f(c) = d$. We then say that f has the *intermediate value property* (IVP) on I.

Functions with this property are called *Darboux* functions after Jean Gaston Darboux (1842–1917), who showed in 1875 that for every differentiable function F on an interval I, the derivative F' has the IVP on

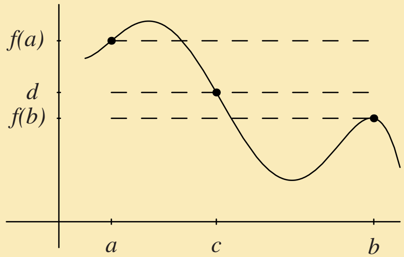

Figure 5.3. At some point between a and b the function assumes any given value d between $f(a)$ and $f(b)$.

I. He is also particularly famous for his 1875 account of the Riemann integral using upper and lower sums; often reference is made to the "Darboux integral," meaning this version of the classical Riemann integral.

Example 5.28: Let

$$F(x) = \begin{cases} \sin x^{-1} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

The graph of F is shown in Figure 5.4. You may wish to verify that F has the IVP. In particular, F assumes every value in the interval $[-1, 1]$ infinitely often in every neighborhood of $x = 0$. ◀

We haven't yet made precise the phrase "the graph has no jumps," but the IVP seems to convey that idea well enough. Since this property is so easy to describe and appears to have content that is easy to visualize, why *not* take it as the definition of continuity?

Before attempting to answer that question, let us offer a competing phrase to capture the idea of continuity: "If x is near x_0, then $f(x)$ is near $f(x_0)$." As stated, this phrase is not precise, but we can make it precise using the limit concept. This phrase could be interpreted really as asserting that

$$f(x_0) = \lim_{x \to x_0} f(x). \tag{2}$$

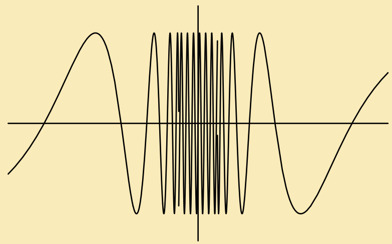

Figure 5.4. Graph of the function $F(x) = \sin x^{-1}$ on $[-\pi/8, \pi/8]$.

According to this criterion our function F of Example 5.28 would *not* be continuous at $x_0 = 0$, because $F(0) = 0$, but $\lim_{x\to 0} F(x)$ does not exist.

We shall see presently that the definition based on limits allows the development of a useful theory. We'll see that the class of continuous functions [as defined using equation (2)] is closed under addition and multiplication, and that such functions have many other desirable properties. For example, the class is closed under certain kinds of limits of sequences, and every continuous function on $[a, b]$ is integrable. On the other hand (as is shown in the exercises), none of the analogous statements is valid for the class of functions defined by IVP.

Thus a theory of continuity based on the limit concept allows a rich structure and enjoys wide applicability, whereas one based on the IVP is rather limited. In addition, the fundamental notion of limit extends to much more general settings than $\mathbb{R}$. In contrast, extensions of IVP, while possible, are peripheral to mathematical analysis.

Exercises

5.4.1 Refer to Example 5.28. Let

$$G(x) = \begin{cases} -F(x) & \text{if } x \neq 0 \\ 1 & \text{if } x = 0. \end{cases}$$

Show that G has the IVP, yet $F + G$ does not. Thus the class of functions with IVP is not closed under addition.

5.4.2 Give an example to show that the class of functions with IVP is not closed under multiplication.

5.4.3 Let

$$H(x) = \begin{cases} x^{-1}\sin x^{-1} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Show that H has IVP on $[0,1]$ but is not bounded there. (This shows that the IVP does not imply boundedness; we shall see that, in contrast, any continuous function on $[0,1]$ would have to be bounded.)

5.4.4 Give an example of a function f with IVP on $[0,1]$ that is bounded but achieves no absolute maximum on [0,1].

5.4.5 Let K be the Cantor set of Exercises 4.3.23 and 4.4.9 and let $\{(a_k, b_k)\}$ be the sequence of intervals complementary to K in (0,1).

(a) Write down a set of equations defining a function f that vanishes at every two-sided point of accumulation of K, is continuous on each interval $[a_k, b_k]$, and for which

$$\lim_{x\to a_k+} f(x) = -1 \text{ and } \lim_{x\to b_k-} f(x) = 1.$$

(See Figure 5.5 for an illustration of one possible choice.)

(b) Verify that f has the intermediate value property.

(c) Verify that f is not continuous in the sense that $f(x_0) = \lim_{x\to x_0} f(x)$ fails at certain points. (Which points?)

5.4.6 ✂ We construct a function with IVP whose graph may be more difficult to visualize. Let $I_0 = (0,1)$. Each $x \in I_0$ has a unique decimal expansion not ending in a string of 9's. For each $n \in \mathbb{N}$ and $x = .a_1a_2\ldots$ in I_0, let

$$f_n(x) = \frac{a_1(x) + a_2(x) + \cdots + a_n(x)}{n}.$$

Thus $f_n(x)$ represents the average of the first n digits of x. For each $x \in I_0$, let $f(x) = \limsup_n f_n(x)$.

(a) Show that $f : I_0 \to [0,10]$.

(b) Describe how to construct $x \in I_0$ such that $f(x) = \pi$.

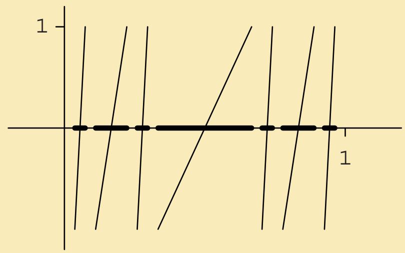

Figure 5.5. Function defined on the complement of the Cantor set, as described in Exercise 5.4.5. The first three stages are shown.

(c) Describe how to construct $x \in (.01, .02)$ such that $f(x) = \pi$.

(d) Show that for each interval $(a, b) \subset I_0$ and each $d \in [0, 10]$ there exists $c \in (a, b)$ such that $f(c) = d$. Thus, f assumes every value in [0,10] in every interval in I_0. In particular, f has IVP.

(e) Let $A = \{x : f(x) = x\}$. Let $g(x) = 0$ if $x \in A$, $g(x) = f(x)$ for $x \notin A$. Show that $g(x)$ has IVP.

(f) Show that $-g(x) + x$ does not have IVP. Thus the sum of a function with IVP with the identity function need not have IVP.

5.4.2 Continuity at a Point

Let us look at Cauchy's concept of *continuous function*. We begin by defining continuity at a point, more specifically continuity at an interior point of the domain of a function f. This way we are assured that if we are interested in what is happening at the point x_0 then f is defined in a neighborhood of x_0; that is, that f is defined in some interval $(x_0 - c, x_0 + c)$ for a positive number c. This simplifies some of the computations.

Definition 5.29: (Continuous) Let f be defined in a neighborhood of x_0. The function f is *continuous* at x_0 provided $\lim_{x\to x_0} f(x) = f(x_0)$.

This means that for each neighborhood V of $f(x_0)$ there is a neighborhood U of x_0 such that $f(U) \subset V$: that is, if $x \in U$, then $f(x) \in V$. We can, of course, state the definition in terms of δ's and ε's: f is continuous at x_0 if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(x_0)| < \varepsilon$ whenever $|x - x_0| < \delta$. In the exercises we ask you to verify that the three formulations, involving the language of limits, of neighborhoods, and of δ's and ε's, are equivalent. We believe that this is an important exercise for readers who do not yet feel comfortable with the limit concept. Feeling comfortable with the various forms that continuity takes is essential to feeling comfortable with many of the arguments that appear in the sections and chapters that follow.

Observe that a function f can fail to be continuous at x_0 in three ways:

1. f is not defined at x_0.

2. $\lim_{x \to x_0} f(x)$ fails to exist.

3. f is defined at x_0 and $\lim_{x \to x_0} f(x)$ exists, but

$$f(x_0) \neq \lim_{x \to x_0} f(x).$$

We leave it to you to provide simple examples of each of these possibilities.

Example 5.30: Let $f : (0, \infty) \to \mathbb{R}$ be defined by $f(x) = 1/x$. We show that if $x_0 \in (0, \infty)$, then f is continuous at x_0.

Let's first try the "neighborhood" definition of continuity. Let V be a neighborhood of $f(x_0)$, say $V = (A, B)$. Thus $A < f(x_0) < B$. We must find a neighborhood $U = (a, b)$ of x_0 such that $f(U) \subset V$. A picture suggests what to do: Let $a = 1/B$, $b = 1/A$. (See Figure 5.6.) But we must be a bit careful here. Nothing in our neighborhood definition of continuity allows us to assume $A > 0$, so b might not be defined (if $A = 0$), or might not be in the domain of f (if $A < 0$). This presents, however, only a minor nuisance. Thus we assume $A > 0$ in our proof.

So, let us assume $A > 0$, $a = 1/B$, $b = 1/A$. Then, since $A < f(x_0) < B$, we have

$$b = \frac{1}{A} > x_0 = \frac{1}{f(x_0)} > \frac{1}{B} = a,$$

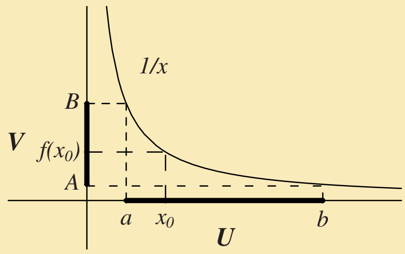

Figure 5.6. Graphical interpretation of the neighborhood definition of continuity for the function $f(x) = x^{-1}$. Note that $a = 1/B$ and $b = 1/A$.

so $x_0 \in (a,b) = U$. Furthermore, if $c \in U$, then $a < c < b$ and

$$B > 1/c = f(c) > A,$$

so $f(c) \in V$. This shows that $f(U) \subset V$ as was required. ◀

Example 5.31: Let us take the same example $f : (0,\infty) \to \mathbb{R}$, defined by $f(x) = 1/x$. We show again that if $x_0 \in (0,\infty)$, then f is continuous at x_0.

We see now how a proof based on the δ-ε definition might look. As with our first proof, we shall provide many details of the proof. After you feel conversant with limits and continuity, you may wish to streamline the proofs somewhat by leaving out details that "any reader finds obvious."

Let $x_0 \in (0,\infty)$ and let $\varepsilon > 0$. We wish to find $\delta > 0$ such that if $|x - x_0| < \delta$ and $x > 0$, then $|1/x - 1/x_0| < \varepsilon$. Rewriting this last inequality as

$$|x - x_0| < \varepsilon x x_0 \tag{3}$$

suggests we try $\delta = \varepsilon x x_0$. But δ should depend only on ε and x_0, not on x. There is no $\delta > 0$ for which the inequality $|x - x_0| < \delta$ implies the inequality

$$|x - x_0| < \varepsilon x x_0$$

for all $x \in (0, \infty)$. We can remove this problem by first requiring x to stay away from 0.

For example, we first require that

$$|x - x_0| < \frac{1}{2}x_0. \tag{4}$$

Then

$$\frac{1}{2}x_0 < x \text{ and} \tag{5}$$

$$\frac{1}{2}\varepsilon x_0^2 < \varepsilon x x_0. \tag{6}$$

The inequalities (3), (4), and (6) suggest taking

$$\delta = \min\left(\frac{1}{2}x_0, \frac{1}{2}x_0^2\varepsilon\right).$$

For this δ , we compute easily that if $|x - x_0| < \delta$, then

$$\left|\frac{1}{x} - \frac{1}{x_0}\right| = \frac{|x - x_0|}{|xx_0|} < \frac{\frac{1}{2}x_0^2\varepsilon}{\frac{1}{2}x_0^2} = \varepsilon,$$

the last inequality being obtained by using (6) on the numerator $|x - x_0|$ and (5) on the denominator $|xx_0|$. ◀

Exercises

5.4.7 Prove that the function $f(x) = x^2$ is continuous at every point of $\mathbb{R}$ using the δ-ε form of continuity,

5.4.8 Prove that the function $f(x) = |x|$ is continuous at every point of $\mathbb{R}$ using the δ-ε form of continuity,

5.4.9 Show that the three formulations of continuity appearing at the beginning of this section are equivalent.

5.4.10 In the δ-ε verification of continuity of the function $1/x$ we obtained a δ that did the job. We made no claim that this δ is the largest possible δ we could have chosen. Show that for $\varepsilon = 1$ and $x_0 \in (0, 1)$ any δ that works must satisfy $\delta < x_0^2/(1 + x_0)$.

5.4.11 **(Sequence Definition of Continuity)** Prove that f is continuous at x_0 if and only if $\lim_{n\to\infty} f(x_n) = f(x_0)$ for every sequence $\{x_n\} \to x_0$. How would you expect this characterization of continuity at x_0 to be modified if x_0 is not an interior point of its domain?

5.4.12 Give three examples of a function f that fails to be continuous at a point x_0. The first should be discontinuous merely because f is not defined at x_0. The second should be discontinuous because $\lim_{x\to x_0} f(x)$ fails to exist. The third should have neither of these defects but should nonetheless be discontinuous.

5.4.13 A function f is said to be *symmetrically continuous* at a point x if

$$\lim_{h\to 0+} [f(x+h) - f(x-h)] = 0.$$

Show that if f is continuous at a point, then it must be symmetrically continuous there and that the converse does not hold.

5.4.3 Continuity at an Arbitrary Point

To this point we have discussed continuity of a function at an interior point of its domain. How should we modify our notions if x_0 is not an interior point?

Continuity at Endpoints For example, if $f : [a, b] \to \mathbb{R}$, how can we define continuity of f at a or at b? Since the function is defined only on the interval $[a, b]$ and we have defined continuity in terms of limits, it seems that we should require, as before, that for any interior point $x_0 \in (a, b)$

$$\lim_{x\to x_0} f(x) = f(x_0)$$

while at the endpoints continuity would be defined by a one-sided limit,

$$\lim_{x\to a+} f(x) = f(a) \quad \text{and} \quad \lim_{x\to b-} f(x) = f(b).$$

We can also reformulate our definition in a way that recognizes that f is defined only on $[a, b]$. In our neighborhood definition we interpret U as a *relative* neighborhood of x_0: We require that $f(U \cap [a, b]) \subset V$. Here by a *relative neighborhood* we mean that part of an ordinary neighborhood that is inside the set where f is defined.

Similarly, for the δ-ε definition, our requirement becomes that

$$|f(x) - f(x_0)| < \varepsilon$$

whenever $|x - x_0| < \delta$ and $x \in [a, b]$. Again we are merely restricting our attention to the set where f is defined.

Continuity on an Arbitrary Set These reformulations would work for any set A, not just an interval. Thus we assume that

$$f : A \to \mathbb{R}$$

so that f is a function with domain A and x_0 is an arbitrary point of A, which need not be an interior point nor even a point of accumulation (it might be isolated in A).

There are four versions of the definition. As before, you should check to see that they are indeed equivalent. Some extra care is needed because x_0 could be any point of A and may be isolated in A.

Definition 5.32: (ε-δ Version) Let f be defined on a set A and let x_0 be any point of A. The function f is *continuous* at x_0 provided for every $\varepsilon > 0$ there is a $\delta > 0$ so that

$$|f(x) - f(x_0)| < \varepsilon$$

for every $x \in A$ for which $|x - x_0| < \delta$.

Definition 5.33: (Limit Version) Let f be defined on a set A and let x_0 be any point of A. The function f is *continuous* at x_0 provided either that x_0 is isolated in A or else that x_0 is a point of accumulation of that set and

$$\lim_{x \to x_0} f(x) = f(x_0).$$

Definition 5.34: (Neighborhood Version) Let f be defined on a set A and let x_0 be any point of A. The function f is *continuous* at x_0 provided that for every open set V containing $f(x_0)$ there is an open set U containing x_0 so that $f(U \cap A) \subset V$.

In other words, the neighborhood version asserts that there is a set $U \cap A$ open relative to A that f maps into V. We recall that a set $B \subset A$ is relatively open relative to A if B is the intersection of some open set (here U) with A.

Definition 5.35: (Sequential Version) Let f be defined on a set A and let x_0 be any point of A. The function f is *continuous* at x_0 provided that for every sequence of points $\{x_n\}$ belonging to A and converging to x_0, it follows that $f(x_n) \to f(x_0)$.

Exercises

5.4.14 Prove the equivalence of the four definitions for the continuity of a function defined on an arbitrary set A.

5.4.15 Let $f : \mathbb{N} \to \mathbb{R}$ by writing $f(n) = 1/n^2$. Is f continuous at any point in its domain?

SEE NOTE 114

5.4.16 Using each of the four versions of continuity, show that any function is automatically continuous at any point of its domain that is isolated.

5.4.17 Let f be defined on the set containing the points

$$0,\ 1,\ 1/2,\ 1/4,\ 1/8, \dots,\ 1/2^n$$

only. What values can you assign at these points that will make this function continuous everywhere where it is defined?

SEE NOTE 115

5.4.18 Let f be defined on the set containing the points $0, \pm 1, \pm 1/2, \pm 1/4, \pm 1/8, \dots, \pm 1/2^n, \dots$ only. What values can you assign at these points that will make this function continuous everywhere where it is defined?

5.4.19 If f is continuous at a point x_0 then is it necessarily true that

$$\lim_{x \to x_0} f(x) = f(x_0)?$$

At what points in the domain of f can you say this?

SEE NOTE 116

5.4.20 A function $f : [a, b] \to \mathbb{R}$ is said to be *Lipschitz* if there is a positive number M so that $|f(x)-f(y)| \leq M|x-y|$ for all x, $y \in [a, b]$. Show that a Lipschitz function must be continuous. Is the converse true? [Rudolf Otto Sigismund Lipschitz (1832–1903) is probably best remembered for this condition, now forever attached to his name, which he used in formulating an existence theorem for differential equations of the form $y' = f(x, y)$.]

SEE NOTE 117

5.4.4 Continuity on a Set

Enrich.

Continuity is defined at points. A function such as $f(x) = x^2$ could be said to be continuous at every real number x_0, meaning only that $\lim_{x\to x_0} x^2 = x_0^2$ for every real number. In many cases the function considered is continuous at every point in its domain. We say simply that f is continuous. But we must remember this is an assertion about every single point where f is defined.

Definition 5.36: Let $f : A \to \mathbb{R}$. Then f is *continuous* (or *continuous on* A) if f is continuous at each point of A.

If we wish to prove directly from this definition that f is continuous, we must show that f is continuous at every $x_0 \in A$. It is sometimes easier to use the *global* characterization of continuity that follows.

Theorem 5.37: *Let $f : A \to \mathbb{R}$. Then f is continuous if and only if for every open set $V \subset \mathbb{R}$, the set $f^{-1}(V) = \{x \in A : f(x) \in V\}$ is open (relative to A).*

Proof. Suppose first that f is continuous. Let V be open, let $x_0 \in f^{-1}(V)$ and choose $\alpha < \beta$ so that $(\alpha, \beta) \subset V$ and so that $x_0 \in f^{-1}((\alpha, \beta))$. Then $\alpha < f(x_0) < \beta$. We will find a neighborhood U of x_0 such that $\alpha < f(x) < \beta$ for all $x \in U \cap A$. Let $\varepsilon = \min(\beta - f(x_0), f(x_0) - \alpha)$. Since f is continuous at x_0, there exists $\delta > 0$ such that if

$$x \in A \cap (x_0 - \delta, x_0 + \delta),$$

then

$$|f(x) - f(x_0)| < \varepsilon.$$

Thus

$$f(x) - f(x_0) < \beta - f(x_0),$$

and so $f(x) < \beta$. Similarly,

$$f(x) - f(x_0) > \alpha - f(x_0),$$

and so $f(x) > \alpha$. Thus the relative neighborhood $U = (x_0 - \delta, x_0 + \delta) \cap A$ is a subset of $f^{-1}((\alpha, \beta))$ and hence also a subset of $f^{-1}(V)$. We have shown that each member of $f^{-1}(V)$ has a relative neighborhood in $f^{-1}(V)$. That is, $f^{-1}(V)$ is open relative to A.

To prove the converse, suppose f satisfies the condition that for each open interval (α, β) with $\alpha < \beta$, the set $f^{-1}((\alpha, \beta))$ is open relative to A. Let $x_0 \in A$. We must show that f is continuous at x_0. Let $\varepsilon > 0$, $\beta = f(x_0) + \varepsilon$, $\alpha = f(x_0) - \varepsilon$. Our hypothesis implies that $f^{-1}((\alpha, \beta))$ is open relative to A. Thus

$$f^{-1}((\alpha, \beta)) = \bigcup (a_i, b_i) \cap A,$$

the union being a finite or countable union of pairwise disjoint open intervals. One of these intervals, say (a_j, b_j), contains x_0. Let

$$\delta = \min(x_0 - a_j, b_j - x_0).$$

For $|x - x_0| < \delta$ and $x \in A$ we find

$$\alpha < f(x) < \beta.$$

Because $\beta = f(x_0) + \varepsilon$ and $\alpha = f(x_0) - \varepsilon$ we must have

$$|f(x) - f(x_0)| < \varepsilon.$$

This shows that f is continuous at x_0. ■

We spelled out the details of the proof of Theorem 5.37. This may have caused it to appear rather lengthy. But the proof is nothing more than writing down in a rigorous way what some intuitive pictures indicate. You might find that the neighborhood notion of continuity is a more natural one to use for proving the theorem. We leave this as Exercise 5.4.23.

As a corollary let us point out that we can replace open sets by open intervals; thus to check continuity of a function f it is enough to show that $f^{-1}((\alpha, \beta))$ is open for every interval (α, β).

Corollary 5.38: *Let $f : A \to \mathbb{R}$. Then f is continuous if and only if for every interval (α, β), $f^{-1}((\alpha, \beta))$ is open (relative to A).*

Proof. We verify that the conditions (i) $f^{-1}(V)$ is relatively open for all open $V \subset \mathbb{R}$ and (ii) $f^{-1}((\alpha, \beta))$ is relatively open for all $\alpha < \beta$ are equivalent. But this is immediate. If (i) is satisfied, then (ii) is also, since the requirement (ii) is just a special case of (i). On the other hand, if (ii) is satisfied and

$$V = \bigcup (\alpha_i, \beta_i),$$

then

$$f^{-1}(V) = \bigcup f^{-1}((\alpha_i, \beta_i)).$$

Each of the sets $f^{-1}((\alpha_i, \beta_i))$ is open by hypothesis, so $f^{-1}(V)$ is also open because it is a union of a family of open sets. ■

Example 5.39: Let $f(x) = 1/x$ $(x > 0)$. We find

$$f^{-1}((\alpha, \beta)) = \left(\frac{1}{\beta}, \frac{1}{\alpha}\right).$$

Since $(1/\beta, 1/\alpha)$ is open it would follow that f is continuous on $(0, \infty)$. ◀

Exercises

5.4.21 Prove that the function $f(x) = x^2$ is continuous on $\mathbb{R}$ by using Theorem 5.37.

5.4.22 Prove that the function $f(x) = |x|$ is continuous on $\mathbb{R}$ by using Theorem 5.37.

5.4.23 Prove Theorem 5.37 using the neighborhood definition of continuity.

5.4.24 Let f be continuous in a neighborhood U of the point x_0. If $f(x) < \beta$ for all $x \in U \setminus \{x_0\}$, prove that $f(x_0) \le \beta$. Show by example that we cannot conclude $f(x_0) < \beta$.

5.4.25 Let f, g be defined on $\mathbb{R}$. Suppose $f(0) = 0$ and f is continuous at $x = 0$. Suppose g is bounded in some neighborhood of zero. Prove that fg is continuous at $x = 0$. Apply this to the function $f(x) = x\sin(1/x)$ ($f(0) = 0$) at $x = 0$.

5.4.26 Let $x_0 \in \mathbb{R}$. Following are four δ-ε conditions on a function $f : \mathbb{R} \to \mathbb{R}$. Which, if any, of these conditions imply continuity of f at x_0? Which, if any, are implied by continuity at x_0?

(a) For every $\varepsilon > 0$ there exists $\delta > 0$ such that if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \varepsilon$.

(b) For every $\varepsilon > 0$ there exists $\delta > 0$ such that if $|f(x) - f(x_0)| < \delta$, then $|x - x_0| < \varepsilon$.

(c) For every $\varepsilon > 0$ there exists $\delta > 0$ such that if $|x - x_0| < \varepsilon$, then $|f(x) - f(x_0)| < \delta$.

(d) For every $\varepsilon > 0$ there exists $\delta > 0$ such that if $|f(x) - f(x_0)| < \varepsilon$, then $|x - x_0| < \delta$.

5.4.27 Let $x_0 \in \mathbb{R}$. Following are four δ-ε conditions on a function $f : \mathbb{R} \to \mathbb{R}$. Which, if any, of these conditions imply continuity of f at x_0? Which, if any, are implied by continuity at x_0?

(a) There exists $\varepsilon > 0$ such that for each $\delta > 0$, if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \varepsilon$.

(b) There exists $\varepsilon > 0$ such that for each $\delta > 0$, if $|f(x) - f(x_0)| < \delta$, then $|x - x_0| < \varepsilon$.

(c) There exists $\varepsilon > 0$ such that for each $\delta > 0$, if $|x - x_0| < \varepsilon$, then $|f(x) - f(x_0)| < \delta$.

(d) There exists $\varepsilon > 0$ such that for each $\delta > 0$, if $|f(x) - f(x_0)| < \varepsilon$, then $|x - x_0| < \delta$.

5.4.28 For each of the eight conditions of Exercises 5.4.26 and 5.4.27, describe in words which functions satisfy the condition. (Some of these conditions characterize familiar classes of functions, including the empty class.)

5.4.29 Let $A \subset \mathbb{R}$, $f : A \to \mathbb{R}$, $g : f(A) \to \mathbb{R}$. Prove that if f is continuous at $x_0 \in A$ and g is continuous at $f(x_0)$, then $g \circ f$ is continuous at x_0. Apply this to prove that if f is continuous at x_0, then $|f|$ is continuous at x_0.

5.4.30 Using the notions of unilateral or one-sided limits, define *left continuity* of a function f at a point x_0. Do the same for *right continuity*. If f is defined in a neighborhood of x_0, prove that f is continuous at x_0 if and only if f is both left continuous and right continuous at x_0.

5.4.31 Let $f : \mathbb{R} \to \mathbb{R}$. Prove that f is continuous if and only if for every closed set $K \subset \mathbb{R}$, the set $f^{-1}(K)$ is closed in $\mathbb{R}$. State carefully and prove the analogous result if $f : A \to \mathbb{R}$, where A is an arbitrary nonempty subset of $\mathbb{R}$.

5.4.32 Suppose f has the IVP on (a, b) and is discontinuous at $x_0 \in (a, b)$. Prove that there exists $y \in \mathbb{R}$ such that $\{x : f(x) = y\}$ is infinite.

5.5 Properties of Continuous Functions

We now present some of the most basic of the properties of continuous functions. The first theorem is an algebraic one; it asserts that the family of continuous functions defined on a set has many of the properties of an *algebra*: elements may be added, subtracted, multiplied, and (under some conditions) divided.

Theorem 5.40: *Let $f, g : A \to \mathbb{R}$ and let $c \in \mathbb{R}$. Suppose f and g are continuous at $x_0 \in A$. Then cf, $f + g$ and fg are continuous at x_0. Furthermore, if $g(x_0) \neq 0$, then f/g is continuous at x_0.*

Proof. The results follow immediately from the limit definition of continuity and the usual algebraic properties of limits. ■

Corollary 5.41: *Every polynomial is continuous on $\mathbb{R}$.*

Proof. The functions $f(x) = 1$ and $g(x) = x$ are continuous on $\mathbb{R}$. The corollary follows from Theorem 5.40. ■

Corollary 5.42: *Every rational function is continuous at each point in its domain (i.e., at each $x \in \mathbb{R}$ at which the denominator does not vanish).*

One of our most important properties allows us to compose two continuous functions. Be careful, though, with the conditions on the domains as they cannot be overlooked.

Theorem 5.43: *Let $f : A \to \mathbb{R}$, $g : B \to \mathbb{R}$ and suppose that $f(A) \subset B$. Suppose that f is continuous at a point $x_0 \in A$ and that g is continuous at the point $y_0 = f(x_0) \in B$. Then the composition function*

$$g \circ f : A \to \mathbb{R}$$

is continuous at x_0.

Proof. This follows from Theorem 5.25. ■

A global version follows as a corollary.

Corollary 5.44: *Let $f : A \to \mathbb{R}$, $g : B \to \mathbb{R}$ and suppose that $f(A) \subset B$. If f is continuous on A and g is continuous on B, then the composition function*

$$g \circ f : A \to \mathbb{R}$$

is continuous on A.

Exercises

5.5.1 If f and g are functions such that $f + g$ is continuous, does it follow that at least one of f or g must be continuous?

5.5.2 If $|f|$ is continuous, does it follow that f is continuous?

5.5.3 If $e^{f(x)}$ is continuous, does it follow that f is continuous?

5.5.4 If $f(f(x))$ is continuous, does it follow that f is continuous?

5.6 Uniform Continuity

Let us take a closer look at the meaning of continuity of a function f on an interval I. The definition asserts that for each $x_0 \in I$ and for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $x \in I$ and $|x - x_0| < \delta$, then

$$|f(x) - f(x_0)| < \varepsilon.$$

Now carefully consider the following statement:

For every $\varepsilon > 0$, there exists $\delta > 0$ such that if x, $x_0 \in I$ and $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \varepsilon$.

This may appear at first sight to be just a restatement of the meaning of continuity expressed in the first paragraph. If you cannot detect the difference, then you are in good company: Cauchy did not see any difference and used the property just quoted incorrectly to prove that a continuous function on an interval $[a, b]$ must be integrable.

We need to focus on the fact that the number δ depends not only on f and on ε, but also on x_0; that is, $\delta = \delta(f, \varepsilon, x_0)$.

Example 5.45: Consider the function $f(x) = 1/x$ on the interval $I = (0, 1)$. We found in Exercise 5.4.10 that if we take $\varepsilon = 1$, we can choose

$$\delta(f, 1, x_0) = \frac{x_0^2}{1 + x_0},$$

but we cannot choose a larger value. Thus if $x_0 \to 0$, then $\delta(f, 1, x_0) \to 0$. No number δ is sufficiently small to "work" for *all* $x_0 \in I$. ◀

It is often important to be able to select δ independently of x_0. When this is possible, we say that f is uniformly continuous on I.

Definition 5.46: (Uniformly Continuous) Let f be defined on a set $A \subset \mathbb{R}$. We say that f is *uniformly continuous* (on A) if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $x, y \in A$ and $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$.

As an illustration of the usefulness of uniform continuity, we note that if f is uniformly continuous on a bounded interval I, then f is bounded on I.

Theorem 5.47: *If a function f is uniformly continuous on a bounded interval I, then f is bounded on I.*

Proof. Here we suppose that I is one of (a, b), $[a, b]$, $[a, b)$, or $(a, b]$. To check that f is bounded, choose δ so that $|f(x) - f(y)| < 1$ whenever $x, y \in I$ and $|x - y| < \delta$. There is a finite set $a = x_0 < x_1 < \cdots < x_n = b$ such that $|x_i - x_{i-1}| < \delta$ for $i = 1, \ldots, n$. Our definition of δ implies that f is bounded on each of the intervals $[x_{i-1}, x_i] \cap I$. Let

$$\begin{aligned} m_i &= \inf\{f(x) : x_{i-1} \le x \le x_i\ ,\ x \in I\}, \\ M_i &= \sup\{f(x) : x_{i-1} \le x \le x_i\ ,\ x \in I\}, \\ m &= \min\{m_1, \ldots, m_n\} \\ M &= \max\{M_1, \ldots, M_n\}. \end{aligned}$$

Then, for every $x \in I$, $m \le f(x) \le M$, so f is bounded on I. ■

Observe that if we tried to present a similar argument for the function $f(x) = 1/x$ on the interval $I = (0, 1)$, the continuity of f would allow us to conclude that every $x \in I$ is in an interval on which f is bounded, but we would be unable to obtain a finite number of such intervals that cover I.

In our illustration that uniform continuity on I implies boundedness, we did not specify whether I contained one or more of its endpoints. Our next objective is to show that when $I = [a, b]$ is a *closed* interval, then every function f that is continuous on I is uniformly continuous on I. (Note also the more general version given in Exercise 5.6.14.)

This result will be of importance in many places. In particular, the important result we will later prove, that a continuous function f on $[a, b]$ is integrable, depends on the uniform continuity of f. Cauchy certainly recognized this fact but failed to distinguish between continuity and uniform continuity.

Theorem 5.48: *Let f be continuous on $[a, b]$. Then f is uniformly continuous.*

Proof. Our proof invokes a compactness argument. We recall from our investigations of compactness in Section 4.5 that there are several equivalent formulations possible. We shall use the Bolzano-Weierstrass property. (Exercise 5.6.2 asks for another proof of this same theorem using Cousin's lemma. In Exercise 5.6.13 you are asked to prove it using the Heine-Borel property.)

We use an indirect proof. If f is not uniformly continuous, then there are sequences $\{x_n\}$ and $\{y_n\}$ so that $x_n - y_n \to 0$ but

$$|f(x_n) - f(y_n)| > c$$

for some positive c. (The verification of this step is left as Exercise 5.6.12.)

Now apply the Bolzano-Weierstrass property to obtain a convergent subsequence $\{x_{n_k}\}$. Write z as the limit of this new sequence $\{x_{n_k}\}$. Observe that $x_{n_k} - y_{n_k} \to 0$ since $x_n - y_n \to 0$. Thus $\{x_{n_k}\}$ and the corresponding subsequence $\{y_{n_k}\}$ of the sequence $\{y_n\}$ both converge to the same limit z, which must be a point in the interval $[a, b]$. By the continuity of f, $f(x_{n_k}) \to f(z)$ and $f(y_{n_k}) \to f(z)$. Since $|f(x_n) - f(y_n)| > c$ for all n, this means from our study of sequence limits that

$$|f(z) - f(z)| \geq c > 0$$

and this is impossible. This contradiction proves the theorem. ■

Boundedness of Continuous Functions As an application of Theorem 5.48 we can now prove that any continuous function on a closed bounded interval $[a, b]$ is bounded. Indeed such a function must be uniformly continuous there, and we have already seen in Theorem 5.47 that a uniformly continuous function on a bounded interval is bounded. Thus we have the following useful theorem.

Theorem 5.49: *Let f be continuous on $[a, b]$. Then f is bounded.*

Exercises

5.6.1 Adjust the proof of Theorem 5.48 to show that if f is continuous on a compact set K, then f is uniformly continuous on K.

SEE NOTE 118

5.6.2 Give another proof of Theorem 5.48 but this time using Cousin's lemma.

SEE NOTE 119

5.6.3 Because of Theorem 5.47 any function that is continuous on $(0, 1)$ but unbounded cannot be uniformly continuous there. Give an example of a continuous function on $(0, 1)$ that is bounded, but not uniformly continuous.

5.6.4 Let $x_1, x_2, \ldots, x_n$ be real numbers, each in the domain of some function f. Show that f is uniformly continuous on the set $X = \{x_1, x_2, \ldots, x_n\}$.

5.6.5 Let $X = \{x_1, x_2, \ldots, x_n, \ldots\}$. What property must X have so that every function continuous on X is uniformly continuous on X?

SEE NOTE 120

5.6.6 Suppose f is uniformly continuous on each of the sets X_1, X_2, $\ldots$, X_n and let $X = \bigcup_{i=1}^n X_i$. Show that f need not be continuous on X. Show that, even if f is continuous on X, f need not be uniformly continuous on X.

5.6.7 Suppose f is uniformly continuous on each of the compact sets

$$X_1, X_2, \ldots, X_n.$$

Prove that f is uniformly continuous on the set $X = \bigcup_{i=1}^{n} X_i$. Show that this need not be the case if the sets X_k are not closed and need not be the case if the sets X_k are not bounded.

See Note 121

5.6.8 Let f be a uniformly continuous function on a set E. Show that if $\{x_n\}$ is a Cauchy sequence in E then $\{f(x_n)\}$ is a Cauchy sequence in $f(E)$. Show that this need not be true if f is continuous but not uniformly continuous.

5.6.9 A function $f : E \to \mathbb{R}$ is said to be Lipschitz if there is a positive number M so that $|f(x) - f(y)| \leq M|x - y|$ for all x, $y \in E$. Show that such a function must be uniformly continuous on E. Is the converse true?

See Note 122

5.6.10 Explain how Exercise 5.6.4 can be deduced from Exercise 5.6.6 or from Exercise 5.6.7.

See Note 123

5.6.11 Give an example of a function f that is continuous on $\mathbb{R}$ and a sequence of compact intervals X_1, X_2, $\ldots$, X_n, $\ldots$ on each of which f is uniformly continuous, but for which f is not uniformly continuous on $X = \bigcup_{i=1}^{\infty} X_i$.

See Note 124

5.6.12 Show that if f is not uniformly continuous on an interval $[a, b]$ then there are sequences $\{x_n\}$ and $\{y_n\}$ chosen from that interval so that $x_n - y_n \to 0$ but $|f(x_n) - f(y_n)| > c$ for some positive c.

See Note 125

5.6.13 ✂ Prove Theorem 5.48 using the Heine-Borel property.

See Note 126

5.6.14 Prove the following more general and complete version of Theorem 5.48.

> Suppose that $f : E \to \mathbb{R}$ is continuous. If E is compact, then f must be uniformly continuous on E. Conversely, if every continuous function $f : E \to \mathbb{R}$ is uniformly continuous, then E must be compact.

5.6.15 Prove Theorem 5.49 without using the fact that such a function is uniformly continuous. Use Cousin's lemma.

SEE NOTE 127

5.6.16 Prove Theorem 5.49 without using the fact that such a function is uniformly continuous. Use the Bolzano-Weierstrass property.

SEE NOTE 128

5.6.17 ✂ Prove Theorem 5.49 without using the fact that such a function is uniformly continuous. Use the Heine-Borel property.

SEE NOTE 129

5.7 Extremal Properties

A familiar kind of problem that we study in elementary calculus involves locating extrema of continuous functions defined on an interval $[a, b]$. The technique entails checking values of the function at points where its derivative is zero, at the endpoints of the interval, and at any points of nondifferentiability. For such a process to work, we must be sure the function *has* a maximum (or minimum) on the interval. We verify this now.

Theorem 5.50: *Let f be continuous on $[a, b]$. Then f possesses both an absolute maximum and an absolute minimum.*

Proof. Let $M = \sup\{f(x) : a \leq x \leq b\}$. By Theorem 5.48, f is uniformly continuous on $[a, b]$. Thus, by Theorem 5.49, $M < \infty$. If there exists x_0 such that $f(x_0) = M$, then f achieves a maximum value M. Suppose, then, that $f(x) < M$ for all $x \in [a, b]$. We show this is impossible.

Let $g(x) = 1/(M - f(x))$. For each $x \in [a, b]$, $f(x) \neq M$; as a consequence, g is continuous and $g(x) > 0$ for all $x \in [a, b]$. From the definition of M we see that

$$\inf\{M - f(x) : x \in [a, b]\} = 0,$$

so

$$\sup\left\{\frac{1}{M-f(x)} : x \in [a,b]\right\} = \infty.$$

This means that g is not bounded on $[a,b]$. This is impossible because, as we saw in Section 5.6, a continuous function defined on a closed interval must be bounded. A similar proof would show that f has an absolute minimum on A. ■

Example 5.51: Does this theorem extend to more general situations? If we replace the interval $[a,b]$ by some other set does the conclusion remain true? The example

$$f(x) = \frac{1}{x} \text{ for } x \in (0,1)$$

shows that the closed interval cannot be replaced by an open one. On the other hand, the example

$$f(x) = x \text{ for } x \in [0,\infty)$$

shows that the bounded closed interval $[a,b]$ cannot be replaced by an unbounded closed one. ◀

From this example the suggestion that we need a closed and bounded set (i.e., a compact set) seems to offer itself. Indeed that is the correct generalization of Theorem 5.50.

Theorem 5.52: *Let f be continuous on a closed and bounded set A. Then f possesses an absolute maximum and an absolute minimum on A.*

Exercises

5.7.1 Give an example of an everywhere discontinuous function that possesses a unique point at which there is an absolute maximum and a unique point at which there is an absolute minimum.

5.7.2 Show that a continuous function maps compact sets to compact sets.

See Note 130

5.7.3 Prove Theorem 5.50 using a Bolzano-Weierstrass argument.

See Note 131

5.7.4 Give an example of a function defined only on the rationals and continuous at each point in its domain and yet does not have an absolute maximum.

5.7.5 Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with the property that

$$\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = 0.$$

Show that f has either an absolute maximum or an absolute minimum but not necessarily both.

See Note 132

5.7.6 Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function that is periodic in the sense that for some number p, $f(x + p) = f(x)$ for all $x \in \mathbb{R}$. Show that f has an absolute maximum and an absolute minimum.

5.8 Darboux Property

We have already observed that the IVP (Darboux property) is not the same as continuity. It is true, however, that if f is continuous on $[a, b]$, then f has the Darboux property. We state Theorem 5.53 in a form that suggests use of Cousin's lemma. (Readers that prefer to use the Bolzano-Weierstrass theorem should see the hint for Exercise 5.8.3.) Expressed this way the theorem asserts that if the graph has no point on some horizontal line $y = c$, then the graph must be entirely above or below that line. Another way to say this (see Exercise 5.8.8) is that the function must assume every value between any two of its values.

Theorem 5.53: *Let f be continuous on $[a, b]$ and let $c \in \mathbb{R}$. If for every $x \in [a, b]$, $f(x) \neq c$, then either $f(x) > c$ for all $x \in [a, b]$ or $f(x) < c$ for all $x \in [a, b]$.*

Proof. Again, as in the proof of Theorem 5.48, we will use a compactness argument. We shall use Cousin's lemma (Lemma 4.26). In the exercises you are asked to prove this same theorem using the Bolzano-Weierstrass property and the Heine-Borel property.

Let $\mathcal{C}$ denote the collection of closed intervals J such that $f(x) < c$ for all $x \in J$ or $f(x) > c$ for all $x \in J$. We verify that $\mathcal{C}$ forms a Cousin cover of $[a, b]$.

If $x \in [a, b]$, then $|f(x) - c| = \varepsilon > 0$, so there exists $\delta > 0$ such that $|f(t) - f(x)| < \varepsilon$ whenever $|t - x| < \delta$ and $t \in [a, b]$. Thus, if $f(x) < c$, then $f(t) < c$ for all $t \in [x - \delta/2, x + \delta/2]$, while if $f(x) > c$, then $f(t) > c$ for all $t \in [x - \delta/2, x + \delta/2]$. By Cousin's lemma there exists a partition of $[a, b]$, $a = x_0 < x_1 < \cdots < x_n = b$ such that for $i = 1, \ldots, n$, $[x_{i-1}, x_i] \in \mathcal{C}$.

Suppose now that $f(a) < c$. The argument is similar if $f(a) > c$. Since $[a, x_1] = [x_0, x_1] \in \mathcal{C}$, $f(x) < c$ for all $x \in [x_0, x_1]$. Analogously, since $[x_1, x_2] \in \mathcal{C}$, and $f(x_1) < c$, $f(x) < c$ for $x \in [x_1, x_2]$. Proceeding in this way, we see that $f(x) < c$ for all $x \in [a, b]$. ■

You may wish to look at Exercise 5.8.8 for other wordings of this theorem that suggest IVP as "connectedness."

Exercises

5.8.1 Show that a nondecreasing function with the Darboux property must be continuous.

5.8.2 Show that a continuous function maps compact intervals to compact intervals.

See Note 133

5.8.3 Prove Theorem 5.53 using the Bolzano-Weierstrass property of sequences rather than Cousin's lemma.

See Note 134

5.8.4 ✂ Prove Theorem 5.53 using the Heine-Borel property.

See Note 135

5.8.5 Prove Theorem 5.53 using the following "last point" argument: suppose that $f(a) < c < f(b)$ and let z be the last point in $[a, b]$ where $f(z) \leq c$, that is, let

$$z = \sup\{x \in [a, b] : f(x) \leq c\}.$$

Show that $f(z) = c$.

See Note 136

5.8.6 A function $f : [a, b] \to [a, b]$ is said to have a *fixed point* $c \in [a, b]$ if $f(c) = c$. Show that every continuous function f mapping $[a, b]$ into itself has at least one fixed point.

See Note 137

5.8.7 Let $f : [a, b] \to [a, b]$ be continuous. Define a sequence recursively by $z_1 = x_1$, $z_n = f(z_{n-1})$ where $x_1 \in [a, b]$. Show that if the sequence $\{z_n\}$ is convergent, then it must converge to a fixed point of f.

5.8.8 Show that Theorem 5.53 can be reworded in the following ways:

(a) Let f be defined and continuous on an interval I, let $a, b \in I$ with $f(a) \neq f(b)$. Let d lie between $f(a)$ and $f(b)$. Then there exists c between a and b such that $f(c) = d$.

(b) A continuous function defined on an interval I maps subintervals of I onto either single points or else subintervals of $\mathbb{R}$. [Singleton points are often considered to be (degenerate) intervals.]

See Note 138

5.8.9 A continuous function maps compact intervals to compact intervals. Is it true that continuous functions map closed sets to closed sets? Is it true that continuous functions map open sets to open sets?

5.8.10 State forms of Theorem 5.53 and its rewordings in Exercise 5.8.8 for continuous functions defined on intervals that need not be closed and/or bounded.

5.9 Points of Discontinuity

In our discussion of continuous functions we have mentioned discontinuities only as a contrast to the notion of continuity. In many applications of mathematics the functions that arise will have discontinuities and it is well to study such functions. We first ask for a language of discontinuity points. Then we investigate an important class of functions, the monotonic functions, and determine just how badly discontinuous they could be.

5.9.1 Types of Discontinuity

Let x_0 be a point of the domain of some function f. If x_0 is a point of discontinuity, then this means that either the limit $\lim_{x\to x_0} f(x)$ fails to exist or else that limit does exist but

$$f(x_0) \neq \lim_{x\to x_0} f(x).$$

Note that when we discuss discontinuity points we are discussing only points at which the function is defined. (Some calculus texts might call x_0 a point of discontinuity even if $f(x_0)$ fails to be defined. This is not our usage here.)

Note, too, that a discontinuity point cannot occur at an isolated point of the domain of the function.

Removable Discontinuities We can separate these cases into situations of increasing severity. The weakest possibility is that $\lim_{x\to x_0} f(x)$ does indeed exist but fails to equal $f(x_0)$. We call this a *removable discontinuity* of f. The word "removable" suggests that were we merely to assign a new value to $f(x_0)$ we would no longer have a discontinuity.

Jump Discontinuities A little more serious case of discontinuity occurs if the limit

$$\lim_{x\to x_0} f(x)$$

does not exist, but it fails to exist only because

$$\lim_{x\to x_0+} f(x) \quad \text{and} \quad \lim_{x\to x_0-} f(x),$$

the two one-sided limits, exist but disagree. In that case, no matter what value $f(x_0)$ assumes, this is a point of discontinuity.

We call this a *jump discontinuity* of f. The difference between the two limits

$$\lim_{x\to x_0+} f(x) - \lim_{x\to x_0-} f(x)$$

is a measure of the "size" of the discontinuity and is called the *jump*.

Essential Discontinuities Finally, the most intractable kind of discontinuity would be the situation in which $\lim_{x\to x_0} f(x)$ does not exist, and at least one of the two right-hand and left-hand limits (perhaps both)

$$\lim_{x\to x_0+} f(x) \text{ and } \lim_{x\to x_0-} f(x)$$

also does not exist. Again, no matter what value $f(x_0)$ assumes, this is a point of discontinuity. We call this an *essential discontinuity* of f.

Example 5.54: Let $f(x) = 0$ for all $x \neq 0$ and let $f(0) = 2$. It is clear that 0 is a removable discontinuity of f. Perhaps this example seems entirely artificial. A more natural example would be the function given by the following formula:

$$f(x) = \frac{x+1}{x^2-1} \ (x \neq \pm 1), \quad f(1) = c_1, \quad f(-1) = c_2.$$

This function is clearly continuous at every point other than $x = \pm 1$ but may have two discontinuities, one at -1 and one at 1. One of these is not, however, a serious discontinuity since it is removable. You should try to determine which one is removable and which one is essential. ◀

Example 5.55: Let $f(x)$ be defined as the linear function $x + 1$ for $x < 0$ and a different linear function $2x - 1$ for $x \geq 0$. Then there is a discontinuity at 0 since

$$\lim_{x\to 0+} f(x) = \lim_{x\to 0+} (2x - 1) = -1$$

but

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0-} (x + 1) = 1.$$

In this case the size of the jump is -2. A picture would show exactly what this jump represents. ◀

Exercises

5.9.1 Show that a function that has the Darboux property cannot have either removable or jump discontinuities.

5.9.2 What kind of discontinuities does the Dirichlet function (see Section 5.2.6) have?

5.9.3 What kind of discontinuities does the characteristic function of the Cantor set (see Section 5.2.6) have?

5.9.4 Let the function $f : \mathbb{R} \to \mathbb{R}$ have just one point of discontinuity and assume only rational values. What kind of discontinuity point must that be?

5.9.5 Classify the discontinuities of the rational function

$$f(x) = \frac{x+1}{x^2-1} \ (x \neq \pm 1), \quad f(1) = c_1, \quad f(-1) = c_2.$$

5.9.6 Give an example of a function continuous at 0 but with an essential discontinuity at each other point.

5.9.7 Give an example of a function f with a jump discontinuity and yet $(f)^2$ is continuous everywhere.

5.9.8 Give an example of a function f with an essential discontinuity everywhere and yet $(f)^2$ is continuous everywhere.

5.9.9 Define a function F by the formula

$$F(x) = \lim_{n\to\infty} \frac{x^n}{1+x^n}.$$

What is the domain of this function? Classify all discontinuities.

5.9.2 Monotonic Functions

In general, there is not too much to say about the continuity of an arbitrary function. It is possible for a function to be discontinuous everywhere. But if the function is monotonic this is not possible. We start with some definitions, needed here and again later in many places.

Definition 5.56: (Nondecreasing) Let f be real valued on an interval I. If $f(x_1) \leq f(x_2)$ whenever x_1 and x_2 are points in I with $x_1 < x_2$, we say f is *nondecreasing* on I.

Definition 5.57: (Increasing) Let f be real valued on an interval I. If the strict inequality $f(x_1) < f(x_2)$ holds whenever x_1 and x_2 are points in I with $x_1 < x_2$, we say f is *increasing* on I.

In the opposite direction we define nonincreasing and decreasing.[1]

Definition 5.58: (Nonincreasing) Let f be real valued on an interval I. If $f(x_1) \geq f(x_2)$ whenever x_1 and x_2 are points in I with $x_1 < x_2$, we say f is *nonincreasing* on I.

Definition 5.59: (Decreasing) Let f be real valued on an interval I. If the strict inequality $f(x_1) > f(x_2)$ holds whenever x_1 and x_2 are points in I with $x_1 < x_2$, we say f is *decreasing* on I.

A function that is either nonincreasing or nondecreasing is said to be *monotonic*. Sometimes, to emphasize that there is a strict inequality, we say that a function that is increasing or decreasing is *strictly monotonic*.

The class of monotonic functions has a particularly interesting structure as regards continuity. Such functions can never have essential discontinuities. This is because if f is monotonic nondecreasing or monotonic nonincreasing, then at any point both one-sided limits $\lim_{x\to x_0+} f(x)$ and $\lim_{x\to x_0-} f(x)$ exist.

Theorem 5.60: *Let f be monotonic on an interval I. If x_0 is interior to I, then the one-sided limits $\lim_{x\to x_0-} f(x)$ and $\lim_{x\to x_0+} f(x)$ both exist.*

Proof. Suppose f is nondecreasing on I; the proof for the case that f is nonincreasing will then follow by noting that in this case $-f$ is nondecreasing. To prove Theorem 5.60 let x_0 be interior to I and let $\{x_k\}$ be an increasing sequence of points in I such that $\lim_{k\to\infty} x_k = x_0$. Then the sequence $\{f(x_k)\}$ is a nondecreasing sequence of numbers bounded from above by $f(x_0)$. Thus by the monotone convergence principle $\{f(x_k)\}$ approaches a limit L.

For $x_k < x < x_0$,

$$f(x_k) \leq f(x) \leq L.$$

Let $\varepsilon > 0$. Since $f(x_k) \to L$, there exists $N \in \mathbb{N}$ such that

$$L - f(x_k) < \varepsilon$$

[1]Some authors prefer the terms "increasing" and "strictly increasing" for what we would call nondecreasing and increasing. See the comments on page 427.

whenever $k \geq N$. For all x satisfying $x_N \leq x \leq x_0$ we thus have

$$L - f(x) \leq L - f(x_N) < \varepsilon.$$

It follows that

$$\lim_{x \to x_0-} f(x) = L,$$

so f has a left-sided limit at x_0. A similar argument shows that f also has a right-sided limit at x_0. ■

Monotonic Functions Have Jump Discontinuities Recall that a function f is said to have a *jump* at x_0 if f has limits from the left and from the right at x_0, but these limits are different. Thus, if f is monotonic nondecreasing, say, then clearly

$$\lim_{x \to x_0-} f(x) \leq f(x_0) \leq \lim_{x \to x_0+} f(x).$$

Thus the only possibility of a discontinuity at the point x_0 is if the jump

$$J(x_0) = \lim_{x \to x_0+} f(x) - \lim_{x \to x_0-} f(x)$$

is positive. Thus monotonic functions do not have removable discontinuities nor do they have essential discontinuities. They have only jump discontinuities.

Monotonic Functions Have Countably Many Discontinuities We can go further than this. We can ask about the set of points at which there can be a discontinuity point. We ask how large this set can be. The answer is "not very."

Theorem 5.61: *Let f be monotonic on an interval $[a, b]$. Then the set of points of discontinuity of f in that interval is countable. In particular, f must be continuous at the points of a set dense in $[a, b]$.*

Proof. We consider again the case that f is nondecreasing since the case that f is nonincreasing follows by considering the function $-f$. If f is nondecreasing and discontinuous at a point x_0 in (a, b), then the open interval

$$I(x_0) = \left(\lim_{x \to x_0-} f(x), \lim_{x \to x_0+} f(x) \right)$$

either contains no points in the range of f or contains only the single point $f(x_0)$ in the range. (To check this statement, see Exercise 5.9.12.) Thus, each point of discontinuity x_0 of f in I corresponds to an interval $I(x_0)$. For two different points of discontinuity x_1 and x_2, the intervals $I(x_1)$ and $I(x_2)$ are disjoint (because f is nondecreasing). But any collection of disjoint intervals in $\mathbb{R}$ can be arranged into a sequence (Exercise 4.6.10) and so there can be only countably many points of discontinuity of f . ■

It is easy to construct monotonic functions with infinitely many points of discontinuity. For example, if $f(x) = n$ on $[n, n+1)$, then f has jumps at all the integers.

It is natural to ask which countable sets can be the set of discontinuities for some monotonic f. For example, does there exist an increasing function that is discontinuous at every rational number in $\mathbb{R}$? (Exercise 5.9.14 provides an answer.)

Example 5.62: Our theorem shows that a monotonic function has a countable set of points at most where it can be discontinuous. It is easy to find examples of monotonic functions with a prescribed set of discontinuities if the set given to us is finite. Could any countable set be given and we then find a monotonic function that has exactly that set as its points of discontinuity?

The answer, remarkably, is yes. Let C be a countable subset of (a, b). List the elements as c_1, c_2, c_3, ... Define the function for $a \leq x \leq b$ as

$$f(x) = \sum_{c_n < x} \frac{1}{2^n}.$$

This function is hard to visualize since it depends on the order of the terms. Clearly, $f(a) = 0$ and $f(b) = 1$. The other values are much less clear. But we can see that there is a jump of magnitude $1/2$ at the point c_1, a jump of magnitude $1/4$ at the point c_2, a jump of magnitude $1/8$ at the point c_3, and so on. The function is strictly increasing on any subinterval in which C is dense and would be constant in any interval that contains no points of C. It can be shown that the only discontinuities occur at the points of C. ◀

Exercises

5.9.10 Construct a function with a jump discontinuity of magnitude -5 at the point $x = 1$ and continuous everywhere else.

5.9.11 Find a monotonic function on $[0, 1]$ with discontinuities at 1/3, 2/3, and 3/4 only.

5.9.12 Suppose f is increasing on an interval I. Let x_0 be an interior point of I. Prove that $\lim_{x\to x_0-} f(x) \leq f(x_0) \leq \lim_{x\to x_0+} f(x)$.

5.9.13 Verify the claims made in Example 5.62 about the function f there.

SEE NOTE 139

5.9.14 Using Example 5.62, show that there is a (strictly) increasing function on $[0, 1]$ that is discontinuous at each rational number in $(0, 1)$ and continuous at each irrational number.

5.9.15 Show that there is no monotonic function on $[0, 1]$ that is discontinuous precisely at each irrational number in $(0, 1)$.

SEE NOTE 140

5.9.16 Show that if $f : [a, b] \to \mathbb{R}$ is continuous and increasing, then the inverse function f^{-1} exists and is also continuous and increasing on the interval on which it is defined.

SEE NOTE 141

5.9.17 Let f be a continuous function on an open interval (a, b). Suppose that f has no local maximum or local minimum at any point. Show that f must be monotonic.

5.9.18 Suppose that $f : \mathbb{R} \to \mathbb{R}$ and that $f(x) + \alpha x$ is monotonic for every $\alpha \in \mathbb{R}$. Show that $f(x) = ax + b$ for some a, b.

5.9.19 Let $\{f_n\}$ be a sequence of monotonic functions defined on the interval $[0, 1]$. Suppose that

$$f(x) = \lim_{n\to\infty} f_n(x)$$

exists for each $0 \leq x \leq 1$. Show that f is monotonic. (If the word "monotonic" is replaced throughout this problem by "continuous," the exercise would be invalid: show this, too.)

5.9.20 Can the range of an increasing function on the interval $[0, 1]$ consist only of rational numbers? Can it consist only of irrational numbers?

5.9.3 How Many Points of Discontinuity?

✂ Adv.

We have already answered the question as to how many points of discontinuity a monotonic function may have. The set of such points must be countable. We know too that all of these are jump discontinuities; a monotonic function has no removable discontinuities and no essential discontinuities.

What is the situation for an arbitrary function? There are three questions. How many removable discontinuities are possible? How many jump discontinuities are possible? How many essential discontinuities are possible?

Example 5.63: One example that we have seen before shows that there can be a great many essential discontinuities. Let f be the characteristic function of the rational numbers; that is, $f(x)$ is 1 if x is a rational number and is 0 if x is irrational. Clearly,

$$\limsup_{x\to x_0} f(x) = 1$$

and

$$\liminf_{x\to x_0} f(x) = 0$$

at every point x_0. In particular, the limit does not exist anywhere and so every point is an essential discontinuity. ◀

Surprisingly, though, this is not the case for the removable discontinuities or the jump discontinuities. No function can have an uncountable number of such discontinuities.

Theorem 5.64: *Let f be a real function defined on an interval $[a, b]$. The sets of points in $[a, b]$ at which f has a removable discontinuity and at which f has a jump discontinuity are both countable.*

Proof. Let J be the set of points at which there is a jump discontinuity. Every point of J is in one of the two sets:

$$J_+ = \{x \in (a, b) : \lim_{y\to x+} f(y) > \lim_{y\to x-} f(y)\}$$

or

$$J_- = \{x \in (a,b) : \lim_{y \to x+} f(y) < \lim_{y \to x-} f(y)\}.$$

We shall show that J_+ is countable.

If $x \in J_+$, then

$$\lim_{y \to x+} f(y) > \lim_{y \to x-} f(y)$$

and so there is for any such x at least one rational number r so that

$$\lim_{y \to x+} f(y) > r > \lim_{y \to x-} f(y).$$

Moreover, there then must exist some integer m (depending on x and r) so that

$$f(z) > r > f(y)$$

whenever $x - 1/m < y < x < z < x + 1/m$.

Let J_{rn}, where r is a rational and n a positive integer, denote the set of all points x with the property that $f(y) < r < f(z)$ whenever

$$x - 1/n < y < x < z < x + 1/n.$$

We claim that this set is countable. If not, then it must have a point of accumulation and, in particular, there would have to be at least three points $a < b < c$, with $c - a < 1/n$, all belonging to J_{rn}. But by the way that J_{rn} was defined this means, since a and $c \in J_{rn}$, that $f(b) < r$ and $r < f(b)$ are both true. Since this is impossible, all points in J_{rn} are isolated and hence J_{rn} is countable. The union

$$\bigcup_{r \in \mathbb{Q}} \bigcup_{n=1}^{\infty} J_{rn}$$

is a countable union of countable sets and is thus also countable. But this set contains every point of J_+ and so that set is also countable. Similarly, it is true that J_- is countable and hence the set of points with jump discontinuities is countable.

That the set of points at which the function has a removable discontinuity is also countable is left as an exercise. The ideas of the proof here can be used to prove it in a similar fashion. Notice especially this technique of inserting a rational number between two unequal numbers. ■

Incidentally, this theorem throws a new light on the theorem about the discontinuity points of monotonic functions. In that proof we used the properties of monotonic functions to show that the collection of discontinuity points was countable. But we know easily that the only such points are the jump discontinuities and any function, monotonic or not, has only countably many of these points by our theorem here. Thus we have another way of looking at Theorem 5.61.

Exercises

5.9.21 Give an example of a function with a dense set of removable discontinuities.

5.9.22 Give an example of a function with a dense set of jump discontinuities.

5.9.23 Prove the remaining statement of Theorem 5.64 that is not proved in the text.

5.10 Challenging Problems for Chapter 5

5.10.1 Suppose that f is a function defined on the real line with the property that $f(x+y) = f(x) + f(y)$ for all x, y. Suppose that f is continuous at 0. Show that f must be continuous everywhere.

See Note 142

5.10.2 Suppose that f is a function defined on the real line with the property that $f(x+y) = f(x) + f(y)$ for all x, y. Suppose that f is continuous at 0. Show that $f(x) = Cx$ for all x and some number C.

See Note 143

5.10.3 Suppose that f is a function defined on the real line with the property that $f(x+y) = f(x)f(y)$ for all x, y. Suppose that f is continuous at 0. Show that f must be continuous everywhere.

See Note 144

5.10.4 Generalize Theorem 5.61 to prove that if a function f (not necessarily monotonic) has left-sided limits and right-sided limits at every point of an open interval I, then f must be continuous except on a countable set.

5.10.5 Determine necessary and sufficient conditions on a pair of sets A and B so that they will have the property that there exists a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = 0$ for all $x \in A$ and $f(x) = 1$ for all $x \in B$.

SEE NOTE 145

5.10.6 Let $f : [1, \infty)$ be continuous, positive and increasing with $f(x) \to \infty$ as $x \to \infty$. Show that

$$\sum_{k=1}^{\infty} \frac{1}{f(k)}$$

is convergent if and only if the series

$$\sum_{k=1}^{\infty} \frac{f^{-1}(k)}{k^2}$$

converges (where f^{-1} denotes the inverse function).

5.10.7 **(Extensions of continuous functions)** If $f : A \to \mathbb{R}$, $g : B \to \mathbb{R}$, $A \subset B$, and $f(x) = g(x)$ for all $x \in A$, then the function g is said to be an *extension* of the function f. Prove each of the following:

(a) A function that is continuous on a closed set A can be extended to a function that is continuous on $\mathbb{R}$.

(b) A function that is uniformly continuous on a set A can be extended to a function that is uniformly continuous on $\overline{A}$.

(c) A function that is uniformly continuous on an arbitrary nonempty subset of $\mathbb{R}$ can be extended to a function that is uniformly continuous on all of $\mathbb{R}$.

(d) Give an example of a function f that is continuous on (0,1) but that cannot be extended to a function continuous on [0,1].

5.10.8 ✂ For an arbitrary function $f : \mathbb{R} \to \mathbb{R}$ show that

$$\{x_0 : \limsup_{x \to x_0-} f(x) > \limsup_{x \to x_0+} f(x)\}$$

is countable.

5.10.9 ✂ Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ such that there are infinitely many points x_0 at which either

$$f(x_0) > \limsup_{x \to x_0} f(x) \text{ or } f(x_0) < \liminf_{x \to x_0} f(x).$$

5.10.10 ✂ For an arbitrary function $f:\mathbb{R}\to\mathbb{R}$ show that the set of points x_0 at which $f(x_0)$ does not lie between

$$\liminf_{x\to x_0} f(x) \quad \text{and} \quad \limsup_{x\to x_0} f(x)$$

is countable.

5.10.11 ✂ Let y be a real number or $\pm\infty$ and let $f : E \to \mathbb{R}$ be a function. If there is a sequence $\{x_n\}$ of numbers in E and converging to a point c with $x_n \neq c$ and with $f(x_n) \to y$ then y is called *a cluster value* of f at c. Show that every cluster value at c lies between $\liminf_{x\to c} f(x)$ and $\limsup_{x\to c} f(x)$. Show that both $\liminf_{x\to c} f(x)$ and $\limsup_{x\to c} f(x)$ are themselves cluster values of f at c.

5.10.12 Is there a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that for every real y there are precisely two solutions to the equation $f(x) = y$?

5.10.13 Is there a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that for every real y there are precisely three solutions to the equation $f(x) = y$?

5.10.14 Prove that if $f:\mathbb{R}\to\mathbb{R}$, then the set

$$\{x : f \text{ is right continuous at } x \text{ but not left continuous at } x\}$$

is countable.

SEE NOTE 146

Notes

[89]Exercise 5.1.1. Model your answer after Example 5.2.

[90]Exercise 5.1.2. Consider the cases $a = 0$ and $a \neq 0$ separately. If it is easier for you, break into the three cases $a > 0$, $a < 0$, and $a = 0$.

[91]Exercise 5.1.3. Model your answer after Example 5.3.

[92]Exercise 5.1.4. Consider the cases $x_0 = 0$ and $x_0 \neq 0$ separately. Use the factoring trick in Example 5.3 and the device of restricting x to be close to x_0 by assuming that $|x - x_0| < 1$ at least.

[93]Exercise 5.1.8. Don't forget to exclude $x_0 < 0$ from your answer since it is not a point of accumulation of the domain of this function. Consider the cases $x_0 = 0$ and $x_0 > 0$ separately.

[94]Exercise 5.1.12. If $B \subset A$, then the existence of $\lim_{x\to x_0} g(x)$ can be deduced from the existence of $\lim_{x\to x_0} f(x)$. Can you find other conditions? If x_0 is a point of accumulation of $A \cap B$, then the equality of the two limits can be deduced, assuming that both exist.

[95]Exercise 5.1.16. Either find a single sequence

$$x_n \to 0$$

with $x_n \neq 0$ so that the limit

$$\lim_{n\to\infty} |x_n|/x_n$$

does not exist or else find two such sequences with different limits.

[96]Exercise 5.1.22. You could assume (i) that $L > 0$ or (ii) that $f(x) \geq 0$ for all x in its domain. Then convert to a statement about sequences.

[97]Exercise 5.1.28. At $x_0 \neq 0$ the two one-sided limits are equal. What are they? At $x_0 = 0$ they differ.

[98]Exercise 5.1.29. On one side the limit is zero and on the other the limit fails to exist. (Look ahead to Exercise 5.1.38, where you are asked to show that the limit is ∞ which means that the limit does not exist.) You may use the elementary inequality

$$0 < z < e^z$$

(which is valid for all $z > 0$) in your argument. Consider the sequences $1/n \to 0$ and $-1/n \to 0$.

[99]Exercise 5.1.30. Check the definition: There would be no distinction. The limit

$$\lim_{x\to 0-} \sqrt{x},$$

however, would be meaningless since 0 is not a point of accumulation of the domain of the square root function on the left.

[100]Exercise 5.1.34. Use the definitions in this section as a model. You will need a replacement for the "x_0 is a point of accumulation" of the domain condition. If you cannot think of anything better, then simply use the assumption that f is defined in some interval (a, ∞).

[101]Exercise 5.1.38. On one side at 0 the limit is zero and on the other the limit is ∞. See Exercise 5.1.29.

[102]Exercise 5.2.1. Model your proof after Theorem 2.8 for sequences.

[103]Exercise 5.2.3. If the theorem were false, then in every interval $(x_0 - 1/n, x_0 + 1/n)$ there would be a point x_n for which $|f(x_n)| > n$.

[104]Exercise 5.2.9. If x_0 is not a point of accumulation of

$$\operatorname{dom}(f) \cap \operatorname{dom}(g),$$

then the statement

$$\lim_{x \to x_0} f(x) + g(x) = L$$

does not have any meaning even though the two statements about

$$\lim_{x \to x_0} f(x) \text{ and } \lim_{x \to x_0} g(x)$$

may have.

[105]Exercise 5.2.11. What exactly is the domain of the function $f(x)/g(x)$? Show that x_0 would be a point of accumulation of that domain provided that $g(x) \to C$ as $x \to x_0$ and $C \neq 0$.

[106]Exercise 5.2.28. It is enough to assume that $\lim_{x \to x_0} f(x)$ exists and to apply Theorem 5.25 with $F(x) = |x|$. Be sure to explain why this function F has the properties expressed in that theorem.

[107]Exercise 5.2.29. It is enough to assume that $\lim_{x \to x_0} f(x)$ exists and is positive and then apply Theorem 5.25 with $F(x) = \sqrt{x}$. Alternatively, assume that $f(x) \geq 0$ for all x in a neighborhood of x_0. Again be sure to explain why this function F has the properties expressed in that theorem.

[108]Exercise 5.2.32. Use the property of exponentials that $e^{a+b} = e^a e^b$ and the product rule for limits.

[109]Exercise 5.2.33. Use a trigonometric identity for $\sin(x - x_0 + x_0)$ and the sum and products rule for limits.

[110]Exercise 5.2.34. Take the function $H(x)$ of the text and consider instead $H(x) + x$.

[111]Exercise 5.2.36. This would be trivial if the sets A_i were disjoint. So it is the case where these are not disjoint that you need to address.

[112]Exercise 5.2.44. If x_0 is not in the Cantor set K, then it is in some open interval complementary to that set. Use that to prove the existence of the limit. If x_0 is in the Cantor set, then there must be sequences $x_n \to x_0$ and $y_n \to x_0$ with $x_n \in K$ and $y_n \notin K$. Use that to prove the nonexistence of the limit.

[113]Exercise 5.3.5. Consider separately the cases $x_0 \in E$ and $x_0 \notin E$. Under what circumstances in the latter case would the lim sup be larger according to this revised definition?

[114]Exercise 5.4.15. One of the definitions treats isolated points in a special way. Note that each point in the domain of f is isolated.

[115]Exercise 5.4.17. You must arrange for $f(0)$ to be the limit of the sequence of values $f(2^{-n})$. No other condition is necessary.

[116]Exercise 5.4.19. At an isolated point x_0 of the domain the limit $\lim_{x\to x_0} f(x)$ has no meaning. But if x_0 is not an isolated point in the domain of f it must be a point of accumulation and then $\lim_{x\to x_0} f(x)$ is defined and it must be equal to $f(x_0)$.

[117]Exercise 5.4.20. For the converse consider the function $f(x) = \sqrt{x}$ on $[0, 1]$.

[118]Exercise 5.6.1. Let $a = \inf K$ and $b = \sup K$ and apply Cousin's lemma to the interval $[a, b]$ by taking the same collection nearly, namely $\mathcal{C}$ consist of all closed subintervals $[t, s]$ such that

$$|f(t') - f(s')| < \varepsilon/2$$

for all t', $s' \in K \cap [t, s]$. You will have to find a different choice of δ to make your argument work.

[119]Exercise 5.6.2. As usual in applications of Cousin's lemma, we should define first our collection of closed subintervals so as to have a desired property that can be extended to the whole interval $[a, b]$. Let $\varepsilon > 0$. Let $\mathcal{C}$ consist

of all closed subintervals $[t, s]$ such that

$$|f(t') - f(s')| < \varepsilon/2$$

for all t', $s' \in [t, s]$. We check that $\mathcal{C}$ satisfies the hypotheses of Lemma 4.26.

For each $x \in [a, b]$ there exists $\delta(x) > 0$ such that if

$$t \in [a, b] \cap (x - \delta(x), x + \delta(x)),$$

then

$$|f(t) - f(x)| < \varepsilon/4.$$

It follows that if t' and s' are in the set

$$[a, b] \cap (x - \delta(x), x + \delta(x)),$$

then

$$\begin{aligned} |f(t') - f(s')| &\le |f(t') - f(x)| + |f(x) - f(s')| \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}. \end{aligned}$$

Consequently, every interval $[t, s]$ inside

$$[a, b] \cap (x - \delta(x), x + \delta(x))$$

belongs to $\mathcal{C}$.

Thus Lemma 4.26 may be applied and there exists a partition

$$a = x_0 < x_1 < \cdots < x_n = b$$

such that if, for some $i = 1, \ldots, n$,

$$x_{i-1} \le x, y \le x_i,$$

then

$$|f(x) - f(y)| < \varepsilon/2.$$

Let

$$\delta = \min_{i=1,\ldots,n} |x_i - x_{i-1}|.$$

If $x < y$ and $|x - y| < \delta$, then either there exists i for which

$$x_{i-1} \leq x, y \leq x_i,$$

in which case

$$|f(x) - f(y) < \varepsilon/2,$$

or there exists i such that

$$x_{i-1} \leq x \leq x_i \leq y \leq x_{i+1},$$

in which case

$$|f(y) - f(x)| \leq |f(y) - f(x_i)|$$
$$+|f(x_i) - f(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since this argument applies to any positive ε, we have proved that f is uniformly continuous on $[a, b]$.

[120]Exercise 5.6.5. If X is compact (closed and bounded) then this property should hold. If the set X has a point of accumulation that does not belong to X, then it is possible to give an example of a continuous function defined on X that is not uniformly continuous on X. Finally consider the situation in which the set is closed but not bounded: are there points x_i and x_j arbitrarily close together?

[121]Exercise 5.6.7. You need consider only two compact sets X_1, X_2. Since they are compact, there is a positive distance between them that you can use to help define your δ. For not closed consider $X_1 = (0, 1)$ and $X_2 = (1, 2)$ and define f appropriately. For not bounded use

$$X_1 = \{1, 2, 3, \dots\}$$

and

$$X_2 = \{1, 2 + 1/2, 3 + 1/3, 4 + 1/4, \dots$$

and define f appropriately.

[122]Exercise 5.6.9. For the converse consider the function $f(x) = \sqrt{x}$ on $[0, 1]$. By Theorem 5.48 we know that this function is uniformly continuous on $[0, 1]$.

[123]Exercise 5.6.10. Show that any function defined on a set X containing just one element is uniformly continuous. Then consider the sequence $X_i = \{x_i\}$, $i = 1, 2, \ldots, n$.

[124]Exercise 5.6.11. For the sequence of intervals you might choose $[1,2]$, $[2,3]$, $[3,4]$, $\ldots$. (Why would you not be able to choose $[1/2,1]$, $[1/4,1/2]$, $[1/8,1/4]$, $\ldots$?)

[125]Exercise 5.6.12. This can be obtained merely by negating the formal statement that f is uniformly continuous on $[a,b]$.

[126]Exercise 5.6.13. Using the local continuity property, claim that there are open intervals I_x containing any point x so that

$$|f(y) - f(x)| < \varepsilon$$

for any $y \in I_x$. Now apply the Heine-Borel property to this open cover. Obtain uniform continuity from the finite subcover.

[127]Exercise 5.6.15. Let $\mathcal{C}$ be the collection of all closed intervals $I \subset [a,b]$ so that f is bounded on I. Use Cousin's lemma to find a partition of $[a,b]$ using intervals in $\mathcal{C}$.

[128]Exercise 5.6.16. Use an indirect proof. Show that if f is not bounded then there is a sequence $\{x_n\}$ of points in $[a,b]$ so that

$$|f(x_n)| > n$$

for all n. Now apply the Bolzano-Weierstrass property to obtain subsequences and get a contradiction.

[129]Exercise 5.6.17. Using the local continuity property, claim that there are open intervals I_x containing any point x so that

$$|f(y) - f(x)| < 1$$

for any $y \in I_x$. Now apply the Heine-Borel property to this open cover. Obtain boundedness of f from the finite subcover.

[130]Exercise 5.7.2. That is, prove that the image set

$$f(K) = \{f(x) : x \in K\}$$

is compact if K is compact and f is a continuous function defined at every point of K. Give a direct proof that uses the fact that a set is compact if and only if every sequence in the set has a subsequence convergent to a point in the set. Start with a sequence of points $\{y_n\}$ in $f(K)$, explain why there must be a sequence $\{x_n\}$ in K with $f(x_n) = y_n$ etc.

[131]Exercise 5.7.3. Let
$$M = \sup\{f(x) : a \le x \le b\}.$$
Explain why you can choose a sequence of points $\{x_n\}$ from $[a, b]$ so that
$$f(x_n) > M - 1/n.$$
Now apply the Bolzano-Weierstrass theorem and use the continuity of f.

[132]Exercise 5.7.5. If $f(x_0) = c > 0$, then there is an interval $[-N, N]$ so that $x_0 \in [-N, N]$ and $|f(x)| < c/2$ for all $x > N$ and $x < -N$.

[133]Exercise 5.8.2. That is, prove that the image set $f([c, d])$ is a compact interval for any interval $[c, d]$ if f is a continuous function defined at every point of $[c, d]$. Apply Theorem 5.52 and Theorem 5.53.

[134]Exercise 5.8.3. Suppose that the theorem is false and explain, then, why there should exist sequences $\{x_n\}$ and $\{y_n\}$ from $[a, b]$ so that $f(x_n) > c$, $f(y_n) < c$ and $|x_n - y_n| < 1/n$.

[135]Exercise 5.8.4. Suppose that the theorem is false and explain, then, why there should exist at each point $x \in [a, b]$ an open interval I_x centered at x so that either $f(t) > c$ for all $t \in I_x \cap [a, b]$ or else $f(t) < c$ for all $t \in I_x \cap [a, b]$.

[136]Exercise 5.8.5. You may take $c = 0$. Show that if $f(z) > 0$, then there is an interval $[z - \delta, z]$ on which f is positive. Show that if $f(z) < 0$, then there is an interval $[z, z + \delta]$ on which f is negative. Explain why each of these two cases is impossible.

[137]Exercise 5.8.6. The function must be onto. Hence there is a point x_1 with $f(x_1) = a$ and a point x_2 with $f(x_2) = b$. Now convince yourself that there is a point on the graph of the function that is also on the line $y = x$.

[138]Exercise 5.8.8. Condition (a) is the intermediate value property (IVP) according to Definition 5.27, while (b) can be interpreted as saying that connectedness is preserved by continuous functions. This latter interpretation requires a careful definition of connectedness in $\mathbb{R}$.

[139]Exercise 5.9.13. You wish to show that (i) f is discontinuous at every point in C, indeed has a jump discontinuity at each such point; (ii) f is continuous at every point not in C; (iii) f is nondecreasing; (iv) f is increasing on any interval in which C is dense; and (v) f is constant on any interval containing no point of C.

The most direct and easiest proof that f is continuous at every point not in C would be to use "uniform convergence" but that is in a later chapter. Here you will have to use an ε-δ argument.

[140]Exercise 5.9.15. How large can the set of discontinuity points be?

[141]Exercise 5.9.16. The function f^{-1} is defined on the interval $J = [f(a), f(b)]$. Explain first why it exists (not all functions must have an inverse). Prove that it is increasing. Prove that it is continuous (using the fact that it is increasing).

[142]Exercise 5.10.1. The equation $f(x + y) = f(x) + f(y)$ is called a functional equation. You are told about this function only that it satisfies such a relationship and has a nice property at one point. Now you must show that this implies more. Show first that $f(0) = 0$ and that $f(x - y) = f(x) - f(y)$.

[143]Exercise 5.10.2. This continues Exercise 5.10.1. Show first that $f(r) = rf(1)$ for all $r = m/n$ rational. Then make use of the continuity of f that you had already established in the other exercise.

[144]Exercise 5.10.3. Show that either f is always zero or else $f(0) = 1$. Establish

$$f(x - y) = f(x)/f(y).$$

[145]Exercise 5.10.5. Consider the intersection

$$\overline{A} \cap \overline{B}.$$

[146]Exercise 5.10.14. You will need to use the fact that

$$\{x : \limsup_{x\to x_0^-} f(x) > \limsup_{x\to x_0^+} f(x)\}$$

is countable. See Exercise 5.10.8.

Chapter 6

MORE ON CONTINUOUS FUNCTIONS AND SETS

✂ This chapter can be considered enrichment material containing also several more advanced topics and may be skipped in its entirety. You can proceed directly to the study of derivatives and integrals with no loss in the continuity of the material.

6.1 Introduction

In this chapter we go much more deeply into the analysis of continuous functions. For this we need some new set theoretic ideas and methods.

6.2 Dense Sets

[This section reviews material from Section 1.9.]

Consider the set $\mathbb{Q}$ of rational numbers and let (a, b) be an open interval in $\mathbb{R}$. How do we show that there is a member of $\mathbb{Q}$ in the interval (a, b); that is, that $(a, b) \cap \mathbb{Q} \neq \emptyset$?

Suppose first that $0 < a$. Since $b - a > 0$, the archimedean property (Theorem 1.11) implies that there is a positive integer q such that

$$q(b-a) > 1.$$

Thus

$$qb > 1 + qa.$$

The archimedean property also implies that the set of integers

$$\{m \in \mathbb{N} : m > qa\}$$

is nonempty. Thus, according to the well-ordering principle, there is a smallest integer p in this set and for this p, it is true that $p - 1 \leq qa < p$. It follows that

$$qa < p \leq 1 + qa < qb,$$

which implies $a < \frac{p}{q} < b$. We have shown that, under the assumption $a > 0$, there exists a rational number $r = p/q$ in the interval (a, b).

The same is true under the assumption $a < 0$. To see this observe first that if $a < 0 < b$, we can take $r = 0$. If $a < b < 0$, then $0 < -b < -a$, so the argument of the previous paragraph shows that there exists $r \in \mathbb{Q}$ such that $-b < r < -a$. In this case $a < -r < b$.

The preceding discussion proves that every open interval contains a rational number. We often express this fact by saying that the set of rational numbers is a *dense* set.

Definition 6.1: A set of real numbers A is said to be *dense* (in $\mathbb{R}$) if for each open interval (a, b) the set $A \cap (a, b)$ is nonempty.

It is important to have a more general concept, that of a set A being dense in a set B.

Definition 6.2: Let A and B be subsets of $\mathbb{R}$. If every open interval that intersects B also intersects A, we say that *A is dense in B*.

Thus Definition 6.1 states the special case of Definition 6.2 that occurs when $B = \mathbb{R}$. We should note that some authors require that $A \subset B$ in their version of Definition 6.2. We find it more convenient not to impose this restriction. Thus, for example, in *our* language $\mathbb{Q}$ is dense in $\mathbb{R} \setminus \mathbb{Q}$.

It is easy to verify that A is dense in B if and only if $\overline{A} \supset B$ (Exercise 6.2.1).

Exercises

6.2.1 Verify that A is dense in B if and only if $\overline{A} \supset B$.

6.2.2 Prove that every set A is dense in its closure $\overline{A}$.

6.2.3 Prove that if A is dense in B and $C \subset B$, then A is dense in C.

6.2.4 Prove that if $A \subset B$ and A is dense in B, then $\overline{A} = \overline{B}$. Is the statement correct without the assumption that $A \subset B$?

6.2.5 Is $\mathbb{R} \setminus \mathbb{Q}$ dense in $\mathbb{Q}$?

6.2.6 The following are several pairs (A, B) of sets. In each case determine whether A is dense in B.

(a) $A = \mathbb{N}, B = \mathbb{N}$

(b) $A = \mathbb{N}, B = \mathbb{Z}$

(c) $A = \mathbb{N}, B = \mathbb{Q}$

(d) $A = \left\{x : x = \frac{m}{2^n},\ m \in \mathbb{Z}, n \in \mathbb{N}\right\}, B = \mathbb{Q}$

6.2.7 Let A and B be subsets of $\mathbb{R}$. Prove that A is dense in B if and only if for every $b \in B$ there exists a sequence $\{a_n\}$ of points from A such that $\lim_{n\to\infty} a_n = b$.

6.2.8 Let B be the set of all irrational numbers. Prove that the set

$$A = \{q + \sqrt{2} : q \in \mathbb{Q}\}$$

is a countable subset of B that is dense in B.

6.2.9 Let $f : \mathbb{R} \to \mathbb{R}$ be a strictly increasing continuous function. Does f map dense sets to dense sets; that is, is it true that

$$f(E) = \{f(x) : x \in E\}$$

is dense if E is dense?
See Note 147

6.2.10 Prove that every set $B \subset \mathbb{R}$ contains a countable set A that is dense in B.

6.3 Nowhere Dense Sets

We might view a set A that is dense in $\mathbb{R}$ as being somehow large: Inside every interval, no matter how small, we find points of A. There is an opposite extreme to this situation: A set is said to be *nowhere dense*, and hence is in some sense small, if it is not dense in any interval at all. The precise definition of this important concept of smallness follows.

Definition 6.3: The set $A \subset \mathbb{R}$ is said to be *nowhere dense* in $\mathbb{R}$ provided every open interval I contains an open subinterval J such that $A \cap J = \emptyset$.

We can state this another way: A is nowhere dense provided $\overline{A}$ contains no open intervals. (See Exercise 6.3.4.)

Example 6.4: It is easy to construct examples of nowhere dense sets.

1. Any finite set
2. $\mathbb{N}$
3. $\{1/n : n \in \mathbb{N}\}$

Each of these sets is nowhere dense, as you can verify. ◀

Each of the sets in Example 6.4 is countable and hence also small in the sense of cardinality. It is hard to imagine an uncountable set that is nowhere dense but, as we shall see in Section 6.5, such sets do exist.

We establish a simple result showing that any finite union of nowhere dense sets is again nowhere dense. It is not true that a countable union of nowhere dense sets is again nowhere dense. Indeed countable unions of nowhere dense sets will be important in our subsequent study.

Theorem 6.5: *Let $A_1, A_2, \ldots, A_n$ be nowhere dense in $\mathbb{R}$. Then $A_1 \cup \cdots \cup A_n$ is also nowhere dense in $\mathbb{R}$.*

Proof. Let I be any open interval in $\mathbb{R}$. We seek an open interval $J \subset I$ such that $J \cap A_i = \emptyset$ for $i = 1, 2, \ldots, n$.

Since A_1 is nowhere dense, there exists an open interval $I_1 \subset I$ such that $I_1 \cap A_1 = \emptyset$. Now A_2 is also nowhere dense in $\mathbb{R}$, so there exists an open interval $I_2 \subset I_1$ such that $A_2 \cap I_2 = \emptyset$. Proceeding in this way we obtain open intervals

$$I_1 \supset I_2 \supset I_3 \supset \cdots \supset I_n$$

such that for $i = 1, ..., n$, $A_i \cap I_i = \emptyset$. It follows from the fact that $I_n \subset I_i$ for $i = 1, \ldots, n$ that $A_i \cap I_n = \emptyset$ for $i = 1, \ldots, n$. Thus

$$\left(\bigcup_{i=1}^{n} A_i\right) \cap I_n = \bigcup_{i=1}^{n} (A_i \cap I_n) = \bigcup_{i=1}^{n} \emptyset = \emptyset,$$

as was to be proved. ■

Exercises

6.3.1 Give an example of a sequence of nowhere dense sets whose union is not nowhere dense.

SEE NOTE 148

6.3.2 Which of the following statements are true?

(a) Every subset of a nowhere dense set is nowhere dense.

(b) If A is nowhere dense, then so too is $A + c = \{t + c : t \in A\}$ for every number c.

(c) If A is nowhere dense, then so too is $cA = \{ct : t \in A\}$ for every positive number c.

(d) If A is nowhere dense, then so too is A', the set of derived points of A.

(e) A nowhere dense set can have no interior points.

(f) A set that has no interior points must be nowhere dense.

(g) Every point in a nowhere dense set must be isolated.

(h) If every point in a set is isolated, then that set must be nowhere dense.

See Note 149

6.3.3 If A is nowhere dense, what can you say about $\mathbb{R} \setminus A$? If A is dense, what can you say about $\mathbb{R} \setminus A$?

6.3.4 Prove that a set $A \subset \mathbb{R}$ is nowhere dense if and only if $\overline{A}$ contains no intervals; equivalently, the interior of $\overline{A}$ is empty.

6.3.5 What should the statement "A is nowhere dense in the interval I" mean? Give an example of a set that is nowhere dense in $[0, 1]$ but is not nowhere dense in $\mathbb{R}$.

6.3.6 Let A and B be subsets of $\mathbb{R}$. What should the statement "A is nowhere dense in the B" mean? Is $\mathbb{N}$ nowhere dense in $[0, 10]$? Is $\mathbb{N}$ nowhere dense in $\mathbb{Z}$? Is $\{4\}$ nowhere dense in $\mathbb{N}$?

6.3.7 Prove that the complement of a dense open subset of $\mathbb{R}$ is nowhere dense in $\mathbb{R}$.

6.3.8 Let $f : \mathbb{R} \to \mathbb{R}$ be a strictly increasing continuous function. Show that f maps nowhere dense sets to nowhere dense sets; that is,

$$f(E) = \{f(x) : x \in E\}$$

is nowhere dense if E is nowhere dense.

6.4 The Baire Category Theorem

Adv.

In this section we shall establish the Baire category theorem, which gives a sense in which nowhere dense sets can be viewed as "small:" A union of a sequence of nowhere dense sets cannot fill up an interval. If we interpret Cantor's theorem (Theorem 2.4) as asserting that a union of a sequence of finite sets cannot fill up an interval, then we see the Baire category theorem as a far-reaching generalization.

We motivate this important theorem by way of a game idea that is due to Stefan Banach (1892–1945) and Stanislaw Mazur (1905–1981). Although the origins of the theorem are due to René Baire, after whom the theorem is named, the game approach helps us see why the Baire category theorem might be true. This Banach-Mazur game is just one of many mathematical games that are used throughout mathematics to develop interesting concepts.

6.4.1 A Two-Player Game

We introduce the Baire category theorem via a game between two players (A) and (B).

Player (A) is given a subset A of $\mathbb{R}$, and player (B) is given the complementary set $B = \mathbb{R} \setminus A$. Player (A) first selects a closed interval $I_1 \subset \mathbb{R}$; then player (B) chooses a closed interval $I_2 \subset I_1$. The players alternate moves, a move consisting of selecting a closed interval inside the previously chosen interval.

The play of the game thus determines a descending sequence of closed intervals

$$I_1 \supset I_2 \supset I_3 \supset \cdots \supset I_n \supset \ldots$$

where player (A) chooses those with odd index and player (B) those with even index. If

$$A \cap \bigcap_{n=1}^{\infty} I_n \neq \emptyset,$$

then player (A) wins; otherwise player (B) wins.

The goal of player (A) is evidently to make sure that the intersection contains a point of A; the goal of player (B) is to ensure that the intersection is empty or contains only points of B. We expect that player (A) should win if his set A is large while player (B) should win if his set is large. It is not, however, immediately clear what "large" might mean for this game.

Example 6.6: If the set A given to player (A) contains an open interval J, then (A) should choose any interval $I_1 \subset J$. No matter how the game continues, player (A) wins. Another way to say this: If the set given to player (B) is not dense, he loses. ◄

Example 6.7: For a more interesting example, let player (A) be dealt the "large" set of all irrational numbers, so that player (B) is dealt the rationals. (Both players have been dealt dense sets now.) Let A consist of the irrational numbers. Player (A) can win by following the strategy we now describe. Let q_1, q_2, q_3, ... be a listing of all of the rational numbers; that is,

$$\mathbb{Q} = \{q_1, q_2, q_3, \ldots\}.$$

Player (A) chooses the first interval I_1 as any closed interval such that $q_1 \notin I_1$. Inductively, suppose $I_1, I_2, \dots, I_{2n}$ have been chosen according to the rules of the game so that it is now time for player (A) to choose I_{2n+1}. The set $\{q_1, q_2, \dots, q_n\}$ is finite, so there exists a closed interval $I_{2n+1} \subset I_{2n}$ such that

$$I_{2n+1} \cap \{q_1, q_2, \dots, q_n\}$$

is empty. Player (A) chooses such an interval.

Since for each $n \in \mathbb{N}$, $q_n \notin I_{2n+1}$, the set $\bigcap_{n=1}^{\infty} I_n$ contains no rational numbers, but, as a descending sequence of closed intervals, $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$. Thus $A \cap \bigcap_{n=1}^{\infty} I_n \neq \emptyset$, and (A) wins. ◀

In these two examples, using informal language, we can say that player (A) has a *strategy* to win: No matter how player (B) proceeds, player (A) can "answer" each move and win the game.

In both examples player (A) had a clear advantage: The set A was larger than the set B. But in what sense is it larger? It is not the fact that A is uncountable while B is countable that matters here. It is something else: The fact that given an interval I_{2n}, player (A) can choose I_{2n+1} inside I_{2n} in such a way that I_{2n+1} misses the set $\{q_1, q_2, \dots, q_n\}$.

Let us try to see in the second example a general strategy that should work for player (A) in some cases. The set B was the union of the singleton sets $\{q_n\}$. Suppose instead that B is the union of a sequence of "small" sets Q_n. Then the same "strategy" will prevail if given any interval J and given any $n \in \mathbb{N}$, there exists an interval $I \subset J$ such that

$$I \cap (Q_1 \cup Q_2 \cup \cdots \cup Q_n) = \emptyset.$$

The set $\bigcap_{n=1}^{\infty} I_n$ will be nonempty, and will miss the set $\bigcup_{n=1}^{\infty} Q_n$. Thus, if $B = \bigcup_{n=1}^{\infty} Q_n$, player (A) has a winning strategy. It is in this sense that the set B is "small." The set A is "large" because the set B is "small". If we look carefully at the requirement on the sets Q_k, we see it is just that each of these sets is nowhere dense in $\mathbb{R}$.

Thus the key to player (A) winning rests on the concept of a nowhere dense set. But note that it rests on the set B being the union of a sequence of nowhere dense sets.

6.4.2 The Baire Category Theorem

Adv.

We can formulate our result from our discussion of the game in several ways:

1. $\mathbb{R}$ cannot be expressed as a countable union of nowhere dense sets.

2. The complement of a countable union of nowhere dense sets is dense.

The second of these provides a sense in which countable unions of nowhere dense sets are "small:" No matter which countable collection of nowhere dense sets we choose, their union leaves a dense set uncovered.

To formulate the Baire category theorem we need some definitions. This is the original language of Baire and it has survived; he simply places sets in two types or categories. Into the first category he places the sets that are to be considered small and into the second category he puts the remaining (not small) sets.

Definition 6.8: Let A be a set of real numbers.

1. A is said to be of the *first category* if it can be expressed as a countable union of nowhere dense sets.

2. A is said to be of the *second category* if it is not of the first category.

3. A is said to be *residual* in $\mathbb{R}$ if the complement $\mathbb{R} \setminus A$ is of the first category.

The following properties of first category sets and their complements, the residual sets, are easily proved and left as exercises.

Lemma 6.9: *A union of any sequence of first category sets is again a first category set.*

Lemma 6.10: *An intersection of any sequence of residual sets is again a residual set.*

Theorem 6.11 (Baire Category Theorem) *Every residual subset of* $\mathbb{R}$ *is dense in* $\mathbb{R}$.

Proof. The discussion in Section 6.4.1 constitutes a proof. Suppose that player (A) is dealt a set $A = X \cap [a, b]$ where X is residual. Then there is a sequence of nowhere dense sets $\{Q_n\}$ so that

$$X = \mathbb{R} \setminus \bigcup_{n=1}^{\infty} Q_n.$$

Then player (A) wins by choosing any interval $I_1 \subset [a, b]$ that avoids Q_1 and continues following the strategy of Section 6.4.1. In particular, X must contain a point of the interval $[a, b]$, and hence a point of any interval. ■

Theorem 6.11 provides a sense of largeness of sets that is not shared by dense sets in general. The intersection of two dense sets might be empty, but the intersection of two, or even countably many, residual sets must still be dense.

Exercises

6.4.1 Show that the union of any sequence of first category sets is again a first category set.

SEE NOTE 150

6.4.2 Show that the intersection of any sequence of residual sets is again a residual set.

SEE NOTE 151

6.4.3 Rewrite the proof of Theorem 6.11 without using the games language.

SEE NOTE 152

6.4.4 Give an example of two dense sets whose intersection is not dense. Does this contradict Theorem 6.11?

SEE NOTE 153

6.4.5 Suppose that $\bigcup_{n=1}^{\infty} A_n$ contains some interval (c, d). Show that there is a set, say A_{n_0}, and a subinterval $(c', d') \subset (c, d)$ so that A_{n_0} is dense in (c', d').

SEE NOTE 154

6.4.3 Uniform Boundedness

There are many applications of the Baire category Theorem in analysis. For now, we present just one application, dealing with the concept of *uniform boundedness*. Suppose we have a collection $\mathcal{F}$ of functions defined on $\mathbb{R}$ with the property that for each $x \in \mathbb{R}$, $\{|f(x)| : f \in \mathcal{F}\}$ is bounded. This means that for each $x \in \mathbb{R}$ there exists a number $M_x \geq 0$ such that $|f(x)| \leq M_x$ for all $f \in \mathcal{F}$. We can describe this situation by saying that $\mathcal{F}$ is *pointwise bounded*. Does this imply that the collection is *uniformly bounded*; that is, that there is a single number M so that $|f(x)| \leq M$ for all $f \in \mathcal{F}$ and every $x \in \mathbb{R}$?

Example 6.12: Let $q_1, q_2, q_3, \ldots$ be an enumeration of $\mathbb{Q}$. For each $n \in \mathbb{N}$ we define a function f_n by $f_n(q_k) = k$ if $n \leq k$, $f_n(x) = 0$ for all other values x. Let $\mathcal{F} = \{f_n : n \in \mathbb{N}\}$. Then if $x \in \mathbb{R} \setminus \mathbb{Q}$, $f(x) = 0$ for all $f \in \mathcal{F}$, and if $x = q_k$, $|f(x)| \leq k$ for all $f \in \mathcal{F}$. Thus, for each $x \in \mathbb{R}$, the set $\{|f(x)| : f \in \mathcal{F}\}$ is bounded. The bounds can be taken to be 0 if $x \in \mathbb{R} \setminus \mathbb{Q}$ ($M_x = 0$ if $x \in \mathbb{R} \setminus \mathbb{Q}$) and we can take $M_{q_k} = k$. But since $\mathbb{Q}$ is dense in $\mathbb{R}$, none of the functions f_n is bounded on any interval. (Verify this.) Thus a collection of functions may be pointwise bounded but not uniformly bounded on any interval. ◀

The functions f_n in Example 6.12 are everywhere discontinuous. Our next theorem shows that if we had taken a collection $\mathcal{F}$ of *continuous* functions, then not only would each $f \in \mathcal{F}$ be bounded on closed intervals (as Theorem 5.49 guarantees), but there would be an interval I on which the entire collection is *uniformly bounded* ; that is, there exists a constant M such that $|f(x)| \leq M$ for all $f \in \mathcal{F}$ and each $x \in I$.

Theorem 6.13: *Let $\mathcal{F}$ be a collection of continuous functions on $\mathbb{R}$ such that for each $x \in \mathbb{R}$ there exists a constant $M_x > 0$ such that $|f(x)| \leq M_x$ for each $f \in \mathcal{F}$. Then there exists an open interval I and a constant $M > 0$ such that $|f(x)| \leq M$ for each $f \in \mathcal{F}$ and $x \in I$.*

Proof. For each $n \in \mathbb{N}$, let $A_n = \{x : |f(x)| \leq n \text{ for all } f \in \mathcal{F}\}$. By hypothesis, $\mathbb{R} = \bigcup_{n=1}^{\infty} A_n$. Also, by hypothesis, each $f \in \mathcal{F}$ is continuous and so it is easy to check that each of the sets

$$\{x : |f(x)| \leq n\}$$

must be closed (e.g., Exercise 5.4.31). Thus

$$A_n = \bigcap_{f \in \mathcal{F}} \{x : |f(x)| \leq n\}$$

is an intersection of closed sets and is therefore itself closed. This expresses the real line $\mathbb{R}$ as a union of the sequence of closed sets $\{A_n\}$.

It now follows from the Baire category theorem that at least one of the sets, say A_{n_0}, must be dense in some open interval I. Since A_{n_0} is closed and dense in the interval I, A_{n_0} must contain I. This means that $|f(x)| \leq n_0$ for each $f \in \mathcal{F}$ and all $x \in I$. ■

Exercises

6.4.6 Let $\{f_n\}$ be a sequence of continuous functions on an interval $[a, b]$ such that

$$\lim_{n\to\infty} f_n(x) = f(x)$$

exists at every point $x \in [a, b]$. Show that f need not be continuous nor even bounded, but that f must be bounded on some subinterval of $[a, b]$.

6.4.7 Let $\{f_n\}$ be a sequence of continuous functions on $[0, 1]$ and suppose that

$$\lim_{n\to\infty} f_n(x) = 0$$

for all $0 \leq x \leq 1$. Show that there must be an interval $[c, d] \subset [0, 1]$ so that, for all sufficiently large n, $|f_n(x)| \leq 1$ for all $x \in [c, d]$.

SEE NOTE 155

6.4.8 Give an example of a sequence of functions on $[0, 1]$ with the property that

$$\lim_{n\to\infty} f_n(x) = 0$$

for all $0 \leq x \leq 1$ and yet for every interval $[c, d] \subset [0, 1]$ and every N there is some $x \in [c, d]$ and $n > N$ with $f_n(x) > 1$.

6.5 Cantor Sets

We say that a set is *perfect* if it is a nonempty closed set with no isolated points. The only examples that might come to mind are sets that are finite unions of intervals. It might be difficult to imagine a perfect subset of $\mathbb{R}$ that is also nowhere dense. In this section we obtain such a set, the very important classical Cantor set. We also discuss some of its variants. Such sets have historical significance and are of importance in a number of areas of mathematical analysis.

6.5.1 Construction of the Cantor Ternary Set

We begin with the closed interval $[0,1]$. From this interval we shall remove a dense open set G. The remaining set $K = [0,1] \setminus G$ will then be closed and nowhere dense in [0,1]. We construct G in such a way that K has no isolated points and is nonempty. Thus K will be a nonempty, nowhere dense perfect subset of [0,1].

It is easiest to understand the set G if we construct it in stages. Let $G_1 = \left(\frac{1}{3}, \frac{2}{3}\right)$, and let $K_1 = [0,1] \setminus G_1$. Thus $K_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$ is what remains when the middle third of the interval [0,1] is removed. This is the first stage of our construction.

We repeat this construction on each of the two component intervals of K_1. Let $G_2 = \left(\frac{1}{9}, \frac{2}{9}\right) \cup \left(\frac{7}{9}, \frac{8}{9}\right)$ and let $K_2 = [0,1] \setminus (G_1 \cup G_2)$. Thus

$$K_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right].$$

This completes the second stage.

We continue inductively, obtaining two sequences of sets, $\{K_n\}$ and $\{G_n\}$ with the following properties: For each $n \in \mathbb{N}$

1. G_n is a union of 2^{n-1} pairwise disjoint open intervals.

2. K_n is a union of 2^n pairwise disjoint closed intervals.

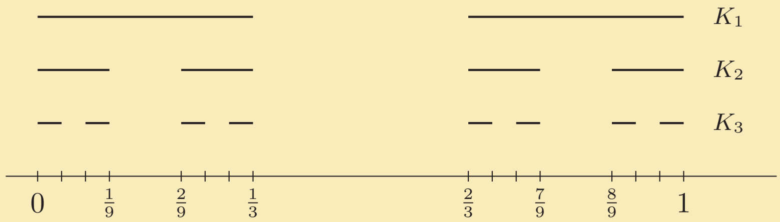

Figure 6.1. The third stage in the construction of the Cantor ternary set.

3. $K_n = [0,1] \setminus (G_1 \cup G_2 \cup \cdots \cup G_n)$.

4. Each component of G_{n+1} is the "middle third" of some component of K_n.

5. The length of each component of K_n is $1/3^n$.

Figure 6.1 shows K_1, K_2, and K_3.

Now let

$$G = \bigcup_{n=1}^{\infty} G_n$$

and let

$$K = [0,1] \setminus G = \bigcap_{n=1}^{\infty} K_n.$$

Then G is open and the set K (our Cantor set) is closed.

To see that K is nowhere dense, it is enough, since K is closed, to show that K contains no open intervals (Exercise 6.3.4). Let J be an open interval in $[0,1]$ and let λ be its length. Choose $n \in \mathbb{N}$ such that $1/3^n < \lambda$. By property 5, each component of K_n has length $1/3^n < \lambda$, and by property 2 the components of K_n are pairwise disjoint. Thus K_n cannot contain J, so neither can $K = \bigcap_1^\infty K_n$. We have shown that the closed set K contains no intervals and is therefore nowhere dense.

It remains to show that K has no isolated points. Let $x_0 \in K$. We show that x_0 is a limit point of K. To do this we show that for every $\varepsilon > 0$ there exists $x_1 \in K$ such that $0 < |x_1 - x_0| < \varepsilon$. Choose n such that $1/3^n < \varepsilon$. There is a component L of K_n that contains x_0. This component is a closed interval of length $1/3^n < \varepsilon$. The set $K_{n+1} \cap L$ has two components L_0 and L_1, each of which contains points of K. The point x_0 is in one of the components, say L_0. Let x_1 be any point of $K \cap L_1$. Then $0 < |x_0 - x_1| < \varepsilon$. This verifies that x_0 is a limit point of K. Thus K has no isolated points.

The set K is called the *Cantor set.* Because of its construction, it is often called the Cantor middle third set. In a moment we shall present a purely arithmetic description of the Cantor set that suggests another common name for K, the "Cantor ternary set". But first, we mention a few properties of K and of its complement G that may help you visualize these sets.

First note that G is an open dense set in $[0,1]$. Write $G = \bigcup_{k=1}^\infty (a_k, b_k)$. (The component intervals (a_k, b_k) of G can be called the intervals *complementary* to K in $(0,1)$. Each is a middle third of a component interval of some K_n.) Observe that no two of these component intervals can have a common endpoint. If, for example, $b_m = a_n$, then this point would be an isolated point of K, and K has no isolated points.

Next observe that for each $k \in \mathbb{N}$, the points a_k and b_k are points of K. But there are other points of K as well. In fact, we shall see presently that K is uncountable. These other points are all limit points of the endpoints of the complementary intervals. The set of endpoints is countable, but the closure of this set is uncountable as we shall see. Thus, in the sense of cardinality, "most" points of the Cantor set are *not* endpoints of intervals complementary to K.

Each component interval of the set G_n has length $1/3^n$; thus the sum of the lengths of these component intervals is

$$\frac{2^{n-1}}{3^n} = \frac{1}{2}\left(\frac{2}{3}\right)^n .$$

It follows that the lengths of all component intervals of G forms a geometric series with sum

$$\sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{2}{3}\right)^n = 1.$$

(This also gives us a clue as to why K cannot contain an interval: After removing from the unit interval a sequence of pairwise disjoint intervals with length-sum one, no room exists for any intervals in the set K that remains.)

Exercises

6.5.1 Let E be the set of endpoints of intervals complementary to the Cantor set K. Prove that $\overline{E} = K$.

6.5.2 Let G be a dense open subset of $\mathbb{R}$ and let $\{(a_k, b_k)\}$ be its set of component intervals. Prove that $H = \mathbb{R} \setminus G$ is perfect if and only if no two of these intervals have common endpoints.

6.5.3 Let K be the Cantor set and let $\{(a_k, b_k)\}$ be the sequence of intervals complementary to K in $[0, 1]$. For each $k \in \mathbb{N}$, let $c_k = (a_k + b_k)/2$ (the midpoint of the interval (a_k, b_k)) and let $N = \{c_k : k \in \mathbb{N}\}$. Prove each of the following:

(a) Every point of N is isolated.

(b) If $c_i \neq c_j$, there exists $k \in \mathbb{N}$ such that c_k is between c_i and c_j (i.e., no point in N has an immediate "neighbor" in N).

(c) Show that there is an *order-preserving mapping* $\phi : \mathbb{Q} \cap (0, 1) \to N$ [i.e., if $x < y \in \mathbb{Q} \cap (0, 1)$, then $\phi(x) < \phi(y) \in N$]. This may seem surprising since $\mathbb{Q} \cap (0, 1)$ has *no* isolated points while N has *only* isolated points.

6.5.4 It is common now to say that a set E of real numbers is a *Cantor set* if it is nonempty, bounded, perfect, and nowhere dense. Show that the union of a finite number of Cantor sets is also a Cantor set.

6.5.5 Show that every Cantor set is uncountable.

6.5.6 ✂ Let A and B be subsets of $\mathbb{R}$. A function h that maps A onto B, is one-to-one, and with both h and h^{-1} continuous is called a *homeomorphism* between A and B. The sets A and B are said to be *homeomorphic*. Prove that a set C is a Cantor set if and only if it is homeomorphic to the Cantor ternary set K.

6.5.2 An Arithmetic Construction of K

We turn now to a purely arithmetical construction for the Cantor set. You will need some familiarity with ternary (base 3) arithmetic here.

Each $x \in [0,1]$ can be expressed in base 3 as

$$x = .a_1a_2a_3\dots,$$

where $a_i = 0$, 1 or 2, $i = 1, 2, 3, \dots$. Certain points have two representations, one ending with a string of zeros, the other in a string of twos. For example, $.1000\cdots = .0222\dots$ both represent the number 1/3 (base ten). Now, if $x \in (1/3, 2/3)$, $a_1 = 1$, thus each $x \in G_1$ must have '1' in the first position of its ternary expansion. Similarly, if

$$x \in G_2 = \left(\frac{1}{9}, \frac{2}{9}\right) \cup \left(\frac{7}{9}, \frac{8}{9}\right),$$

it must have a 1 in the second position of its ternary expansion (i.e., $a_2 = 1$). In general, each point in G_n must have $a_n = 1$. It follows that every point of $G = \bigcup_1^\infty G_n$ must have a 1 someplace in its ternary expansion.

Now endpoints of intervals complementary to K have two representations, one of which involves no 1's. The remaining points of K never fall in the middle third of a component of one of the sets K_n, and so have ternary expansions of the form

$$x = .a_1a_2\dots \quad a_i = 0 \text{ or } 2.$$

We can therefore describe K arithmetically as the set

$$\{x = .a_1a_2a_3\dots \ \text{(base three)} \ : a_i = 0 \text{ or } 2 \text{ for each } i \in \mathbb{N}\}.$$

As an immediate result, we see that K is uncountable. In fact, K can be put into 1-1 correspondence with [0,1]: For each

$$x = .a_1a_2a_3\dots \text{ (base 3)}, \ a_i = 0, 2,$$

in the set K, let there correspond the number

$$y = .b_1b_2b_3\ldots \text{ (base 2)},\ b_i = a_i/2.$$

This provides a 1-1 correspondence between K (minus endpoints of complementary intervals) and $[0,1]$ (minus the countable set of numbers with two base 2 representations). By allowing these two countable sets to correspond to each other, we obtain a 1-1 correspondence between K and $[0,1]$.

Note. We end this section by mentioning that variations in the constructions of K can lead to interesting situations. For example, by changing the construction slightly, we can remove intervals in such a way that

$$G' = \bigcup_{k=1}^{\infty}(a'_k, b'_k)$$

with

$$\sum_{k=1}^{\infty}(b'_k - a'_k) = 1/2$$

(instead of 1), while still keeping $K' = [0,1] \setminus G'$ nowhere dense and perfect. The resulting set K' created problems for late nineteenth-century mathematicians trying to develop a theory of measure. The "measure" of G' should be 1/2; the "measure" of [0,1] should be 1. Intuition requires that the measure of the nowhere dense set K' should be $1 - \frac{1}{2} = \frac{1}{2}$. How can this be when K' is so "small?"

Exercises

6.5.7 Find a specific irrational number in the Cantor ternary set.

SEE NOTE 156

6.5.8 Show that the Cantor ternary set can be defined as

$$K = \left\{x \in [0,1] : x = \sum_{n=1}^{\infty} \frac{i_n}{3^n} \text{ for } i_n = 0 \text{ or } 2\right\}.$$

6.5.9 Let

$$D = \left\{ x \in [0,1] : x = \sum_{n=1}^{\infty} \frac{j_n}{3^n} \text{ for } j_n = 0 \text{ or } 1 \right\}.$$

Show that $D + D = \{x + y : x, y \in D\} = [0,1]$. From this deduce, for the Cantor ternary set K, that $K + K = [0,2]$.

6.5.10 A careless student makes the following argument. Explain the error.

> "If $G = (a,b)$, then $\overline{G} = [a,b]$. Similarly, if $G = \bigcup_{i=1}^{\infty}(a_i, b_i)$ is an open set, then $\overline{G} = \bigcup_{i=1}^{\infty}[a_i, b_i]$. It follows that an open set G and its closure $\overline{G}$ differ by at most a countable set."

See Note 157

6.5.3 The Cantor Function

The Cantor set allows the construction of a rather bizarre function that is continuous and nondecreasing on the interval $[0,1]$. It has the property that it is constant on every interval complementary to the Cantor set and yet manages to increase from $f(0) = 0$ to $f(1) = 1$ by doing all of its increasing on the Cantor set itself. It has sometimes been called "the devil's staircase."

Define the function f in the following way. On $(1/3, 2/3)$, let $f = 1/2$; on $(1/9, 2/9)$, let $f = 1/4$; on $(7/9, 8/9)$, let $f = 3/4$. Proceed inductively. On the 2^{n-1} open intervals appearing at the nth stage, define f to satisfy the following conditions:

1. f is constant on each of these intervals.
2. f takes the values
$$\frac{1}{2^n}, \frac{3}{2^n}, \ldots, \frac{2^n - 1}{2^n}$$
on these intervals.
3. If x and y are members of different nth-stage intervals with $x < y$, then $f(x) < f(y)$.

This description defines f on $G = [0,1] \setminus K$. Extend f to all of $[0,1]$ by defining $f(0) = 0$ and, for $0 < x \leq 1$,

$$f(x) = \sup\{f(t) : t \in G, t < x\}.$$

In order to check that this defines the function that we want, we need to check each of the following.

1. $f(G)$ is dense in $[0,1]$.

2. f is nondecreasing on $[0,1]$.

3. f is continuous on $[0,1]$.

4. $f(K) = [0,1]$.

These have been left as exercises.

Figure 6.2 illustrates the construction. The function f is called the *Cantor function.* Observe that f “does all its rising” on the set K.

The Cantor function allows a negative answer to many questions that might be asked about functions and derivatives and, hence, has become a popular counterexample. For example, let us follow this kind of reasoning. If f is a continuous function on $[0,1]$ and $f'(x) = 0$ for every $x \in (0,1)$ then f is constant. (This is proved in most calculus courses by using the mean value theorem.) Now suppose that we know less, that $f'(x) = 0$ for every $x \in (0,1)$ excepting a “small” set E of points at which we know nothing. If E is finite it is still easy to show that f must be constant. If E is countable it is possible, but a bit more difficult, to show that it is still true that f must be constant. The question then arises, just how small a set E can appear here; that is, what would we have to know about a set E so that we could say $f'(x) = 0$ for every $x \in (0,1) \setminus E$ implies that f is constant?

The Cantor function is an example of a function constant on every interval complementary to the Cantor set K (and so with a zero derivative at those points) and yet is not constant. The Cantor set, since it is nowhere dense, might be viewed as extremely small, but even so it is not insignificant for this problem.

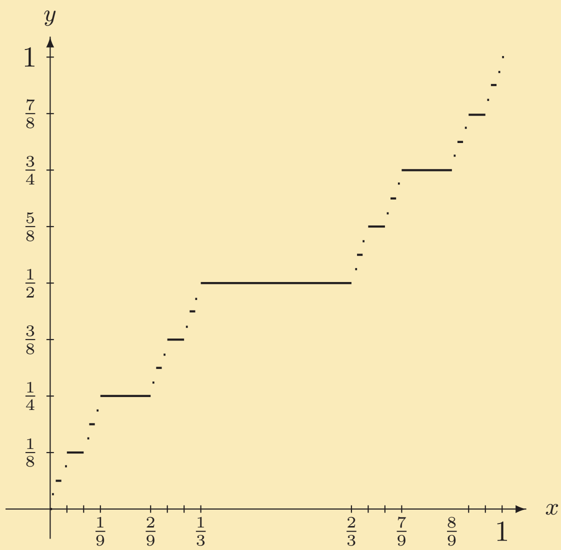

Figure 6.2. The third stage in the construction of the Cantor function.

Exercises

6.5.11 In the construction of the Cantor function complete the verification of details.

(a) Show that $f(G)$ is dense in $[0, 1]$.

(b) Show that f is nondecreasing on $[0, 1]$.

(c) Infer from (a) and (b) that f is continuous on $[0, 1]$.

(d) Show that $f(K) = [0, 1]$ and thus (again) conclude that K is uncountable.

6.5.12 Find a calculus textbook proof for the statement that a continuous function f on an interval $[a, b]$ that has a zero derivative on (a,b) must be constant. Improve the proof to allow a finite set of points on which f is not known to have a zero derivative.

6.6 Borel Sets

Adv.

In our study of continuous functions we have seen that the classes of open sets and closed sets play a significant role. But the class of sets that are of importance in analysis goes beyond merely the open and closed sets. E. Borel (1871–1956) recognized that for many operations of analysis we need to form countable intersections and countable unions of classes of sets. The collection of Borel sets was introduced exactly to allow these operations. We recall that a countable union of closed sets may not be closed (or open) and that a countable intersection of open sets, also, may not be open (or closed).

In this section we introduce two additional types of sets of importance in analysis, sets of type $\mathcal{G}_\delta$ and sets of type $\mathcal{F}_\sigma$. These classes form just the beginning of the large class of Borel sets. We shall find that they are precisely the right classes of sets to solve some fundamental questions about real functions.

6.6.1 Sets of Type G_δ

Adv.

Recall that the union of a collection of open sets is open (regardless of how many sets are in the collection), but the intersection of a collection of open sets need not be open if the collection has infinitely many sets. For example,

$$\bigcap_{n=1}^{\infty}\left(-\frac{1}{n},\frac{1}{n}\right)=\{0\}.$$

Similarly, if $q_1, q_2, q_3, \ldots$ is an enumeration of $\mathbb{Q}$, then

$$\bigcap_{k=1}^{\infty}(\mathbb{R}\setminus\{q_k\})=\mathbb{R}\setminus\mathbb{Q},$$

the set of irrational numbers. The set $\{0\}$ is closed (not open), and $\mathbb{R} \setminus \mathbb{Q}$ is neither open nor closed. The set $\mathbb{R} \setminus \mathbb{Q}$ is a countable intersection of open sets. Such sets are of sufficient importance to give them a name.

Definition 6.14: A subset H of $\mathbb{R}$ is said to be of *type* $\mathcal{G}_\delta$ (or a $\mathcal{G}_\delta$ *set*) if it can be expressed as a countable intersection of open sets, that is, if there exist open sets $G_1, G_2, G_3, \dots$ such that $H = \bigcap_{k=1}^{\infty} G_k$.

Example 6.15: A closed interval $[a, b]$ or a half-open interval $(a, b]$ is of type $\mathcal{G}_\delta$ since

$$[a, b] = \bigcap_{n=1}^{\infty} \left(a - \frac{1}{n}, b + \frac{1}{n}\right)$$

and

$$(a, b] = \bigcap_{n=1}^{\infty} \left(a, b + \frac{1}{n}\right).$$

◀

Theorem 6.16: *Every open set and every closed set in* $\mathbb{R}$ *is of type* $\mathcal{G}_\delta$.

Proof. Let G be an open set in $\mathbb{R}$. It is clear that G is of type $\mathcal{G}_\delta$. We also show that G can be expressed as a countable union of closed sets. Express G in the form

$$G = \bigcup_{k=1}^{\infty} (a_k, b_k)$$

where the intervals (a_k, b_k) are pairwise disjoint. Now for each $k \in \mathbb{N}$ there exist sequences $\{c_{k_j}\}$ and $\{d_{k_j}\}$ such that the sequence $\{c_{k_j}\}$ decreases to a_k, the sequence $\{d_{k_j}\}$ increases to b_k and $c_{k_j} < d_{k_j}$ for each $j \in \mathbb{N}$. Thus

$$(a_k, b_k) = \bigcup_{j=1}^{\infty} [c_{k_j}, d_{k_j}].$$

We have expressed each component interval of G as a countable union of closed sets. It follows that

$$G = \bigcup_{k=1}^{\infty} \bigcup_{j=1}^{\infty} [c_{k_j}, d_{k_j}] = \bigcup_{j,k=1}^{\infty} [c_{k_j}, d_{k_j}]$$

is also a countable union of closed sets. Now take complements. This shows that $\mathbb{R} \setminus G$ can be expressed as a countable intersection of open sets (by using the de Morgan laws). Since every closed set F can be written

$$F = \mathbb{R} \setminus G$$

for some open set G, we have shown that any closed set is of type $\mathcal{G}_\delta$. ■

We observed in Section 6.4 that a dense set can be small in the sense of category. For example, $\mathbb{Q}$ is a first category set. Our next result shows that a dense set *of type* $\mathcal{G}_\delta$ must be large in the sense of category.

Theorem 6.17: *Let H be of type $\mathcal{G}_\delta$ and be dense in $\mathbb{R}$. Then H is residual.*

Proof. Write

$$H = \bigcap_{k=1}^{\infty} G_k$$

with each of the sets G_k open. Since H is dense by hypothesis and $H \subset G_k$ for each $k \in \mathbb{N}$, each of the open sets G_k is also dense. Thus $\mathbb{R} \setminus G_k$ is nowhere dense for every $k \in \mathbb{N}$, and so each G_k is residual. The result now follows from Lemma 6.10. ■

Exercises

6.6.1 Which of the following sets are of type $\mathcal{G}_\delta$?

(a) $\mathbb{N}$

(b) $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$

(c) The set $\{C_n : n \in \mathbb{N}\}$ of midpoints of intervals complementary to the Cantor set

(d) A finite union of intervals (that need not be open or closed)

6.6.2 Prove Theorem 6.17 for the interval $[a, b]$ in place of $\mathbb{R}$.

6.6.3 Prove that a set E of type $\mathcal{G}_\delta$ in $\mathbb{R}$ is either residual or else there is an interval containing no points of E.

6.6.2 Sets of Type F_σ

Just as the countable intersections of open sets form a larger class of sets, the $\mathcal{G}_\delta$ sets, so also the countable unions of closed sets form a larger class of sets.

The complements of open sets are closed. By dealing with complements of $\mathcal{G}_\delta$ sets we arrive at the dual notion of a set of type $\mathcal{F}_\sigma$.

Definition 6.18: A subset E of $\mathbb{R}$ is said to be of *type* $\mathcal{F}_\sigma$ (or an $\mathcal{F}_\sigma$ set) if it can be expressed as a countable union of closed sets; that is, if there exist closed sets $F_1, F_2, F_3, \ldots$ such that $E = \bigcup_{k=1}^{\infty} F_k$.

Using the de Morgan laws, we verify easily that the complement of a $\mathcal{G}_\delta$ set is an $\mathcal{F}_\sigma$ and vice versa (Exercise 6.6.4). This is closely related to the fact that a set is open if and only if its complement is closed.

Example 6.19: The set of rational numbers, $\mathbb{Q}$ is a set of type $\mathcal{F}_\sigma$. This is clear since it can be expressed as

$$\mathbb{Q} = \bigcup_{n=1}^{\infty} \{r_n\}$$

where $\{r_n\}$ is any enumeration of the rationals. The singleton sets $\{r_n\}$ are clearly closed. But note that $\mathbb{Q}$ is *not* of type $\mathcal{G}_\delta$ also. It follows from Theorem 6.17 that a dense set of type $\mathcal{G}_\delta$ must be uncountable (because a countable set is first category). In particular, $\mathbb{Q}$ is not of type $\mathcal{G}_\delta$ (and therefore $\mathbb{R} \setminus \mathbb{Q}$ is not of type $\mathcal{F}_\sigma$). ◀

Theorem 6.20: *A set is of type $\mathcal{G}_\delta$ if and only if its complement is of type $\mathcal{F}_\sigma$.*

Example 6.21: A half-open interval $(a,b]$ is both of type $\mathcal{G}_\delta$ and of type $\mathcal{F}_\sigma$:

$$(a,b] = \bigcap_{n=1}^{\infty}\left(a, b+\frac{1}{n}\right) = \bigcup_{n=1}^{\infty}\left[a+\frac{b-a}{n}, b\right].$$

◀

Note. The only subsets of $\mathbb{R}$ that are both open and closed are the empty set and $\mathbb{R}$ itself. There are, however, many sets that are of type $\mathcal{G}_\delta$ and also of type $\mathcal{F}_\sigma$. See Exercise 6.6.1.

We can now enlarge on Theorem 6.16. There we showed that all open sets and all closed sets are in the class $\mathcal{G}_\delta$. We now show they are also in the class $\mathcal{F}_\sigma$.

Theorem 6.22: *Every open set and every closed set in $\mathbb{R}$ is both of type $\mathcal{F}_\sigma$ and $\mathcal{G}_\delta$.*

Proof. In the proof of Theorem 6.16 we showed explicitly how to express any open set as an $\mathcal{F}_\sigma$. Thus open sets are of type $\mathcal{F}_\sigma$ as well as of type $\mathcal{G}_\delta$ (the latter being trivial). The part pertaining to closed sets now follows by considering complements and using the de Morgan laws. The complement of a closed set is open and therefore the complement of an $\mathcal{F}_\sigma$ set is a $\mathcal{G}_\delta$ set. ■

Exercises

6.6.4 Verify that a subset A of $\mathbb{R}$ is an $\mathcal{F}_\sigma$ ($\mathcal{G}_\delta$) if and only if $\mathbb{R}\setminus A$ is a $\mathcal{G}_\delta$ ($\mathcal{F}_\sigma$).

6.6.5 Which of the following sets are of type $\mathcal{F}_\sigma$?

(a) $\mathbb{N}$

(b) $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$

(c) The set $\{C_n : n \in \mathbb{N}\}$ of midpoints of intervals complementary to the Cantor set

(d) A finite union of intervals (that need not be open or closed)

6.6.6 ✂ Prove that a set of type $\mathcal{F}_\sigma$ in $\mathbb{R}$ is either first category or contains an open interval.

6.6.7 ✂ Let $\{f_n\}$ be a sequence of real functions defined on $\mathbb{R}$ and suppose that $f_n(x) \to f(x)$ at every point x. Show that

$$\{x : f(x) > \alpha\} = \bigcup_{m=1}^{\infty} \bigcup_{r=1}^{\infty} \bigcap_{n=r}^{\infty} \{x : f_n(x) \geq \alpha + 1/m\}.$$

If each function f_n is continuous, what can you assert about the set

$$\{x : f(x) > \alpha\}?$$

SEE NOTE 158

6.7 Oscillation and Continuity

In this section we return to a problem that we began investigating in Section 5.9 about the nature of the set of discontinuity points of a function. To discuss this set we shall need the notions of $\mathcal{F}_\sigma$ and $\mathcal{G}_\delta$ sets and we need to introduce a new tool, the oscillation of a function.

We begin with an example of a function f that is discontinuous at every rational number and continuous at every irrational number.

Example 6.23: Let $q_1, q_2, q_3, \ldots$ be an enumeration of $\mathbb{Q}$. Define a function f by

$$f(x) = \begin{cases} \frac{1}{k}, & \text{if } x = q_k \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Since $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$, f can be continuous at a point x only if $f(x) = 0$; that is, only if $x \in \mathbb{R} \setminus \mathbb{Q}$. Thus f is discontinuous at every $x \in \mathbb{Q}$. To check that f is continuous at each point of $\mathbb{R} \setminus \mathbb{Q}$, let $x_0 \in \mathbb{R} \setminus \mathbb{Q}$ and let $\varepsilon > 0$. Choose $k \in \mathbb{N}$ such that $1/k < \varepsilon$. Since the set $q_1, q_2, \ldots, q_k$ is a finite set not containing x_0, there exists $\delta > 0$ such that $|q_i - x_0| \geq \delta$ for each $i = 1, \ldots, k$. Thus if $x \in \mathbb{R}$ and $|x - x_0| < \delta$, then either $x \in \mathbb{R} \setminus \mathbb{Q}$ or $x = q_j$ for some $j > k$. In either case $|f(x) - f(x_0)| \leq \frac{1}{k} < \varepsilon$. This verifies the continuity of f at x_0. Since x_0 was an arbitrary irrational point, we see that f is continuous at every irrational. ◀

Our example shows that it is possible for a function to be continuous at every irrational number and discontinuous at every rational number. Is it possible for the opposite to occur? Does there exist a function f continuous on $\mathbb{Q}$ and discontinuous on $\mathbb{R} \setminus \mathbb{Q}$? More generally, what sets can be the set of points of continuity of some function f defined on an interval.

We answer this question in this section. The principal tool is that of *oscillation* of a function at a point.

6.7.1 Oscillation of a Function

✂ *Adv.*

In order to describe a point of discontinuity we need a way of measuring that discontinuity. For monotonic functions the jump was used previously for such a measure. For general, nonmonotonic, functions a different tool is used.

Definition 6.24: Let f be defined on a nondegenerate interval I. We define the *oscillation of f on I* as the quantity

$$\omega f(I) = \sup_{x,y \in I} |f(x) - f(y)|.$$

Let's see how oscillation relates to continuity. Suppose f is defined in a neighborhood of x_0, and f is continuous at x_0. Then

$$\inf_{\delta>0} \omega f((x_0 - \delta, x_0 + \delta)) = 0. \tag{1}$$

To see this, let $\varepsilon > 0$. Since f is continuous at x_0, there exists $\delta_0 > 0$ such that

$$|f(x) - f(x_0)| < \varepsilon/2$$

if $|x - x_0| < \delta_0$. If

$$x_0 - \delta_0 < x_1 \leq x_2 < x_0 + \delta_0,$$

then

$$|f(x_1) - f(x_2)| \leq |f(x_1) - f(x_0)| + |f(x_0) - f(x_2)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \tag{2}$$

Since (2) is valid for all $x_1, x_2 \in (x_0 - \delta_0, x_0 + \delta_0)$, we have

$$\sup\{|f(x_1) - f(x_2)| : x_0 - \delta_0 < x_1 \le x_2 < x_0 + \delta_0\} \le \varepsilon. \tag{3}$$

But (3) implies that if $0 < \delta < \delta_0$, then

$$\omega f([x_0 - \delta, x_0 + \delta]) \le \varepsilon.$$

Since ε was arbitrary, the result follows.

The converse is also valid. Suppose (1) holds. Let $\varepsilon > 0$. Choose $\delta > 0$ such that

$$\omega f(x_0 - \delta, x_0 + \delta) < \varepsilon.$$

Then

$$\sup\{|f(x) - f(x_0)| : x \in (x_0 - \delta, x_0 + \delta)\} < \varepsilon,$$

so $|f(x) - f(x_0)| < \varepsilon$ whenever $|x - x_0| < \delta$. This implies continuity of f at x_0.

We summarize the preceding as a theorem.

Theorem 6.25: *Let f be defined on an interval I and let $x_0 \in I$. Then f is continuous at x_0 if and only if*

$$\inf_{\delta>0} \omega f((x_0 - \delta, x_0 + \delta)) = 0.$$

The quantity in the statement of the theorem is sufficiently important to have a name.

Definition 6.26: Let f be defined in a neighborhood of x_0. The quantity

$$\omega_f(x_0) = \inf_{\delta>0} \omega f((x_0 - \delta, x_0 + \delta))$$

is called the *oscillation of f at x_0*.

Theorem 6.25 thus states that a function f is continuous at a point x_0 if and only if $\omega_f(x_0) = 0$. Returning to the function that introduced this section, we see that

$$\omega_f(x) = \begin{cases} 1/k, & \text{if } x = q_k \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Let's now see how the concept of oscillation relates to the set of points of continuity of a function.

Theorem 6.27: *Let f be defined on a closed interval I (which may be all of $\mathbb{R}$). Let $\gamma > 0$. Then the set*

$$\{x : \omega_f(x) < \gamma\}$$

is open and the set

$$\{x : \omega_f(x) \geq \gamma\}$$

is closed.

Proof. Let $A = \{x : \omega_f(x) < \gamma\}$ and let $x_0 \in A$. We wish to find a neighborhood U of x_0 such that $U \subset A$; that is, such that $\omega_f(x) < \gamma$ for all $x \in U$.

Let $\omega_f(x_0) = \alpha < \gamma$ and let $\beta \in (\alpha, \gamma)$. From Definition 6.26 we infer the existence of a number $\delta > 0$ such that

$$|f(u) - f(v)| \leq \beta$$

for $u, v \in (x_0 - \delta, x_0 + \delta)$. Let

$$U = (x_0 - \delta, x_0 + \delta)$$

and let $x \in U$. Since U is open, there exists $\delta_1 < \delta$ such that

$$(x - \delta_1, x + \delta_1) \subset U.$$

Then

$$\begin{aligned} \omega_f(x) &\leq \sup\{|f(t) - f(s)| : t, s \in (x - \delta_1, x + \delta_1)\} \\ &\leq \sup\{|f(u) - f(v)| : u, v \in U\} \leq \beta < \gamma, \end{aligned}$$

so $x \in A$. This proves A is open. It follows then that the complement of A in I, the set

$$\{x : \omega_f(x) \geq \gamma\},$$

must be closed. ■

We use the oscillation in the next subsection to answer a question about the nature of the set of points of continuity of a function.

Exercises

6.7.1 Suppose that f is bounded on an interval I. Prove that

$$\omega f(I) = \sup_{x \in I} f(x) - \inf_{x \in I} f(x).$$

6.7.2 A careless student believes that the oscillation can be written as

$$\omega_f(x_0) = \limsup_{x \to x_0} f(x) - \liminf_{x \to x_0} f(x).$$

Show that this is not true, even for bounded functions.

6.7.3 Prove that

$$\omega_f(x_0) = \lim_{\delta \to 0+} \omega f((x_0 - \delta, x_0 + \delta)).$$

SEE NOTE 159

6.7.4 Calculate $\omega_f(0)$ for each of the following functions.

(a) $f(x) = \begin{cases} x, & \text{if } x \neq 0 \\ 4, & \text{if } x = 0 \end{cases}$

(b) $f(x) = \begin{cases} 0, & \text{if } x \in \mathbb{Q} \\ 1, & \text{if } x \notin \mathbb{Q} \end{cases}$

(c) $f(x) = \begin{cases} n, & \text{if } x = \frac{1}{n} \\ 0, & \text{otherwise} \end{cases}$

(d) $f(x) = \begin{cases} \sin\frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$

(e) $f(x) = \begin{cases} \sin\frac{1}{x}, & \text{if } x \neq 0 \\ 7, & \text{if } x = 0 \end{cases}$

(f) $f(x) = \begin{cases} \frac{1}{x}\sin\frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$

6.7.5 In the proof of Theorem 6.27 we let $\omega_f(x_0) = \alpha < \gamma$ and let $\beta \in (\alpha, \gamma)$. Why was the β introduced? Would the proof have worked if we had used $\beta = \gamma$?

6.7.2 The Set of Continuity Points

Adv.

Given an arbitrary function, how can we describe the nature of the set of points where f is continuous? Can it be any set? Given a set E, how can we know whether there is a function that is continuous at every point of E and discontinuous at every point not in E?

We saw in Example 6.23 that a function exists whose set of continuity points is exactly the irrationals. Can a function exist whose set of continuity points is exactly the rationals? By characterizing the set of such points we can answer this and other questions about the structure of functions.

We now prove the main result of this section using primarily the notion of oscillation introduced in Section 6.7.1.

Theorem 6.28: *Let f be defined on a closed interval I (which may be all of $\mathbb{R}$). Then the set C_f of points of continuity of f is of type $\mathcal{G}_\delta$, and the set D_f of points of discontinuity of f is of type $\mathcal{F}_\sigma$. Conversely, if H is a set of type $\mathcal{G}_\delta$, then there exists a function f defined on $\mathbb{R}$ such that $C_f = H$.*

Proof. To prove the first part, let $f : I \to \mathbb{R}$. We show that the set

$$C_f = \{x : \omega_f(x) = 0\}$$

is of type $\mathcal{G}_\delta$. For each $k \in \mathbb{N}$, let

$$B_k = \left\{x : \omega_f(x) \geq \frac{1}{k}\right\}.$$

By Theorem 6.27, each of the sets B_k is closed. Thus the set

$$B = \bigcup_{k=1}^{\infty} B_k$$

is of type $\mathcal{F}_\sigma$. By Theorem 6.25, $D_f = B$. Therefore, $C_f = I \setminus B$. Since the complement of an $\mathcal{F}_\sigma$ is a $\mathcal{G}_\delta$, the set C_f is a $\mathcal{G}_\delta$.

To prove the converse, let H be any subset of $\mathbb{R}$ of type $\mathcal{G}_\delta$. Then H can be expressed in the form

$$H = \bigcap_{k=1}^{\infty} G_k$$

with each of the sets G_k being open. We may assume without loss of generality that $G_1 = \mathbb{R}$ and that $G_i \supset G_{i+1}$ for each $i \in \mathbb{N}$. (Verify this.)

Let $\{\alpha_k\}$ and $\{\beta_k\}$ be sequences of positive numbers, each converging to zero, with

$$\alpha_k > \beta_k > \alpha_{k+1},$$

for all $k \in \mathbb{N}$. Define a function $f:\mathbb{R}\to\mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \in H \\ \alpha_k & \text{if } x \in (G_k \setminus G_{k+1}) \cap \mathbb{Q} \\ \beta_k & \text{if } x \in (G_k \setminus G_{k+1}) \cap (\mathbb{R} \setminus \mathbb{Q}). \end{cases}$$

We show that f is continuous at each point of H and discontinuous at each point of $\mathbb{R} \setminus H$.

Let $x_0 \in H$ and let $\varepsilon > 0$. Choose n such that $\alpha_n < \varepsilon$. Since

$$x_0 \in H = \bigcap_{k=1}^{\infty} G_k,$$

we see that $x_0 \in G_n$. The set G_n is open, so there exists $\delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \subset G_n$. From the definition of f on G_n, we see that

$$0 \leq f(x) \leq \alpha_n < \varepsilon$$

for all $x \in (x_0 - \delta, x_0 + \delta)$. Thus

$$|f(x) - f(x_0)| = |f(x) - 0| = |f(x)| < \varepsilon$$

if $|x - x_0| < \delta$, so f is continuous at x_0.

Now let $x_0 \in \mathbb{R} \setminus H$. Then there exists $k \in \mathbb{N}$ such that x_0 belongs to the set $G_k \setminus G_{k+1}$. Thus $f(x_0) = \alpha_k$ or $f(x_0) = \beta_k$. Let us suppose that $f(x_0) = \alpha_k$. If x_0 is an interior point of $G_k \setminus G_{k+1}$, then x_0 is a limit point of

$$\{x : x \in (G_k \setminus G_{k+1}) \cap (\mathbb{R} \setminus \mathbb{Q})\} = \{x : f(x) = \beta_k\},$$

so f is discontinuous at x_0.

The argument is similar if x_0 is a boundary point of $G_k \setminus G_{k+1}$. Again, assume $f(x_0) = \alpha_k$. Arbitrarily close to x_0 there are points of the set

$$\mathbb{R} \setminus (G_k \setminus G_{k+1}).$$

At these points, f takes on values in the set

$$S = \{0\} \cup \bigcup_{i \neq k} \alpha_i \cup \bigcup_{j \neq k} \beta_j.$$

The only limit point of this set is zero and so S is closed. In particular, α_k is *not* a limit point of this set and does not belong to the set. Let ε be half the distance from the point α_k to the closed set S; that is, let

$$\varepsilon = \frac{1}{2} d(\alpha_k, S).$$

Arbitrarily close to x_0 there are points x such that $f(x) \in S$. For such a point,

$$|f(x) - f(x_0)| = |f(x) - \alpha_k| > \varepsilon,$$

so f is discontinuous at x_0. ■

Observe that Theorem 6.28 answers a question we asked earlier: Is there a function f continuous on $\mathbb{Q}$ and discontinuous at every point of $\mathbb{R} \setminus \mathbb{Q}$? The answer is negative, since $\mathbb{Q}$ is not of type $\mathcal{G}_\delta$.

Exercises

6.7.6 In the second part of the proof of Theorem 6.28 we provided a construction for a function f with $C_f = H$, where H is an arbitrary set of type $\mathcal{G}_\delta$. Exhibit explicitly sets G_k that will give rise to a function f such that $C_f = \mathbb{R} \setminus \mathbb{Q}$. Can you do this in such a way that the resulting function is the one we obtained at the beginning of this section?

6.7.7 In the proof of Theorem 6.28 we took $\varepsilon = \frac{1}{2} d(\alpha_k, S)$. Show that this number equals

$$\frac{1}{2} \min_{i \neq k} \left\{ \min\{|\alpha_i - \alpha_k|, |\beta_i - \beta_k|\} \right\}.$$

6.8 Challenging Problems for Chapter 6

6.8.1 Show that a function is discontinuous except at the points of a first category set if and only if it is continuous at a dense set of points.

6.8.2 Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that for every positive number ε the sequence $\{f(n\varepsilon)\}$ converges to zero as $n \to \infty$. Prove that

$$\lim_{x \to \infty} f(x) = 0.$$

See Note 160

6.8.3 Let f_n be a sequence of continuous functions defined on an interval $[a, b]$ such that $\lim_{n \to \infty} f_n(x) = 0$ for each $x \in [a, b]$. Show that for any $\varepsilon > 0$ there is an interval $[c, d] \subset [a, b]$ and an integer N so that

$$|f_n(x)| < \varepsilon$$

for every $n \geq N$ and every $x \in [c, d]$. Show that this need not be true for $[c, d] = [a, b]$.

6.8.4 Let f_n be a sequence of continuous functions defined on an interval $[a, b]$ such that $\lim_{n \to \infty} f_n(x) = \infty$ for each $x \in [a, b]$. Show that for any $M > 0$ there is an interval $[c, d] \subset [a, b]$ and an integer N so that

$$f_n(x) > M$$

for every $n \geq N$ and every $x \in [c, d]$. Show that this need not be true for $[c, d] = [a, b]$.

Notes

[147]Exercise 6.2.9. To make this true, assume that f is onto or else show that if E is dense then $f(E)$ is dense in the set (interval) $f(\mathbb{R})$.

[148]Exercise 6.3.1. If $q_1, q_2, q_3, \dots$ is an enumeration of the rationals, then each of the sets $\{q_i\}$, $i \in \mathbb{N}$, is nowhere dense, but

$$\bigcup_{i=1}^{n} \{q_i\} = \mathbb{Q}$$

is not nowhere dense. (Indeed it is dense.)

[149]Exercise 6.3.2. All of (a)–(e) and (h) are true. Find counterexamples for (f) and (g). The proofs that the others are true follow routinely from the definition.

[150]Exercise 6.4.1. Suppose that

$$A_n = \bigcup_{k=1}^{\infty} A_{nk}$$

with each of the sets A_{nk} nowhere dense. Then

$$\bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} A_{nk} = \bigcup_{n,k=1}^{\infty} A_{nk}$$

expresses that union as a first category set.

[151]Exercise 6.4.2. Let $\{B_n\}$ be a sequence of residual subsets of $\mathbb{R}$. Thus each of the sets B_n is the complement of a first category set A_n. For each n write

$$A_n = \bigcup_{k=1}^{\infty} A_{nk}$$

with each of the sets A_{nk} nowhere dense. Then

$$B_n = \mathbb{R} \setminus \bigcup_{k=1}^{\infty} A_{nk}.$$

Now use De Morgan's laws.

[152]Exercise 6.4.3. Suppose that X is residual, that is,

$$X = \mathbb{R} \setminus \bigcup_{n=1}^{\infty} Q_n$$

where each Q_n is nowhere dense. Show that for any interval $[a,b]$ there is a point in $X \cap [a,b]$ by constructing an appropriate descending sequence of closed subintervals of $[a,b]$.

[153]Exercise 6.4.4. Make sure your sets are dense but not both residual (e.g., $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$).

[154]Exercise 6.4.5. This follows, with the correct interpretation, directly from the Baire category theorem.

[155]Exercise 6.4.7. Consider the sequence

$$A_N = \{x \in [0,1] : |f_n(x)| \leq 1, \text{ all } n \geq N\}.$$

Check that

$$\bigcup_{N=1}^{\infty} A_N = [0,1].$$

[156]Exercise 6.5.7. It is clear that there must be many irrational numbers in the Cantor ternary set, since that set is uncountable and the rationals are countable. Your job is to find just one.

[157]Exercise 6.5.10. Consider $G = (0,1) \setminus C$ where C is the Cantor ternary set.

[158]Exercise 6.6.7. Often to prove a set identity such as this the best way is to start with a point x that belongs to the set on the right and then show that point must be in the set on the left. After that is successful start with a point

x that belongs to the set on the left. For example, if $f(x) > \alpha$, then

$$f(x) \geq \alpha + 1/m$$

for some integer m. But

$$f_n(x) \to f(x)$$

and so there must be an integer R so that $f_n(x) > \alpha + 1/m$ for all $n \geq R$, etc.

This exercise shows how unions and intersections of sequences of open and closed sets might arise in analysis. Note that the sets

$$\{x : f_n(x) \geq \alpha + 1/m\}$$

would be closed if the functions f_n are continuous. Thus it would follow that the set

$$\{x : f(x) > \alpha\}$$

must be of type $\mathcal{F}_\sigma$. This says something interesting about a function f that is the limit of a sequence of continuous functions $\{f_n\}$.

[159]Exercise 6.7.3. You need to recall Theorem 5.60, which asserts that monotone functions have left- and right-hand limits.

[160]Exercise 6.8.2. This is from the 1964 Putnam Mathematical Competition.

Chapter 7

DIFFERENTIATION

7.1 Introduction

Calculus courses succeed in conveying an idea of what a derivative is, and the students develop many technical skills in computations of derivatives or applications of them. We shall return to the subject of derivatives but with a different objective.

Now we wish to see a little deeper and to understand the basis on which that theory develops. Much of this chapter will appear to be a review of the subject of derivatives with more attention paid to the details now and less to the applications. Some of the more advanced material will be, however, completely new.

We start at the beginning, at the rudiments of the theory of derivatives.

7.2 The Derivative

Let f be a function defined on an interval I and let x_0 and x be points of I. Consider the *difference quotient* determined by the points x_0 and x:

$$\frac{f(x)-f(x_0)}{x-x_0}, \tag{1}$$

representing the average rate of change of f on the interval with endpoints at x and x_0.

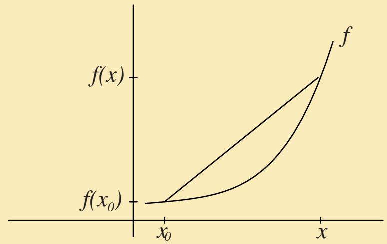

Figure 7.1. The chord determined by $(x, f(x))$ and $(x_0, f(x_0))$.

In Figure 7.1 this difference quotient represents the slope of the chord (or secant line) determined by the points $(x, f(x))$ and $(x_0, f(x_0))$. This same picture allows a physical interpretation. If $f(x)$ represents the distance a point moving on a straight line has moved from some fixed point in time x, then $f(x) - f(x_0)$ represents the (net) distance it has moved in the time interval $[x_0, x]$, and the difference quotient (1) represents the average velocity in that time interval.

Suppose now that we fix x_0, and allow x to approach x_0. We learn in elementary calculus that if

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists, then the limit represents the slope of the tangent line to the graph of the function f at the point $(x_0, f(x_0))$. In the setting of motion, the limit represents instantaneous velocity at time x_0.

The derivative owes its origins to these two interpretations in geometry and in the physics of motion, but now completely transcends them; the derivative finds applications in nearly every part of mathematics and the sciences.

We shall study the structure of derivatives, but with less concern for computations and applications than we would have seen in our calculus courses. Now we wish to understand the notion and see why it has the properties used in the many computations and applications of the calculus.

7.2.1 Definition of the Derivative

We begin with a familiar definition.

Definition 7.1: Let f be defined on an interval I and let $x_0 \in I$. The *derivative* of f at x_0, denoted by $f'(x_0)$, is defined as

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}, \tag{2}$$

provided either that this limit exists or is infinite. If $f'(x_0)$ is finite we say that f is *differentiable* at x_0. If f is differentiable at every point of a set $E \subset I$, we say that f is *differentiable* on E. When E is all of I, we simply say that f is a *differentiable* function.

Note. We have allowed infinite derivatives and they do play a role in many studies, but differentiable always refers to a finite derivative. Normally the phrase "a derivative exists" also means that that derivative is finite.

Example 7.2: Let $f(x) = x^2$ on $\mathbb{R}$ and let $x_0 \in \mathbb{R}$. If $x \in \mathbb{R}$, $x \neq x_0$, then

$$\frac{f(x) - f(x_0)}{x - x_0} = \frac{x^2 - x_0^2}{x - x_0} = \frac{(x - x_0)(x + x_0)}{(x - x_0)}.$$

Since $x \neq x_0$, the last expression equals $x + x_0$, so

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} (x + x_0) = 2x_0,$$

establishing the formula, $f'(x_0) = 2x_0$ for the function $f(x) = x^2$. ◀

Let us take a moment to clarify the definition when the interval I contains one or both of its endpoints. Suppose $I = [a, b]$. For $x_0 = a$ (or $x_0 = b$), the limit in (2) is just a one-sided, or unilateral, limit. The function f is defined only on $[a, b]$ so we cannot consider points outside of that interval.

This brings us to another point. It can happen that a function that is *not* differentiable at a point x_0 does satisfy the requirement of (2) from one side of x_0. This means that the limit in (2) exists as $x \to x_0$ from that side. We present a formal definition.

Definition 7.3: Let f be defined on an interval I and let $x_0 \in I$. The *right-hand derivative* of f at x_0, denoted by $f'_+(x_0)$ is the limit

$$f'_+(x_0) = \lim_{x \to x_0+} \frac{f(x) - f(x_0)}{x - x_0},$$

provided that one-sided limit exists or is infinite. Similarly, the *left-hand derivative* of f at x_0, $f'_-(x_0)$, is the limit

$$f'_-(x_0) = \lim_{x \to x_0-} \frac{f(x) - f(x_0)}{x - x_0}.$$

Observe that, if x_0 is an interior point of I, then $f'(x_0)$ exists if and only if $f'_+(x_0) = f'_-(x_0)$. (See Exercise 7.2.8)

Example 7.4: Let $f(x) = |x|$ on $\mathbb{R}$. Let us consider the differentiability of f at $x_0 = 0$. The difference quotient (1) becomes

$$\frac{f(x) - f(0)}{x - 0} = \frac{|x|}{x} = \begin{cases} 1, & \text{if } x > 0 \\ -1, & \text{if } x < 0. \end{cases}$$

Thus

$$f'_+(0) = \lim_{x \to x_0+} \frac{|x|}{x} = 1$$

while

$$f'_-(0) = \lim_{x \to x_0-} \frac{|x|}{x} = -1.$$

The function has different right-hand and left-hand derivatives at $x_0 = 0$ so is not differentiable at $x_0 = 0$.
◀

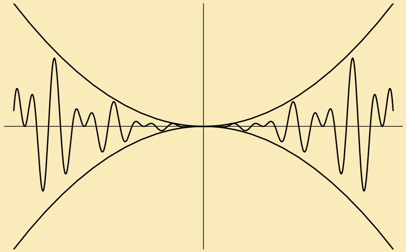

Figure 7.2. A function trapped between x^2 and $-x^2$.

Example 7.5: (A "trapping principle")

Let f be any function defined in a neighborhood I of zero. Suppose f satisfies the inequality $|f(x)| \leq x^2$ for all $x \in I$. Thus, the graph of f is "trapped" between the parabolas $y = x^2$ and $y = -x^2$. In particular, $f(0) = 0$. The difference quotient computed for $x_0 = 0$ becomes

$$\frac{f(x) - f(0)}{x - 0} = \frac{f(x)}{x},$$

from which we calculate

$$\left|\frac{f(x)}{x}\right| \leq \left|\frac{x^2}{x}\right| = |x|$$

so

$$\lim_{x \to 0} \left|\frac{f(x)}{x}\right| \leq \lim_{x \to 0} |x| = 0.$$

Thus

$$\lim_{x \to 0} \frac{f(x)}{x} = 0.$$

As a result, $f'(0) = 0$. Figure 7.2 illustrates the principle. ◀

Higher-Order Derivatives When a function f is differentiable on I, it is possible that its derivative f' is also differentiable. When this is the case, the function $f'' = (f')'$ is called the *second derivative* of the function f. Inductively, we can define derivatives of all orders: $f^{(n+1)} = (f^{(n)})'$ (provided $f^{(n)}$ is differentiable). When n is small, it is customary to use the convenient notation f'' for $f^{(2)}$, f''' for $f^{(3)}$ etc.

Notation It is useful to have other notations for the derivative of a function f. Common notations are $\frac{df}{dx}$ and $\frac{dy}{dx}$ (when the function is expressed in the form $y = f(x)$). Another notation that is useful is Df. These alternate notations along with slight variations are useful for various calculations. You are no doubt familiar with such uses—the convenience of writing

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

when using the chain rule, or viewing D as an operator in solving linear differential equations. Notation *can* be important at times. Consider, for example, how difficult it would be to perform a simple arithmetic calculation such as the multiplication (104)(90) using Roman numerals (CIV)(XC)!

Exercises

7.2.1 You might be familiar with a slightly different formulation of the definition of derivative. If x_0 is interior to I, then for h sufficiently small, the point $x_0 + h$ is also in I. Show that expression (2) then reduces to

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

Repeat Examples 7.2 and 7.4 using this formulation of the derivative.

See Note 161

7.2.2 Let $c \in \mathbb{R}$. Calculate the derivatives of the functions $g(x) = c$ and $k(x) = x$ directly from the definition of derivative.

7.2.3 Check the differentiability of each of the functions below at $x_0 = 0$.

(a) $f(x) = x|x|$

(b) $f(x) = x \sin x^{-1}$ $(f(0) = 0)$

(c) $f(x) = x^2 \sin x^{-1}$ $(f(0) = 0)$

(d) $f(x) = \begin{cases} x^2, & \text{if } x \text{ rational} \\ 0, & \text{if } x \text{ irrational} \end{cases}$

7.2.4 Let $f(x) = \begin{cases} x^2, & \text{if } x \geq 0 \\ ax, & \text{if } x < 0 \end{cases}$

(a) For which values of a is f differentiable at $x = 0$?

(b) For which values of a is f continuous at $x = 0$?

(c) When f is differentiable at $x = 0$, does $f''(0)$ exist?

7.2.5 For what positive values of p is the function $f(x) = |x|^p$ differentiable at 0?

7.2.6 A function f has a *symmetric derivative* at a point if

$$f'_s(x) = \lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h}$$

exists. Show that $f'_s(x) = f'(x)$ at any point at which the latter exists but that $f'_s(x)$ may exist even when f is not differentiable at x.

SEE NOTE 162

7.2.7 Find all points where $f(x) = \sqrt{1 - \cos x}$ is not differentiable and at those points find the one-sided derivatives.

SEE NOTE 163

7.2.8 Prove that if x_0 is an interior point of an interval I, then $f'(x_0)$ exists or is infinite if and only if $f'_+(x_0) = f'_-(x_0)$.

7.2.9 Let a function $f : \mathbb{R} \to \mathbb{R}$ be defined by setting $f(1/n) = c_n$ for $n = 1, 2, 3, \ldots$ where $\{c_n\}$ is a given sequence and elsewhere $f(x) = 0$. Find a condition on that sequence so that $f'(0)$ exists.

7.2.10 Let a function $f : \mathbb{R} \to \mathbb{R}$ be defined by setting $f(1/n^2) = c_n$ for $n = 1, 2, 3, \ldots$ where $\{c_n\}$ is a given sequence and elsewhere $f(x) = 0$. Find a condition on that sequence so that $f'(0)$ exists.

7.2.11 Give an example of a function with an infinite derivative at some point. Give an example of a function f with $f'_+(x_0) = \infty$ and $f'_-(x_0) = -\infty$ at some point x_0.

7.2.12 If $f'(x_0) > 0$ for some point x_0 in the interior of the domain of f show that there is a $\delta > 0$ so that

$$f(x) < f(x_0) < f(y)$$

whenever $x_0 - \delta < x < x_0 < y < x_0 + \delta$. Does this assert that f is increasing in the interval $(x_0 - \delta, x_0 + \delta)$?

SEE NOTE 164

7.2.13 Let f be increasing and differentiable on an interval. Does this imply that $f'(x) \geq 0$ on that interval? Does this imply that $f'(x) > 0$ on that interval?

SEE NOTE 165

7.2.14 Suppose that two functions f and g have the following properties at a point x_0: $f(x_0) = g(x_0)$ and $f(x) \leq g(x)$ for all x in an open interval containing the point x_0. If both $f'(x_0)$ and $g'(x_0)$ exist show that they must be equal. How does this compare to the trapping principle used in Example 7.5, where it seems much more is assumed about the function f.

SEE NOTE 166

7.2.15 Suppose that f is a function defined on the real line with the property that $f(x + y) = f(x)f(y)$ for all x, y. Suppose that f is differentiable at 0 and that $f'(0) = 1$. Show that f must be differentiable everywhere and that $f'(x) = f(x)$.

SEE NOTE 167

7.2.2 Differentiability and Continuity

A continuous function need not be differentiable (Example 7.4) but the converse is true. Every differentiable function is continuous.

Theorem 7.6: *Let f be defined in a neighborhood I of x_0. If f is differentiable at x_0, then f is continuous at x_0.*

Proof. It suffices to show that $\lim_{x\to x_0}(f(x)-f(x_0))=0$. For $x\neq x_0$,

$$f(x)-f(x_0)=\left(\frac{f(x)-f(x_0)}{x-x_0}\right)(x-x_0).$$

Now

$$\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}=f'(x_0)$$

and $\lim_{x\to x_0}(x-x_0)=0$. We then obtain

$$\lim_{x\to x_0}(f(x)-f(x_0))=(f'(x_0))(0)=0$$

by the product rule for limits. ■

We can use this theorem in two ways. If we know that a function has a discontinuity at a point, then we know immediately that there is no derivative there. On the other hand, if we have been able to determine by some means that a function is differentiable at a point then we know automatically that the function must also be continuous at that point.

Exercises

7.2.16 Construct a function on the interval $[0,1]$ that is continuous and is not differentiable at each point of some infinite set.
See Note 168

7.2.17 Suppose that a function has both a right-hand and a left-hand derivative at a point. What, if anything, can you conclude about the continuity of that function at that point?

7.2.18 Suppose that a function has an infinite derivative at a point. What, if anything, can you conclude about the continuity of that function at that point?
See Note 169

7.2.19 Show that if a function f has a symmetric derivative $f'_s(x_0)$ (see Exercise 7.2.6), then f must be symmetrically continuous at x_0 in the sense that $\lim_{h\to 0}[f(x_0+h)-f(x_0-h)]=0$. Must f in fact be continuous?

See Note 170

7.2.20 If $f'(x_0)=\infty$, does it follow that f must be continuous at x_0 on one side at least?

7.2.21 Find an example of an everywhere differentiable function f so that f' is not everywhere continuous.

7.2.22 Show that a function f that satisfies an inequality of the form

$$|f(x)-f(y)| \le M\sqrt{|x-y|}$$

for some constant M and all x, y must be everywhere continuous but need not be everywhere differentiable.

7.2.23 The Dirichlet function (see Section 5.2.6) is discontinuous at each rational number. By Theorem 7.6 it follows that this function has no derivative at any rational number. Does it have a derivative at any irrational number?

7.2.3 The Derivative as a Magnification

✂ *Enrich.*

We offer now one more interpretation of the derivative, this time as a magnification factor. In elementary calculus one often makes use of the geometric content of the graph of a function f. In particular, we can view the derivative in terms of slopes of tangent lines to the graph. But the graph of f is a subset of two-dimensional space, while the range of f is a subset of one-dimensional space and, as such, has some additional geometric content.

Suppose f is differentiable on an interval I, and let J be a closed sub-interval of I. The range of f on J will also be a closed interval, because f is differentiable and hence continuous on J, and continuous functions map closed intervals onto closed intervals (Exercise 5.8.2). If we compare the length $|J|$ of the interval J to the length $|f(J)|$ of the interval $f(J)$ the expression

$$\frac{|f(J)|}{|J|}$$

represents the amount that the interval J has been expanded (or contracted) under the mapping f.

For example, if $f(x) = x^2$ and $J = [2, 3]$, then

$$\frac{|f(J)|}{|J|} = \frac{|[4,9]|}{|[2,3]|} = \frac{5}{1} = 5.$$

Thus the interval $[2, 3]$ has been expanded by f to an interval of 5 times its size. If we look only at small intervals then the derivative offers a clue to the size of the magnification factor.

If J is a sufficiently small interval having x_0 as an endpoint, then the ratio $|f(J)|/|J|$ is approximately $|f'(x_0)|$, the approximation becoming "exact in the limit." Thus $|f'(x_0)|$ can be viewed as a "magnification factor" of small intervals containing the point x_0. In our illustration with the function $f(x) = x^2$, the magnification factor at $x_0 = 2$ is $f'(2) = 4$. Small intervals about x_0 are magnified by a factor of about 4. At the other endpoint $x_0 = 3$, small intervals about x_0 are magnified by a factor of about 6.

In Exercise 7.2.26 we ask you to prove a precise statement covering the preceding discussion.

Exercises

7.2.24 What is the ratio

$$\frac{|f(J)|}{|J|}$$

for the function $f(x) = x^2$ if $J = [2, 2.001]$, $J = [2, 2.0001]$, $J = [2, 2.00001]$?

7.2.25 In this section we have interpreted $f'(x_0)$ as a magnification factor. If $f'(x_0) = 0$, does this mean that small intervals containing the point x_0 are magnified by a factor of 0 when mapped by f?

7.2.26 Let f be differentiable on an interval I and let x_0 be an interior point of I. Make precise the following statement and prove it:

$$\lim_{J \to x_0} \frac{|f(J)|}{|J|} = |f'(x_0)|.$$

7.3 Computations of Derivatives

Example 7.2 provides a calculation of the derivative of the function $f(x) = x^2$. The calculation involved direct evaluation of the limit of an appropriate difference quotient. For the function $f(x) = x^2$, this evaluation was straightforward. But limits of difference quotients can be quite complicated. You are familiar with certain rules that are useful in calculating derivatives of functions that are “built up” from functions whose derivatives are known.

In this section we review some of the calculus rules that are commonly used to compute derivatives. We need first to prove the algebraic rules: The sum rule, the product rule, and the quotient rule. Then we turn to the chain rule. Finally, we look at the power rule. Our viewpoint here is not to practice the computation of derivatives but to build up the theory of derivatives, making sure to see how it depends on work on limits that we proved earlier on.

The various rules we shall obtain in this section should be viewed as aids for computations of derivatives. An understanding of these rules is, of course, necessary for various calculations. But they in no way can substitute for an understanding of the derivative. And they might not be useful in calculating certain derivatives. (For example, derivatives of the functions of Exercise 7.2.3 cannot be calculated at $x_0 = 0$ by using these rules.)

Nonetheless, it is true that one often has a function that can be expressed in terms of several functions via the operations we considered in this section, functions whose derivatives we know. In those cases, the techniques of this section might be useful.

7.3.1 Algebraic Rules

Functions can be combined algebraically by multiplying by constants, by addition and subtraction, by multiplication, and by division. To each of these there is a calculus rule for computing the derivative. We recall that the limit of a sum (a difference, a product, a quotient) is the sum (difference, product, quotient) of the limits. Perhaps we might have thought the same kind of rule would apply to derivatives. The derivative of the sum is indeed the sum of the derivatives, but the derivative of the product is not the product of the derivatives. Nor do quotients work in such a simple way. The reasons for the form of the various rules can be found by writing out the definition of the derivative and following through on the computations.

Theorem 7.7: *Let f and g be defined on an interval I and let $x_0 \in I$. If f and g are differentiable at x_0 then $f+g$ and fg are differentiable at x_0. If $g(x_0) \neq 0$, then f/g is differentiable at x_0. Furthermore, the following formulas are valid:*

(i) $(cf)'(x_0) = cf'(x_0)$ *for any real number c.*

(ii) $(f+g)'(x_0) = f'(x_0) + g'(x_0)$.

(iii) $(fg)'(x_0) = f(x_0)g'(x_0) + g(x_0)f'(x_0)$.

(iv) $\left(\dfrac{f}{g}\right)'(x_0) = \dfrac{g(x_0)f'(x_0) - f(x_0)g'(x_0)}{(g(x_0))^2}$ *(if $g(x_0) \neq 0$).*

Proof. Parts (i) and (ii) follow easily from the definition of the derivative and appropriate limit theorems.

To verify part (iii), let $h = fg$. Then for each $x \in I$ we have

$$h(x) - h(x_0) = f(x)[g(x) - g(x_0)] + g(x_0)[f(x) - f(x_0)]$$

so

$$\frac{h(x) - h(x_0)}{x - x_0} = f(x)\frac{g(x) - g(x_0)}{x - x_0} + g(x_0)\frac{f(x) - f(x_0)}{x - x_0}. \tag{3}$$

As $x \to x_0$, $f(x) \to f(x_0)$ since f being differentiable is also continuous. By the definition of the derivative we also know that

$$\frac{g(x) - g(x_0)}{x - x_0} \to g'(x_0)$$

and

$$\frac{f(x) - f(x_0)}{x - x_0} \to f'(x_0)$$

as $x \to x_0$. We now see from equation (3) that

$$\lim_{x\to x_0} \frac{h(x)-h(x_0)}{x-x_0} = f(x_0)g'(x_0) + g(x_0)f'(x_0),$$

verifying part (iii).

Finally, to establish part (iv) of the theorem, let $h = f/g$. Straightforward algebraic manipulations show that

$$\frac{h(x)-h(x_0)}{x-x_0} =$$
$$\frac{1}{g(x)g(x_0)}\left[g(x_0)\left(\frac{f(x)-f(x_0)}{x-x_0}\right) - f(x)\left(\frac{g(x)-g(x_0)}{x-x_0}\right)\right]. \tag{4}$$

Now let $x \to x_0$. Since f and g are continuous at x_0, $f(x) \to f(x_0)$ and $g(x) \to g(x_0)$. Thus part (iv) of the theorem follows from equation (4), the definition of derivative, and basic limit theorems. ■

Example 7.8: To calculate the derivative of $h(x) = (x^3+1)^2$ we have several ways to proceed.

1. Apply the definition of derivative. You may wish to set up the difference quotient and see that a calculation of its limit is a formidable task.

2. Write $h(x) = x^6 + 2x^3 + 1$ and apply the formula $\frac{d}{dx}x^n = nx^{n-1}$ (Exercise 7.3.5) and the rule for sums. Thus we get
$$h'(x) = 6x^5 + 6x^2.$$

3. Use the product rule to obtain
$$h'(x) = (x^3+1)\frac{d}{dx}(x^3+1) + (x^3+1)\frac{d}{dx}(x^3+1).$$
Then, again, use the formula $\frac{d}{dx}x^n = nx^{n-1}$ and the rule for sums to continue:
$$h'(x) = (x^3+1)3x^2 + (x^3+1)3x^2 = 6x^5 + 6x^2.$$

◄

Exercises

7.3.1 Give the details needed in the proof of Theorem 7.7 for the sum rule for derivatives; that is, $(f+g)'(x_0) = f'(x_0) + g'(x_0)$.

7.3.2 The table shown in Figure fig–table2 gives the values of two functions f and g at certain points. Calculate $(f+g)'(1)$, $(fg)'(1)$ and $(f/g)'(1)$. What can you assert about $(f/g)'(3)$? Is there enough information to calculate $f''(3)$?

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	3	3	2	2
2	4	4	4	0
3	6	1	1	0
4	-1	0	1	1
5	2	5	3	3

Figure 7.3. Values of f and g at several points.

7.3.3 Obtain the rule

$$\frac{d}{dx}\frac{1}{f(x)} = -\frac{f'(x)}{f(x)^2}$$

from Theorem 7.7 and also directly from the definition of the derivative.

7.3.4 Obtain the rule for

$$\frac{d}{dx}\left(f(x)\right)^2 = 2f(x)f'(x)$$

from Theorem 7.7 and also directly from the definition of the derivative.

7.3.5 Obtain the formula

$$\frac{d}{dx}x^n = nx^{n-1}$$

for $n = 1, 2, 3, \ldots$ by induction.

See Note 171

7.3.6 State and prove a theorem that gives a formula for $f'(x_0)$ when

$$f = f_1 + f_2 + \cdots + f_n$$

and each of the functions $f_1, \dots, f_n$ is differentiable at x_0.

7.3.7 State and prove a theorem that gives a formula for $f'(x_0)$ when

$$f = f_1 f_2 \dots f_n$$

and each of the functions $f_1, \dots, f_n$ is differentiable at x_0.

7.3.8 Show that

$$(fg)''(x_0) = f''(x_0)g(x_0) + 2f'(x_0)g'(x_0) + f(x_0)g''(x_0)$$

under appropriate hypotheses.

7.3.9 Extend Exercise 7.3.8 by obtaining a similar formula for $(fg)'''(x_0)$.

7.3.10 Obtain a formula for $(fg)^{(n)}(x_0)$ valid for $n = 1, 2, 3, \dots$.

See Note 172

7.3.2 The Chain Rule

There is another, nonalgebraic, interpretation of Example 7.8 that you may recall from calculus courses.

Example 7.9: We can view the function $h(x) = (x^3+1)^2$ as a *composition* of the function $f(x) = x^3 + 1$ and $g(u) = u^2$. Thus

$$h(x) = g \circ f(x).$$

You are familiar with the *chain rule* that is useful in calculating derivatives of composite functions. In this case the calculation would lead to

$$\begin{aligned} h'(x) &= g'(f(x))f'(x) = g'(x^3+1)3x^2 \\ &= 2(x^3+1)3x^2 = 6x^5 + 6x^2. \end{aligned}$$

In elementary calculus you might have preferred to obtain

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 2(x^3+1)(3x^2) = 6x^5 + 6x^2$$

by making the substitution $u = x^3 + 1, y = u^2$. ◀

The chain rule is the familiar calculus formula

$$\frac{d}{dx}g(f(x)) = g'(f(x))f'(x)$$

for the differentiation of the composition of two functions $g \circ f$ under appropriate assumptions. Calculus students often memorize this in the form

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

by using the new variables $y = g(u)$ and $u = f(x)$.

Let us first try to see why the chain rule should work. Then we'll provide a precise statement and proof of the chain rule. Perhaps the easiest way to "see" the chain rule is by interpreting the derivative as a magnification factor.

Let f be defined in a neighborhood of x_0 and let g be defined in a neighborhood of $f(x_0)$. If f is differentiable at x_0, then f maps each small interval J containing x_0 onto an interval $f(J)$ containing $f(x_0)$ with $|f(J)|/|J|$ approximately $|f'(x_0)|$. If, also, g is differentiable at $f(x_0)$, then g will map a small interval $f(J)$ containing $f(x_0)$ onto an interval $g(f(J))$ with $|g(f(J))|/|f(J)|$ approximately $|g'(f(x_0))|$. Thus $h = g \circ f$ maps J onto the interval $h(J) = g(f(J))$ and

$$\frac{|h(J)|}{|J|} = \frac{|g(f(J))|}{|f(J)|}\frac{|f(J)|}{|J|}$$

and this is approximately equal to

$$|g'(f(x_0))||f'(x_0)|.$$

In short, the magnification factors $|f'(x_0)|$ and $|g'(f(x_0))|$ multiply to give the magnification factor $|h'(x_0)|$.

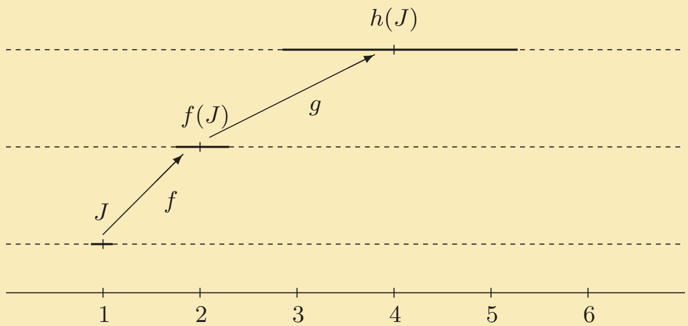

Figure 7.4. f maps J to $f(J)$ and g maps that to $h(J)$. Here $h = g \circ f$, $x_0 = 1$, and $J = [.9, 1.1]$.

Example 7.10: Let us relate this discussion to our example $h(x) = (x^3 + 1)^2$. Here $f(x) = x^3 + 1$, $g(x) = x^2$. At $x_0 = 1$ we obtain $f(x_0) = 2$, $f'(x_0) = 3$, $g(f(x_0)) = 4$ and $g'(f(x_0)) = 4$. The function f maps small intervals about $x_0 = 1$ onto ones about three times as long, and in turn, the function g maps those intervals onto ones about four times as long, so the total magnification factor for the function $h = g \circ f$ is about 12 at $x_0 = 1$ (Fig. 7.4). ◀

Proof of the Chain Rule If we wished to formulate a proof of the chain rule based on the preceding discussion we could begin by writing

$$\frac{g(f(x)) - g(f(x_0))}{x - x_0} = \left(\frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)}\right)\left(\frac{f(x) - f(x_0)}{x - x_0}\right) \tag{5}$$

which compares to our formula

$$\frac{|h(J)|}{|J|} = \frac{|g(f(J))|}{|f(J)|}\frac{|f(J)|}{|J|}.$$

If we let $x \to x_0$ in (5), we would expect to get the desired result

$$(g \circ f)'(x_0) = g'(f(x_0))f'(x_0).$$

And this argument would be valid if f were, for example, increasing. But in order for equation (5) to be valid, we must have $x \neq x_0$ and $f(x) \neq f(x_0)$. When computing the limit of a difference quotient, we can assume $x \neq x_0$, but we can't assume, without additional hypotheses, that if $x \neq x_0$ then $f(x) \neq f(x_0)$. Yet the chain rule applies nonetheless.

The proof is clearer if we separate these two cases. In the simpler case the function does not repeat the value $f(x_0)$ in some neighborhood of x_0. In the harder case the function repeats the value $f(x_0)$ in *every* neighborhood of x_0. Exercise 7.3.11 shows that in that case we must have $f'(x_0) = 0$ and so the chain rule reduces to showing that the composite function $g \circ f$ also has a zero derivative.

Theorem 7.11 (Chain Rule) *Let f be defined on a neighborhood U of x_0 and let g be defined on a neighborhood V of $f(x_0)$ for which*

$$f(x_0) \in f(U) \subset V.$$

Suppose f is differentiable at x_0 and g is differentiable at $f(x_0)$. Then the composite function $h = g \circ f$ is differentiable at x_0 and

$$h'(x_0) = (g \circ f)'(x_0) = g'(f(x_0))f'(x_0).$$

Proof. Consider any sequence of distinct points x_n different from x_0 and converging to x_0. If we can show that the sequence

$$S_n = \frac{g(f(x_n)) - g(f(x_0))}{x_n - x_0}$$

converges to $g'(f(x_0))f'(x_0)$ for every such sequence then we have obtained our required formula.

Note that if $f(x_n) \neq f(x_0)$, then we can write $y_n = f(x_n)$, $y_0 = f(x_0)$ and display S_n as

$$S_n = \left(\frac{g(y_n) - g(y_0)}{y_n - y_0}\right)\left(\frac{f(x_n) - f(x_0)}{x_n - x_0}\right). \tag{6}$$

Seen in this form it becomes obvious that

$$S_n \to g'(y_0)f'(x_0) = g'(f(x_0))f'(x_0)$$

except for the problem that we cannot (as we remarked before beginning our proof) assume that in all cases $f(x_n) \neq f(x_0)$.

Thus we consider two cases. In the first case we assume that for any sequence of distinct points x_n converging to x_0 there cannot be infinitely many terms with $f(x_n) = f(x_0)$. In that case the chain rule formula is evidently valid.

In the second case we assume that there does exist a sequence of distinct points x_n converging to x_0 with $f(x_n) = f(x_0)$ for infinitely many terms. In that case (Exercise 7.3.11) we must have $f'(x_0) = 0$ and so, to establish the chain rule, we need to prove that $h'(x_0) = 0$. But in this case for any sequence x_n converging to x_0 either $S_n = 0$ [when $f(x_n) = f(x_0)$] or else S_n can be written in the form of equation (6) [when $f(x_n) \neq f(x_0)$]. It is then clear that $S_n \to 0$ and the proof is complete. ■

Exercises

7.3.11 Show that if for each neighborhood U of x_0 there exists $x \in U$, $x \neq x_0$ for which $f(x) = f(x_0)$, then either $f'(x_0)$ does not exist or else $f'(x_0) = 0$.

SEE NOTE 173

7.3.12 Give an explicit example of functions f and g such that the "proof" of the chain rule based on equation (5) fails.

SEE NOTE 174

7.3.13 The heuristic discussion preceding Theorem 7.11 dealt with $|h'(x_0)|$, not with $h'(x_0)$. Explain how the signs of $f'(x_0)$ and $g'(f(x_0))$ affect the discussion. In particular, how can we modify the discussion to get the correct sign for $h'(x_0)$?

7.3.14 Most calculus texts use a proof of Theorem 7.11 based on the following ideas. Define a function G in the neighborhood V of $f(x_0)$ by

$$G(v) = \begin{cases} [g(v) - g(f(x_0))]/[v - f(x_0)], & \text{if } v \neq f(x_0) \\ g'(f(x_0)), & \text{if } v = f(x_0). \end{cases} \tag{7}$$

(a) Show that G is continuous at $f(x_0)$.

(b) Show that $G(v)(v - f(x_0)) = g(v) - g(f(x_0))$ for every $v \in V$, regardless of whether or not $f(x_0) = v$.

(c) Prove that $\lim_{x \to x_0} \frac{h(x)-h(x_0)}{x-x_0} = g'(f(x_0))f'(x_0)$.

7.3.15 State and prove a theorem that gives a formula for $f'(x_0)$ when

$$f = f_n \circ f_{n-1} \circ \cdots \circ f_2 \circ f_1.$$

(Be sure to state all the hypotheses that you need.)

7.3.16 The table in Figure 7.3.16 gives the values of two functions f and g at certain points. Calculate $(f \circ g)'(1)$ and $(g \circ f)'(1)$. Is there enough information to calculate $(f \circ g)'(3)$ and/or $(g \circ f)'(3)$? How about $\dfrac{d}{dx}(f^2)(1)$ and $(f \circ f)'(1)$?

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	3	3	2	2
2	4	4	4	0
3	6	1	1	0
4	-1	0	1	1
5	2	5	3	3

Figure 7.5. Values of f and g at several points.

7.3.3 Inverse Functions

Suppose that a function $f : I \to J$ has an inverse. This simply means that there is a function g (called the *inverse* of f) that reverses the mapping: If $f(a) = b$ then $g(b) = a$. We can assume that I and J are intervals. Thus f maps the interval I onto the interval J and the inverse function g then maps J back to I. Not all functions have an inverse, but we are supposing that this one does.

Suppose too that f is differentiable at a point $x_0 \in I$. Then we would expect from geometric considerations that that the inverse function g should be differentiable at the image point $z_0 = f(x_0) \in J$.

This is entirely elementary. The connection between a function f and its inverse g is given by

$$f(g(x)) = x \text{ for all } x \in J$$

or

$$g(f(x)) = x \text{ for all } x \in I.$$

Using the chain rule on the second of these immediately gives

$$g'(f(x))f'(x) = 1$$

and hence we have the connection

$$g'(f(x)) = \frac{1}{f'(x)},$$

which a geometrical argument could also have found.

Example 7.12: Suppose that the exponential function e^x has been developed and that we have proved that it is differentiable for all values of x and we have the usual formula $\frac{d}{dx}e^x = e^x$. Then, provided we can be sure there is an inverse, a formula for the derivative of that inverse can be found. Let $L(x)$ be the inverse function of $f(x) = e^x$. Then, since we know that $f'(x) = f(x)$

$$L'(f(x)) = \frac{1}{f'(x)} = \frac{1}{f(x)}$$

or, replacing $f(x)$ by another letter, say z, we have

$$L'(z) = \frac{1}{z}.$$

This must be valid for every value z in the domain of L, that is, for every value in the range of f. You should recognize the derivative of the function $\ln z$ here. Even so, we would still need to justify the existence of the inverse function before we could properly claim to have proved this formula. ◀

We would like a better way to handle inverse functions than presented here. Our observations here allow us to compute the derivative of an inverse but do not assure us that an inverse will exist. For a theorem that allows us merely to look at the derivative and determine that an inverse exists and has a derivative, see Theorem 7.32.

Exercises

7.3.17 Find a formula for the derivative of the function $\sin^{-1} x$ assuming that the usual formula for

$$\frac{d}{dx}\sin x = \cos x$$

has been found.

SEE NOTE 175

7.3.18 Find a formula for the derivative of the function $\tan^{-1} x$ assuming that the usual formula for $\frac{d}{dx}\tan x = \sec^2 x$ has been found.

7.3.19 Give a geometric interpretation of the relationship between the slope of the tangent at a point (x_0, y_0) on the graph of $y = f(x)$ and the slope of the tangent at the point (y_0, x_0) on the graph of $y = g(x)$ where g is the inverse of f.

SEE NOTE 176

7.3.20 What facts about the function $f(x) = e^x$ would need to be established in order to claim that there is indeed an inverse function? What is the domain and range of that inverse function?

7.3.4 The Power Rule

The power rule is the formula

$$\frac{d}{dx}x^p = px^{p-1}$$

which is the basis for many calculus problems. We have already shown (in Exercise 7.3.5) that

$$\frac{d}{dx}x^n = nx^{n-1}$$

for $n = 1, 2, 3, \ldots$ and for every value of x.

This is easy enough to extend to negative integers. Just interpret for $n = 1, 2, 3, \ldots$ and for every value of $x \neq 0$,

$$\frac{d}{dx}x^{-n} = \frac{d}{dx}\frac{1}{x^n}$$

and, using the quotient rule, we find that again the power rule formula is valid for $p = -1, -2, -3, \ldots$ and any value of x other than 0.

The formula also works for $p = 0$ since we interpret x^0 as the constant 1 (although for $x = 0$ we prefer not to make any claims). Is the formula indeed valid for every value of p, not just for integer values?

Example 7.13: We can verify the power rule formula for $p = 1/2$; that is, we prove that

$$\frac{d}{dx}\sqrt{x} = \frac{d}{dx}x^{1/2} = \frac{1}{2}x^{1/2-1} = \frac{1}{2\sqrt{x}}.$$

First we must insist that $x > 0$ otherwise $\sqrt{x}$ and the fraction in our formula would not be defined. Now interpret $\sqrt{x}$ as the inverse of the square function $f(x) = x^2$. Specifically let $f(x) = x^2$ for $x > 0$ and $g(x) = \sqrt{x}$ for $x > 0$ and note that $f(g(x)) = g(f(x)) = x$. Thus

$$\frac{d}{dx}f(g(x)) = \frac{d}{dx}x = 1$$

and so, since $f'(x) = 2x$ and $f'(g(x))g'(x) = 1$ we obtain $2\sqrt{x}g'(x) = 1$ and finally that

$$g'(x) = \frac{1}{2\sqrt{x}}$$

as required if the power rule formula is valid. ◀

Is the power rule

$$\frac{d}{dx}x^p = px^{p-1}$$

valid for all rational values of p? We can handle the case $p = m/n$ for integer m and n by essentially the same methods. We state this as a theorem whose proof is left as an exercise. For irrational p there is also a discussion in the exercises.

Theorem 7.14: *Let $f(x) = x^{\frac{m}{n}}$ for $x > 0$ and integers m, n. Then*

$$f'(x) = \frac{m}{n} x^{\frac{m}{n}-1}.$$

Example 7.15: Every polynomial is differentiable on $\mathbb{R}$ and its derivative can be calculated via term by term differentiation; that is,

$$\frac{d}{dx}(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n) = a_1 + 2a_2 x + \cdots + n a_n x^{n-1}.$$

This follows from the power rule formula and the rule for sums. Note that the derivative of a polynomial is again a polynomial. ◀

Example 7.16: A rational function is a function $R(x)$ that can be expressed as the quotient of two polynomials,

$$R(x) = \frac{p(x)}{q(x)}.$$

This would be defined at every point at which the denominator $q(x)$ is not equal to zero. Every rational function is differentiable except at those points at which the denominator vanishes. This follows from the previous example, which showed how to differentiate a polynomial, and from the quotient rule. Thus

$$\frac{d}{dx}\left(\frac{p(x)}{q(x)}\right) = \frac{p'(x)q(x) - p(x)q'(x)}{q^2(x)}.$$

Notice that the derivative is another rational function with the same domain since both numerator and denominator are again polynomials. ◀

Exercises

7.3.21 Prove Theorem 7.14.

SEE NOTE 177

7.3.22 Show that the power formula is available for all values of p once the formula $\frac{d}{dx}e^x = e^x$ is known.

SEE NOTE 178

7.3.23 Let

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n.$$

Compute the sequence of values $p(0)$, $p'(0)$, $p''(0)$, $p'''(0)$, $\ldots$.

SEE NOTE 179

7.3.24 Determine the coefficients of the polynomial

$$p(x) = (1+x)^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

by using the formulas that you obtained in Exercise 7.3.23.

SEE NOTE 180

7.4 Continuity of the Derivative?

We have already observed (Theorem 7.6) that if a function f is differentiable on an interval I, then f is also continuous on I. This statement should not be confused with the (incorrect) statement that the derivative, f', is continuous.

Example 7.17: Consider the function f defined on $\mathbb{R}$ by

$$f(x) = \begin{cases} x^2 \sin x^{-1}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

Since $|\sin x^{-1}| \leq 1$ for all $x \neq 0$, $|f(x)| \leq x^2$ for all $x \in \mathbb{R}$. We can now conclude (e.g., from Example 7.5) that $f'(0) = 0$. For $x \neq 0$, we can calculate, as in elementary calculus, that

$$f'(x) = -\cos x^{-1} + 2x \sin x^{-1}.$$

This function f' is continuous at every point $x_0 \neq 0$. At $x_0 = 0$ it is discontinuous. To see this we need only consider an appropriate sequence $x_n \to 0$ and see what happens to $f'(x_n)$. For example, try the sequence

$$x_n = \frac{1}{\pi n}.$$

Since

$$\cos\left(\frac{1}{x_n}\right) = \cos(\pi n)$$

and these numbers are alternately $+1$ and -1 it is clear that $f'(x_n)$ cannot converge. Consequently, f' is discontinuous at 0. ◀

Observe that the function f provides an example of a function that is differentiable on all of $\mathbb{R}$, yet f' is discontinuous at a point. It is possible to modify this function to obtain a differentiable function g whose derivative g' is discontinuous at infinitely many points, and even at all the points of the Cantor set (see Exercise 7.4.2).

You might wonder, then, if anything positive could be said about the properties of a derivative f'. It is possible for the derivative of a differentiable function to be discontinuous on a dense set[1]: An example is given later in Section 14.8. We will also show, in Section 7.9, that the function f', while perhaps discontinuous, nonetheless shares one significant property of continuous functions: It has the intermediate value property (Darboux property).

Exercises

7.4.1 Give a simple example of a function f differentiable in a deleted neighborhood of x_0 such that $\lim_{x\to x_0} f'(x)$ does not exist.

7.4.2 ✂ Let P be a Cantor subset of $[0,1]$ (i.e., P is a nonempty, nowhere dense perfect subset of $[0,1]$) and let $\{(a_n, b_n)\}$ be the sequence of intervals complementary to P in $(0,1)$. (See Section 6.5.1.)

[1] It is not possible for a derivative to be discontinuous at every point. See Corollary 14.41.

(a) On each interval $[a_n, b_n]$ construct a differentiable function such that

$$f_n(a_n) = f_n(b_n) = (f_n')_+(a_n) = (f_n')_-(b_n) = 0,$$

$$\limsup_{x \to a_n^+} f'(x) = \limsup_{x \to b_n^-} f_n'(x) = 1,$$

$$\liminf_{x \to a_n^+} f'(x) = \liminf_{x \to b_n^-} f_n'(x) = -1,$$

and $|f_n(x)| \le (x - a_n)^2 (x - b_n)^2$ and $|f_n'(x)|$ is bounded by 1 in each interval $[a_n, b_n]$.

(b) Let g be defined on $[0, 1]$ by

$$g(x) = \begin{cases} f_n(x), & \text{if } x \in (a_n, b_n), n = 1, 2, \dots \\ 0, & \text{if } x \in P. \end{cases}$$

Sketch a picture of the graph of g.

(c) Prove that g is differentiable on $[0, 1]$.

(d) Prove that $g'(x) = 0$ for each $x \in P$.

(e) Prove that g' is discontinuous at every point of P.

7.5 Local Extrema

We have seen in Section 5.7 that a continuous function defined on a closed interval $[a, b]$ achieves an absolute maximum value and an absolute minimum value on the interval. These are called *extreme values* or *extrema*. There must be points where the maximum and minimum are attained, but how do we go about finding such points? One way is to find all points that may not be themselves extrema, but are *local* extreme points. A function defined on an interval I is said to have a *local maximum* at x_0 in the interior of I, if there exists $\delta > 0$ such that $[x_0 - \delta, x_0 + \delta] \subset I$ and $f(x) \le f(x_0)$ for all x in the smaller interval. A local minimum is similarly defined.

A familiar process studied in elementary calculus is sometimes useful for locating these extrema when the function is differentiable on (a, b): We look for critical points (i.e., points where the derivative is zero). We begin with the theorem that forms the basis for this process.

Theorem 7.18: *Let f be defined on an interval I. If f has a local extremum at a point x_0 in the interior of I and f is differentiable at x_0, then $f'(x_0) = 0$.*

Proof. Suppose f has a local maximum at x_0 in the interior of I, the proof for a local minimum being similar. Then there exists $\delta > 0$ such that

$$[x_0 - \delta, x_0 + \delta] \subset I \quad \text{and} \quad f(x) \leq f(x_0)$$

for all $x \in [x_0 - \delta, x_0 + \delta]$. Thus

$$\frac{f(x) - f(x_0)}{x - x_0} \leq 0 \quad \text{for} \quad x \in (x_0, x_0 + \delta) \tag{8}$$

and

$$\frac{f(x) - f(x_0)}{x - x_0} \geq 0 \quad \text{for} \quad x \in (x_0 - \delta, x_0). \tag{9}$$

If $f'(x_0)$ exists, then

$$f'(x_0) = \lim_{x \to x_0+} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0-} \frac{f(x) - f(x_0)}{x - x_0}. \tag{10}$$

By (8), the first of these limits is at most zero; by (9), the second is at least zero. By (10), these limits are equal and are therefore equal to zero. ■

It follows from Theorem 7.18 that a function f that is continuous on $[a, b]$ must achieve its maximum at one (or more) of these types of points:

1. Points $x_0 \in (a, b)$ at which $f'(x_0) = 0$

2. Points $x_0 \in (a, b)$ at which f is not differentiable

3. The points a or b

We leave it to you to provide simple examples of each of these possibilities.

The usual process for locating extrema in elementary calculus thus involves locating points at which f has a zero derivative and comparing the values of f at those points and the points of nondifferentiability (if any) and at the endpoints a and b. In the setting of elementary calculus the situation is usually relatively simple: The function is differentiable, the set on which $f'(x) = 0$ is finite (or contains an interval), and the equation $f'(x) = 0$ is easily solved. Much more complicated situations can occur, of course. The following exercises provide some examples and theorems that indicate just how complicated the set of extrema can be.

Exercises

7.5.1 Give an example of a differentiable function on $\mathbb{R}$ for which $f'(0) = 0$ but 0 is not a local maximum or minimum of f.

7.5.2 Let

$$f(x) = \begin{cases} x^4(2 + \sin x^{-1}), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

(a) Prove that f is differentiable on $\mathbb{R}$.

(b) Prove that f has an absolute minimum at $x = 0$.

(c) Prove that f' takes on both positive and negative values in every neighborhood of 0.

7.5.3 ✂ Let K be the Cantor set and let $\{(a_k, b_k)\}$ be the sequence of intervals complementary to K in $[0, 1]$. For each k, let $c_k = (a_k + b_k)/2$. Define f on $[0, 1]$ to be zero on K, $1/k$ at c_k, linear and continuous on each of the intervals. (See Figure 7.6.)

(a) Write equations that represent f on the intervals $[a_k, c_k]$ and $[c_k, b_k]$.

(b) Show that f is continuous on $[0, 1]$.

(c) Verify that f has minimum zero, achieved at each $x \in K$.

(d) Verify that f has a local maximum at each of the points c_k.

(e) Modify f to a differentiable function with the same set of extrema.

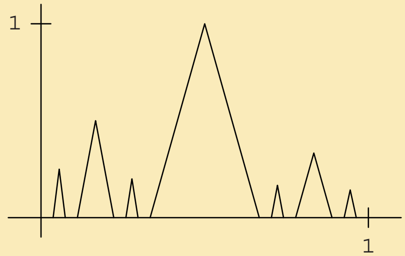

Figure 7.6. Part of the graph of the function in Exercise 7.5.3.

7.5.4 Find all local extrema of the Dirichlet function (see Section 5.2.6) defined on $[0,1]$ by

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is irrational or } x = 0 \\ 1/q, & \text{if } x = p/q,\ \ p,q \in \mathbb{N},\ \ p/q \text{ in lowest terms.} \end{cases}$$

7.5.5 Show that the functions in Exercises 7.5.3 and 7.5.4 have infinitely many maxima, all of them strict. Show that the sets of points at which these functions have a strict maximum is countable.

7.5.6 Prove that if $f:\mathbb{R}\to\mathbb{R}$, then $\{x : f \text{ achieves a strict maximum at } x\}$ is countable.

See Note 181

7.5.7 Let $f:\mathbb{R}\to\mathbb{R}$ have the following property: For each $x \in \mathbb{R}$, f achieves a local maximum (not necessarily strict) at x.

(a) Give an example of such an f whose range is infinite.

(b) Prove that for every such f, the range is countable.

See Note 182

7.5.8 There are continuous functions $f:\mathbb{R}\to\mathbb{R}$, even differentiable functions, that are nowhere monotonic. This means that there is *no interval* on which the function is increasing, decreasing, or constant. For such functions, the set of maxima as well as the set of minima is dense in $\mathbb{R}$. Construction of such functions is given later in Section **??**. Show that such a function f maps its set of extrema onto a dense subset of the range of f.

7.6 Mean Value Theorem

There is a close connection between the values of a function and the values of its derivative. In one direction this is trivial since the derivative is defined in terms of the values of the function. The other direction is more subtle. How does information about the derivative provide us with information about the function? One of the keys to providing that information is the mean value theorem.

Suppose f is continuous on an interval $[a, b]$ and is differentiable on (a, b). Consider a point x in (a, b). For $y \in (a, b)$, $y \neq x$, the difference quotient

$$\frac{f(y) - f(x)}{y - x}$$

represents the slope of the chord determined by the points $(x, f(x))$ and $(y, f(y))$. This slope may or may not be a good approximation to $f'(x)$. If y is sufficiently near x, the approximation will be good; otherwise it may not be. The mean value theorem asserts that somewhere in the interval determined by x and y there will be a point at which the derivative is exactly the slope of the given chord. It is the existence of such a point that provides a connection between the values of the function [in this case the value $(f(y) - f(x))/(y - x)$] and the value of the derivative (in this case the value at *some* point between x and y).

7.6.1 Rolle's Theorem

We begin with a preliminary theorem that provides a special case of the mean value theorem. This derives its name from Michel Rolle (1652–1719) who has little claim to fame other than this. Indeed Rolle's name was only attached to this theorem because he had published it in a book in 1691; the method itself he did not discover. Perhaps his greatest real contribution is the invention of the notation $\sqrt[n]{x}$ for the nth root of x.

Theorem 7.19 (Rolle's Theorem) *Let f be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$ then there exists $c \in (a, b)$ such that $f'(c) = 0$.*

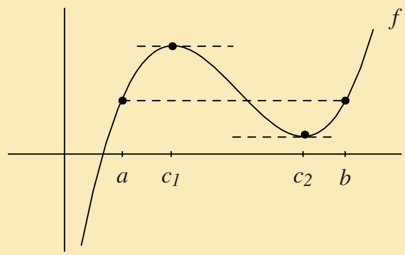

Figure 7.7. Rolle's theorem [note that $f(a) = f(b)$].

Proof. If f is constant on $[a, b]$, then $f'(x) = 0$ for all $x \in (a, b)$, so c can be taken to be any point of (a, b).

Suppose then that f is not constant. Because f is continuous on the compact interval $[a, b]$, f achieves a maximum value M and a minimum value m on $[a, b]$ (Theorem 5.50). Because f is not constant, one of the values M or m is different from $f(a)$ and $f(b)$, say $M > f(a)$. Choose $c \in (a, b)$ such that $f(c) = M$. Since $M > f(a) = f(b)$, $c \neq a$ and $c \neq b$, so $c \in (a, b)$. By Theorem 7.18, $f'(c) = 0$. ■

Observe that Rolle's theorem asserts that under our hypotheses, there is a point at which the tangent to the graph of the function is horizontal, and therefore has the same slope as the chord determined by the points $(a, f(a))$ and $(b, f(b))$. (See Figure 7.7.)

There may, of course, be many such points; Rolle's theorem just guarantees the existence of at least one such point. Observe also that we did not require that f be differentiable at the endpoints a and b. The theorem applies to such functions as $f(x) = x \sin x^{-1}$, $f(0) = 0$, on the interval $[0, 1/\pi]$. This function is not differentiable at zero, but it does have an infinite number of points between 0 and $1/\pi$ where the derivative is zero.

Exercises

7.6.1 Apply Rolle's theorem to the function $f(x) = \sqrt{1-x^2}$ on $[-1, 1]$. Observe that f fails to be differentiable at the endpoints of the interval.

7.6.2 Use Rolle's theorem to explain why the cubic equation

$$x^3 + \alpha x^2 + \beta = 0$$

cannot have more than one solution whenever $\alpha > 0$.

7.6.3 If the nth-degree equation

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = 0$$

has n distinct real roots, then how many distinct real roots does the $(n-1)$st degree equation $p'(x) = 0$ have?

See Note 183

7.6.4 Suppose that $f'(x) > c > 0$ for all $x \in [0, \infty)$. Show that $\lim_{x\to\infty} f(x) = \infty$.

7.6.5 Suppose that $f : \mathbb{R} \to \mathbb{R}$ and both f' and f'' exist everywhere. Show that if f has three zeros, then there must be some point ξ so that $f''(\xi) = 0$.

See Note 184

7.6.6 Let f be continuous on an interval $[a, b]$ and differentiable on (a, b) with a derivative that never is zero. Show that f maps $[a, b]$ one-to-one onto some other interval.

See Note 185

7.6.7 Let f be continuous on an interval $[a, b]$ and twice differentiable on (a, b) with a second derivative that never is zero. Show that f maps $[a, b]$ two-one onto some other interval; that is, there are at most two points in $[a, b]$ mapping into any one value in the range of f.

See Note 186

7.6.2 Mean Value Theorem

If we drop the requirement in Rolle's theorem that $f(a) = f(b)$, we now obtain the result that there is a $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

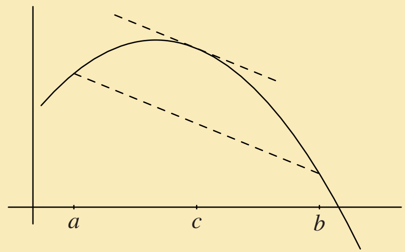

Figure 7.8. Mean value theorem [$f'(c)$ is slope of the chord].

Geometrically, this states that there exists a point $c \in (a,b)$ for which the tangent to the graph of the function at $(c, f(c))$ is parallel to the chord determined by the points $(a, f(a))$ and $(b, f(b))$. (See Figure 7.8.)

This is the mean value theorem, also known as the law of the mean or the first mean value theorem (because there are other mean value theorems).

Theorem 7.20 (Mean Value Theorem) *Suppose that f is a continuous function on the closed interval $[a,b]$ and differentiable on (a,b). Then there exists $c \in (a,b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof. We prove this theorem by subtracting from f a function whose graph is the straight line determined by the chord in question and then applying Rolle's theorem. Let

$$L(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a).$$

We see that $L(a) = f(a)$ and $L(b) = f(b)$. Now let

$$g(x) = f(x) - L(x). \tag{11}$$

Then g is continuous on $[a, b]$, differentiable on (a, b), and satisfies the condition $g(a) = g(b) = 0$.

By Rolle's theorem, there exists $c \in (a, b)$ such that $g'(c) = 0$. Differentiating (11), we see that $f'(c) = L'(c)$. But

$$L'(c) = \frac{f(b) - f(a)}{b - a},$$

so

$$f'(c) = \frac{f(b) - f(a)}{b - a},$$

as was to be proved. ■

Rolle's theorem and the mean value theorem were easy to prove. The proofs relied on the geometric content of the theorems. We suggest that you take the time to understand the geometric interpretation of these theorems.

Exercises

7.6.8 A function f is said to satisfy a *Lipschitz condition* on an interval $[a, b]$ if

$$|f(x) - f(y)| \leq M|x - y|$$

for all x, y in the interval. Show that if f is assumed to be continuous on $[a, b]$ and differentiable on (a, b) then this condition is equivalent to the derivative f' being bounded on (a, b).

See Note 187

7.6.9 Suppose f satisfies the hypotheses of the mean value theorem on $[a, b]$. Let S be the set of all slopes of chords determined by pairs of points on the graph of f and let

$$D = \{f'(x) : x \in (a, b)\}.$$

(a) Prove that $S \subset D$.

(b) Give an example to show that D can contain numbers not in S.

See Note 188

7.6.10 Interpreting the slope of a chord as an average rate of change and the derivative as an instantaneous rate of change, what does the mean value theorem say? If a car travels 100 miles in 2 hours, and the position $s(t)$ of the car at time t satisfies the hypotheses of the mean value theorem, can we be sure that there is at least one instant at which the velocity is 50 mph?

7.6.11 Give an example to show that the conclusion of the mean value theorem can fail if we drop the requirement that f be differentiable at every point in (a, b). Give an example to show that the conclusion can fail if we drop the requirement of continuity at the endpoints of the interval.

7.6.12 Suppose that f is differentiable on $[0, \infty)$ and that

$$\lim_{x\to\infty} f'(x) = C.$$

Determine

$$\lim_{x\to\infty} [f(x+a) - f(x)].$$

See Note 189

7.6.13 Suppose that f is continuous on $[a, b]$ and differentiable on (a, b). If

$$\lim_{x\to a+} f'(x) = C$$

what can you conclude about the right-hand derivative of f at a?

See Note 190

7.6.14 Suppose that f is continuous and that

$$\lim_{x\to x_0} f'(x)$$

exists. What can you conclude about the differentiability of f? What can you conclude about the continuity of f'?

See Note 191

7.6.15 Let $f : [0,\infty) \to \mathbb{R}$ so that f' is decreasing and positive. Show that the series

$$\sum_{i=1}^{\infty} f'(i)$$

is convergent if and only if f is bounded.

See Note 192

7.6.16 Prove a second-order version of the mean value theorem.

Let f be continuous on $[a,b]$ and twice differentiable on (a,b). Then there exists $c \in (a,b)$ such that

$$f(b) = f(a) + (b-a)f'(a) + (b-a)^2 \frac{f''(c)}{2!}.$$

See Note 193

7.6.17 Determine all functions $f : \mathbb{R} \to \mathbb{R}$ that have the property that

$$f'\left(\frac{x+y}{2}\right) = \frac{f(x) - f(y)}{x-y}$$

for every $x \neq y$.

7.6.18 A function is said to be *smooth* at a point x if

$$\lim_{h\to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = 0.$$

Show that a smooth function need not be continuous. Show that if f'' is continuous at x, then f is smooth at x.

See Note 194

7.6.3 Cauchy's Mean Value Theorem

Enrich.

We can generalize the mean value theorem to curves given parametrically. Suppose f and g are continuous on $[a,b]$ and differentiable on (a,b). Consider the curve given parametrically by

$$x = g(t)\,, \qquad y = f(t) \qquad (t \in [a,b]).$$

As t varies over the interval $[a, b]$, the point (x, y) traces out a curve C joining the points $(g(a), f(a))$ and $(g(b), f(b))$. If $g(a) \neq g(b)$, the slope of the chord determined by these points is

$$\frac{f(b) - f(a)}{g(b) - g(a)}.$$

Cauchy's form of the mean value theorem asserts that there is a point (x, y) on C at which the tangent is parallel to the chord in question. We state and prove this theorem.

Theorem 7.21 (Cauchy Mean Value Theorem) *Let f and g be continuous on $[a, b]$ and differentiable on (a, b). Then there exists $c \in (a, b)$ such that*

$$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c). \tag{12}$$

Proof. Let

$$\phi(x) = [f(b) - f(a)]g(x) - [g(b) - g(a)]f(x).$$

Then ϕ is continuous on $[a, b]$ and differentiable on (a, b). Furthermore,

$$\phi(a) = f(b)g(a) - f(a)g(b) = \phi(b).$$

By Rolle's theorem, there exists $c \in (a, b)$ for which $\phi'(c) = 0$. It is clear that this point c satisfies (12). ■

Exercises

7.6.19 Use Cauchy's mean value theorem to prove any simple version of L'Hôpital's rule that you can remember from calculus.

7.6.20 Show that the conclusion of Cauchy's mean value can be put into determinant form as

$$\begin{vmatrix} f(a) & g(a) & 1 \\ f(b) & g(b) & 1 \\ f'(c) & g'(c) & 0 \end{vmatrix} = 0.$$

7.6.21 Formulate and prove a generalized version of Cauchy's mean value whose conclusion is the existence of a point c such that

$$\begin{vmatrix} f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \\ f'(c) & g'(c) & h'(c) \end{vmatrix} = 0.$$

SEE NOTE 195

7.7 Monotonicity

In elementary calculus one learns that if $f' \geq 0$ on an interval I, then f is nondecreasing on I. We use this and related results for a variety of purposes: sketching graphs of functions, locating extrema, etc. In this section we take a closer look at what's involved. We recall some definitions.

Definition 7.22: Let f be real valued on an interval I.

1. If $f(x_1) \leq f(x_2)$ whenever x_1 and x_2 are points in I with $x_1 < x_2$, we say f is *nondecreasing* on I.
2. If the strict inequality $f(x_1) < f(x_2)$ holds, we say f is *increasing*.

A similar definition was given for *nonincreasing* and *decreasing* functions.

Note. Some authors prefer the terms "increasing" and "strictly increasing" for what we would call nondecreasing and increasing. This has the unfortunate result that constant functions are then considered to be both increasing and decreasing. According to our definition we must say that they are both nondecreasing and nonincreasing, which sounds more plausible—if something stays constant it is neither going up nor going down). The disadvantage of our usage is the discomfort you may at first feel in using the terms (which disappears with practice). It is always safe to say "strictly increasing" for increasing even though it is redundant according to the definition.

By a monotonic function we mean a function that is increasing, decreasing, nondecreasing, or nonincreasing.

The theorems involving monotonicity of functions that one encounters in elementary calculus usually are stated for differentiable functions. But a monotonic function need not be differentiable, or even continuous.

Example 7.23: For example, if

$$f(x) = \begin{cases} x, & \text{for } x < 0 \\ x+1, & \text{for } x \geq 0, \end{cases}$$

then f is increasing on $\mathbb{R}$, but is not continuous at $x = 0$. (For more on discontinuities of monotonic functions, see Section 5.9.2.) ◀

Let us now address the role of the derivative in the study of monotonicity. We prove a familiar theorem that is the basis for many calculus applications. Note that the proof is an easy consequence of the mean value theorem.

Theorem 7.24: *Let f be differentiable on an interval I.*

(i) *If $f'(x) \geq 0$ for all $x \in I$, then f is nondecreasing on I.*

(ii) *If $f'(x) > 0$ for all $x \in I$, then f is increasing on I.*

(iii) *If $f'(x) \leq 0$ for all $x \in I$, then f is nonincreasing on I.*

(iv) *If $f'(x) < 0$ for all $x \in I$, then f is decreasing on I.*

(v) *If $f'(x) = 0$ for all $x \in I$, then f is constant on I.*

Proof. To prove (i), let $x_1, x_2 \in I$ with $x_1 < x_2$. By the mean value theorem (7.20) there exists $c \in (x_1, x_2)$ such that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1).$$

If $f'(c) \geq 0$, then $f(x_2) \geq f(x_1)$. Thus, if $f'(x) \geq 0$ for all $x \in I$, f is nondecreasing on I.

Parts (ii), (iii) and (iv) have similar arguments, and (v) follows immediately from parts (i) and (iii). ■

Exercises

7.7.1 Establish the inequality $e^x \leq \frac{1}{1-x}$ for all $x < 1$.

SEE NOTE 196

7.7.2 Suppose that f and g are differentiable functions such that $f' = g$ and $g' = -f$. Show that there exists a number C with the property that

$$[f(x)]^2 + [g(x)]^2 = C$$

for all x.

7.7.3 Suppose f is continuous on (a, c) and $a < b < c$. Suppose also that f is differentiable on (a, b) and on (b, c). Prove that if $f' < 0$ on (a, b) and $f' > 0$ on (b, c), then f has a minimum at b.

SEE NOTE 197

7.7.4 The hypotheses of Theorem 7.24 require that f be differentiable on all of the interval I. You might think that a positive derivative at a single point also implies that the function is increasing, at least in a neighborhood of that point. This is not true. Consider the function

$$f(x) = \begin{cases} x/2 + x^2 \sin x^{-1}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

(a) Show that the function $g(x) = x^2 \sin x^{-1}$ ($g(0) = 0$) is everywhere differentiable and that $g'(0) = 0$.

(b) Show that g' is discontinuous at $x = 0$ and that g' takes on values close to ± 1 arbitrarily near 0.

(c) Show that f' takes on both positive and negative values in every neighborhood of zero.

(d) Show that $f'(0) = \frac{1}{2} > 0$ but that f is not increasing in any neighborhood of zero.

(e) Prove that if a function F is differentiable on a neighborhood of x_0 with $F'(x_0) > 0$ and F' is *continuous* at x_0, then F is increasing on some neighborhood of x_0.

(f) Why does the example $f(x)$ given here not contradict part (e)?

7.7.5 Let f be differentiable on $[0, \infty)$ and suppose that $f(0) = 0$ and that the derivative f' is an increasing function on $[0, \infty)$. Show that

$$\frac{f(x)}{x} < \frac{f(y)}{y}$$

for all $0 < x < y$.

SEE NOTE 198

7.7.6 Suppose that f, $g : \mathbb{R} \to \mathbb{R}$ and both have continuous derivatives and the determinant

$$\phi(x) = \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix}$$

is never zero. Show that between any two zeros of f there must be a zero of g.

SEE NOTE 199

7.8 Dini Derivates

We observed in Example 7.4 that the function $f(x) = |x|$ does not have a derivative at the point $x = 0$ but does have the one-sided derivatives $f'_+(0) = 1$ and $f'_-(0) = -1$. It is not difficult to construct continuous functions that don't have even one-sided derivatives at a point.

Example 7.25: Consider the function

$$f(x) = \begin{cases} |x| \left|\cos x^{-1}\right|, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

(See Figure 7.25). Since $|\cos x^{-1}| \leq 1$ for all $x \neq 0$,

$$\lim_{x \to 0} f(x) = 0 = f(0)$$

so f is continuous at $x = 0$. It is clear that f is continuous at all other points in $\mathbb{R}$, so f is a continuous function.

The oscillatory behavior of f is such that the sets

$$\left\{x : \left|\cos x^{-1}\right| = 1\right\} \quad \text{and} \quad \left\{x : \left|\cos x^{-1}\right| = 0\right\}$$

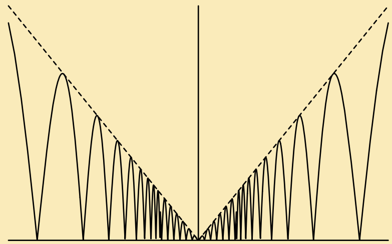

Figure 7.9. Graph of $f(x) = |x \cos x^{-1}|$.

both have zero as a two-sided limit point. Thus each of the sets

$$\{x : f(x) = |x|\} \quad \text{and} \quad \{x : f(x) = 0\}$$

has zero as two-sided limit point. Inspection of the difference quotient reveals that

$$\limsup_{x\to 0+} \frac{f(x) - f(0)}{x - 0} = 1, \text{ while } \liminf_{x\to 0+} \frac{f(x) - f(0)}{x - 0} = 0,$$

so f'_+ does not exist at $x = 0$. Similarly, $f'_-(0)$ does not exist. The limits that are required to exist for f to have a derivative, or a one-sided derivative, don't exist at $x = 0$. ◀

Example 7.26: A function defined on an interval I may fail to have a derivative, even a one-sided derivative, at every point. Let

$$g(x) = \begin{cases} 0, & \text{if } x \text{ is rational,} \\ 1, & \text{if } x \text{ is irrational.} \end{cases}$$

Since g is everywhere discontinuous on both sides, g has no derivative and no one-sided derivative at any point. ◀

There are, also, *continuous* functions that fail to have a one-sided derivative, finite or infinite, at even a single point. Such functions are difficult to construct, the first construction having been given by Besicovitch in 1925.

Now the derivative, when it exists, plays an important role in analysis, and it is useful to have a substitute when it doesn't exist. Many good substitutes have been developed for certain situations. Perhaps the simplest such substitutes are the Dini derivates. These exist at every point for every function defined on an open interval. They are named after the Italian mathematician Ulisse Dini (1845–1918).

Definition 7.27: Let f be real valued in a neighborhood of x_0. We define the four *Dini derivates* of f at x_0 by

1. [Upper right Dini derivate]
$$D^+f(x_0) = \limsup_{x\to x_0+} \frac{f(x)-f(x_0)}{x-x_0}$$
2. [Lower right Dini derivate]
$$D_+f(x_0) = \liminf_{x\to x_0+} \frac{f(x)-f(x_0)}{x-x_0}$$
3. [Upper left Dini derivate]
$$D^-f(x_0) = \limsup_{x\to x_0-} \frac{f(x)-f(x_0)}{x-x_0}$$
4. [Lower left Dini derivate]
$$D_-f(x_0) = \liminf_{x\to x_0-} \frac{f(x)-f(x_0)}{x-x_0}.$$

Example 7.28: For the function $f(x) = |x|\,\left|\cos x^{-1}\right|,\ f(0)=0$, we calculate that

$$D^+f(0) = 1,\ D_+f(0) = 0,\ D^-f(0) = 0,\ D_-f(0) = -1.$$

Elsewhere $f'(x)$ exists and all four Dini derivatives have that value. ◀

Example 7.29: The function

$$g(x) = \begin{cases} 0, & \text{if } x \text{ is rational,} \\ 1, & \text{if } x \text{ is irrational.} \end{cases}$$

has at every rational x

$$D^{+}g(x) = 0, D_{+}g(x) = -\infty, D^{-}g(x) = \infty, D_{-}g(x) = 0.$$

For x irrational there are similar values for the Dini derivates (see Exercise 7.8.1a). ◀

It is easy to check that a function f has a derivative at a point x_0 if and only if all four Dini derivates are equal at that point, and a one-sided derivative at x_0 if the two Dini derivates from that side are equal (see Exercise 7.8.2).

We end this section with an illustration of the way in which knowledge about a Dini derivate can substitute for that of the ordinary derivative. We prove a theorem about monotonicity. You are familiar with the fact that if f is differentiable on an interval $[a, b]$ and $f'(x) > 0$ for all $x \in [a, b]$, then f is an increasing function on $[a, b]$. (We provided a formal proof in Section 7.7.)

Here is a generalization of that theorem.

Theorem 7.30: *Let f be continuous on $[a, b]$. If $D^{+}f(x) > 0$ at each point $x \in [a, b)$, then f is increasing on $[a, b]$.*

Proof. Let us first show that f is nondecreasing on $[a, b]$. We prove this by contradiction. If f fails to be nondecreasing on $[a, b]$, there exist points c and d such that $a \leq c < d \leq b$ and $f(c) > f(d)$. Let y be any point in the interval $(f(d), f(c))$.

Since f is continuous on $[a, b]$, it possesses the intermediate value property. Thus from Theorem 5.53 [or more precisely from the version of that theorem given as Exercise 5.8.8(a)] there exists a point $t \in (c, d)$ such that $f(t) = y$. Thus the set

$$\{x : f(x) = y)\} \cap [c, d]$$

is nonempty. Let

$$x_0 = \sup\{x : c \le x \le d \text{ and } f(x) = y\}.$$

Now, $f(d) < y$ and f is continuous, from which it follows that $x_0 < d$. Thus $f(x) < y$ for $x \in (x_0, d]$. Furthermore, the set $\{x : f(x) = y\}$ is closed (because f is continuous), so $f(x_0) = y$.

But this implies that $D^+f(x_0) \le 0$.This contradicts our hypothesis that $D^+f(x) > 0$ for all $x \in [a, b)$. This contradiction completes the proof that f is nondecreasing.

Now we wish to show that it is in fact increasing. If not, then there must be some subinterval in which the function is constant. But at every point interior to that interval we would have $f'(x) = 0$ and so it would be impossible for $D^+f(x) > 0$ at such points. ■

Exercises

7.8.1 Calculate the four Dini derivates for each of the following functions at the given point.

(a)

$$g(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

for $x = \pi$.

(b) $h(x) = x \sin x^{-1}$ $(h(0) = 0)$ at $x = 0$

(c) $f(x) = x \sin x^{-1}$ $(f(0) = 5)$ at $x = 0$

(d)

$$u(x) = \begin{cases} x^2, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

at $x = 0$ and at $x = 1$

7.8.2 Prove that f has a derivative at x_0 if and only if

$$D^+f(x_0) = D_+f(x_0) = D^-f(x_0) = D_-f(x_0).$$

In that case, $f'(x_0)$ is the common value of the Dini derivates at x_0. (We assume that f is defined in a neighborhood of x_0.)

7.8.3 **(Derived Numbers)** The Dini derivates are sometimes called "extreme unilateral derived numbers." Let $\lambda \in [-\infty, \infty]$. Then λ is a *derived number* for f at x_0 if there exists a sequence $\{x_k\}$ with $\lim_{k\to\infty} x_k = x_0$ such that

$$\lambda = \lim_{k\to\infty} \frac{f(x_k) - f(x_0)}{x_k - x_0}.$$

(a) For the function $f(x) = \left|x \cos x^{-1}\right|$, $f(0) = 0$, show that every number in the interval $[-1, 1]$ is a derived number for f at $x = 0$. Show that the two extreme derived numbers from the right are 0 and 1, and the two from the left are -1 and 0.

(b) Show that a function has a derivative at a point if and only if all derived numbers at that point coincide.

(c) Let $f : \mathbb{R} \to \mathbb{R}$ and let $x_0 \in \mathbb{R}$. Prove that if f is continuous on $\mathbb{R}$, then the set of derived numbers of f at x_0 consists of either one or two closed intervals (that might be degenerate or unbounded). Give examples to illustrate the various possibilities.

7.8.4 Let $f, g : \mathbb{R} \to \mathbb{R}$.

(a) Prove that $D^+(f + g)(x) \le D^+f(x) + D^+g(x)$.

(b) Give an example to illustrate that the inequality in (a) can be strict.

(c) State and prove the analogue of part (a) for the lower right derivate D_+f.

7.8.5 Generalize Theorem 7.18 to the following:

> *If f achieves a local maximum at x_0, then $D^+f(x_0) \le 0$ and $D_-f(x_0) \ge 0$.*

Illustrate the result with a function that is not differentiable at x_0.

7.8.6 Prove a variant of Theorem 7.30 that assumes that, for all x in $[a, b)$ except for x in some countable set, the Dini derivate $D^+f(x) > 0$.

7.8.7 Prove a variant of Theorem 7.30: If f is continuous and $D^+f(x) \ge 0$ for all $x \in [a, b)$, then f is nondecreasing on $[a, b]$.

See Note 200

7.8.8 Prove yet another (more subtle) variant of Theorem 7.30: If f is continuous and $D^+f(x) > 0$ for all $x \in [a, b)$ except for x in some countable set, then f is increasing on $[a, b]$.

7.8.9 Prove that no continuous function can have $D^+f(x) = \infty$ for all $x \in \mathbb{R}$. Give an example of a function $f:\mathbb{R}\to\mathbb{R}$ such that $D^+f(x) = \infty$ for all $x \in \mathbb{R}$.

See Note 201

7.8.10 Show that the set

$$\{x : D^+f(x) < D_-f(x)\}$$

cannot be uncountable. Give an example of a function f such that $D^+f < D_-f$ on an infinite set.

See Note 202

7.9 The Darboux Property of the Derivative

Suppose f is differentiable on an interval $[a, b]$. We argued in the proof of Rolle's theorem (7.19) that if $f(a) = f(b)$, then there exists a point $c \in (a, b)$ at which f achieves an extremum. At this point c we have $f'(c) = 0$.

A different hypothesis can lead to the same conclusion. Suppose f is differentiable on $[a, b]$ and $f'(a) < 0 < f'(b)$ (or $f'(b) < 0 < f'(a)$). Once again, the extreme value f achieves must occur at a point c in the *interior* of $[a, b]$, (why?), and at this point we must have $f'(c) = 0$. This observation is a special case of the following theorem first proved by Darboux in 1875.

Theorem 7.31: *Let f be differentiable on an interval I. Suppose $a, b \in I$, $a < b$, and $f'(a) \neq f'(b)$. Let γ be any number between $f'(a)$ and $f'(b)$. Then there exists $c \in (a, b)$ such that $f'(c) = \gamma$.*

Proof. Let $g(x) = f(x) - \gamma x$. If $f'(a) < \gamma < f'(b)$, then $g'(a) = f'(a) - \gamma < 0$ and $g'(b) = f'(b) - \gamma > 0$. The discussion preceding the statement of the theorem shows that there exists $c \in (a, b)$ such that $g'(c) = 0$. For this c we have

$$f'(c) = g'(c) + \gamma = \gamma,$$

completing the proof for the case $f'(a) < f'(b)$.

The proof when $f'(a) > f'(b)$ is similar. ■

You might have noted that Theorem 7.31 is exactly the statement that the derivative of a differentiable function has the Darboux property (i.e., the intermediate value property) that we established for continuous functions in Section 5.8. The derivative f' of a differentiable function f need not be continuous, of course. The result does imply, however, that f' cannot have jump discontinuities and cannot have removable discontinuities.

Both the mean value theorem and Theorem 7.31 give information about the range of the derivative f' of a differentiable function f. The mean value theorem implies that the range of f' includes all slopes of chords determined by the graph of f on the interval of definition of f. Theorem 7.31 tells us that this range is actually an interval. This interval may be unbounded and, if bounded, may or may not contain its endpoints. (See Exercise 7.9.1.)

Derivative of an Inverse Function Theorem 7.31 allows us to establish a familiar theorem about differentiating inverse functions.

Theorem 7.32: *Suppose f is differentiable on an interval I and for each $x \in I$, $f'(x) \neq 0$. Then*

(i) *f is one-to-one on I,*

(ii) *f^{-1} is differentiable on $J = f(I)$,*

(iii) $(f^{-1})'(f(x)) = \dfrac{1}{f'(x)}$ *for all $x \in I$.*

Proof. By Theorem 7.31 either $f'(x) > 0$ for all $x \in I$ or $f'(x) < 0$ for all $x \in I$. In either case, f is either increasing or decreasing on I, and is thus one-to-one, establishing (i).

To verify (ii) and (iii), observe first that f^{-1} is continuous, since f is continuous and strictly monotonic (see Exercise 5.9.16). Let $y_0 \in J$ and let $x_0 = f^{-1}(y_0)$. We wish to show that $(f^{-1})'(y_0)$ exists and has value $1/(f'(x_0))$. For $x \in I$, write $y = f(x)$, so $x = f^{-1}(y)$.

Consider the difference quotient

$$\frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \frac{x - x_0}{f(x) - f(x_0)}.$$

As $y \to y_0$, $x \to x_0$, because the function f^{-1} is continuous. Thus

$$\lim_{y \to y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \lim_{x \to x_0} \frac{1}{\left(\frac{f(x) - f(x_0)}{x - x_0}\right)} = \frac{1}{f'(x_0)}.$$

■

Exercises

7.9.1 Let f be differentiable on $[a, b]$ and let $\mathcal{R}(f')$ denote the range of f' on $[a, b]$. Give examples to illustrate that $\mathcal{R}(f')$ can be

(a) a closed interval

(b) an open interval

(c) a half-open interval

(d) an unbounded interval

See Note 203

7.9.2 Give an example of a differentiable function f such that

$$f'(x_0) \neq \lim_{x \to x_0} f'(x).$$

Show that if f is defined and continuous in a neighborhood of x_0 and if the limit

$$\lim_{x \to x_0} f'(x)$$

exists and is finite, then f is differentiable at x_0 and f' is continuous at x_0.

7.9.3 Most classes of functions we have encountered are closed under the operations of addition and multiplication (e.g., polynomials, continuous functions, differentiable functions). The class of derivatives is closed under addition, but behaves badly with respect to multiplication. Consider, for example, the pair of functions F and G defined on $\mathbb{R}$ by

$$F(x) = x^2 \sin \frac{1}{x^3}, \ (F(0) = 0), \text{ and}$$
$$G(x) = x^2 \cos \frac{1}{x^3}, \ (G(0) = 0).$$

Verify each of the following statements:

(a) F and G are differentiable on $\mathbb{R}$.

(b) The functions FG' and GF' are bounded functions.

(c) $F(x)G'(x) - F'(x)G(x) = \begin{cases} 3, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$

(d) At least one of the functions FG' or GF' must fail to be a derivative.

Thus, even the product of a differentiable function F with a derivative G' need not be a derivative.

See Note 204

7.9.4 Show, in contrast to Exercise 7.9.3, that if a function f has a continuous derivative on $\mathbb{R}$ and g is differentiable, then fg' is a derivative.

See Note 205

7.9.5 Let f be a differentiable function on an interval $[a, b]$. Show that f' is continuous if and only if the set

$$E_\alpha = \{x : f'(x) = \alpha\}$$

is closed for each real number α.

See Note 206

7.9.6 Let $f : [0, 1] \to \mathbb{R}$ be a continuous function that is differentiable on $(0, 1)$ and with $f(0) = 0$ and $f(1) = 1$. Show there must exist distinct numbers ξ_1 and ξ_2 in that interval such that

$$f'(\xi_1)f'(\xi_2) = 1.$$

7.9.7 Prove or disprove that if $f : \mathbb{R} \to \mathbb{R}$ is differentiable and monotonic, then f' must be continuous on $\mathbb{R}$.

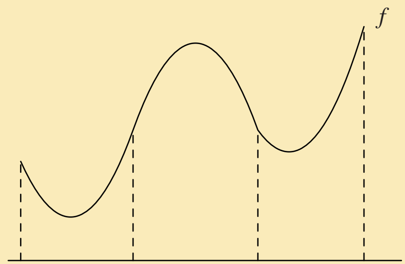

Figure 7.10. Concave up/down/up.

7.10 Convexity

In elementary calculus one studies functions that are concave-up or concave-down on an interval. A knowledge of the intervals on which a function is concave-up or concave-down is useful for such purposes as sketching the graph of the equation $y = f(x)$ and studying extrema of the function (Fig. 7.10).

In the setting of elementary calculus the functions usually have second derivatives on the intervals involved. In that setting we define a function as being concave-up on an interval I if $f'' \geq 0$ on I, and concave-down if $f'' \leq 0$ on I. Definitions involving the first derivative, but not the second, can also be given: f is concave-up on I if f' is increasing on I, concave-down if f' is decreasing on I. Equivalently, f is concave-up if the graph of f lies "above" (more precisely "not below") each of its tangent lines, concave-down if the graph lies below (not above) each of its tangent lines.

The geometric properties we wish to capture when we say a function is concave-up or concave-down do not depend on differentiability properties. The condition is that the graph should lie below (or above) all its chords. The following definitions make this concept precise. We shall follow the common practice of using the terms "convex" and "concave" in place of the terms "concave-up" and "concave-down."

Definition 7.33: Let f be defined on an interval I. If for all $x_1, x_2 \in I$ and $\alpha \in [0,1]$ the inequality

$$f(\alpha x_1 + (1-\alpha)x_2) \leq \alpha f(x_1) + (1-\alpha)f(x_2) \tag{13}$$

is satisfied, we say that f is *convex* on I. If the reverse inequality in (13) applies, we say that f is *concave* on I. If the inequalities are strict for all $\alpha \in (0,1)$ we say f is *strictly convex* or *strictly concave* on I.

For example, the function $f(x) = |x|$ is convex, but not strictly convex on $\mathbb{R}$. Strict convexity implies that the graph of f has no line segments in it. Note that the function $f(x) = |x|$ is not differentiable at $x = 0$.

The geometric condition defining convexity does imply a great deal of regularity of a function. Our first objective is to address this issue. We begin with some simple geometric considerations.

Suppose f is convex on an open interval I. Let x_1 and x_2 be points in I with $x_1 < x_2$. The chord determined by the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ defines a linear function M on $[x_1, x_2]$: If $x = \alpha x_1 + (1-\alpha)x_2$, then

$$M(x) = \alpha f(x_1) + (1-\alpha)f(x_2).$$

The definition of "convex" states that

$$f(x) \leq M(x)$$

for all $x \in [x_1, x_2]$ and that

$$M(x_1) = f(x_1) \ \text{ and } \ M(x_2) = f(x_2).$$

Now let $z \in (x_1, x_2)$ Then

$$\frac{f(z) - f(x_1)}{z - x_1} \leq \frac{M(z) - M(x_1)}{z - x_1} = \frac{M(x_2) - M(z)}{x_2 - z} \leq \frac{f(x_2) - f(z)}{x_2 - z} \tag{14}$$

(Fig. 7.11).

Thus, the chord determined by f and the points x_1 and x_2 has a slope between the slopes of the chord determined by x_1 and z and the chord determined by z and x_2.

The inequalities (14) have a number of useful consequences:

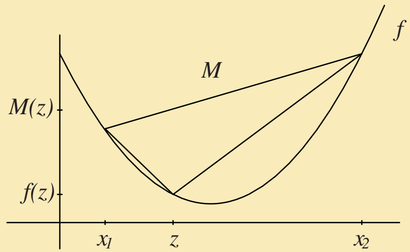

Figure 7.11. Comparison of the three slopes in the inequalities (14).

1. For fixed $x \in I$,
$$(f(x+h)-f(x))/h$$
is a nondecreasing function of h on some interval $(0,\delta)$. Thus
$$\lim_{h\to 0+}\frac{f(x+h)-f(x)}{h} = \inf_{h>0}\frac{f(x+h)-f(x)}{h}$$
exists or possibly is $-\infty$. That it is in fact finite can be shown by using (14) again to get a finite lower bound, since
$$(f(x')-f(x))/(x'-x) \le (f(x+h)-f(x))/h$$
for any $x' \in I$ with $x' < x$. Thus f has a right-hand derivative $f'_+(x)$ at x. Similarly, f has a finite left-hand derivative at x.

2. If $x, y \in I$ and $x < y$, then
$$f'_+(x) \le f'_+(y).$$
From observation 1 we infer that
$$(f(x+h)-f(x))/h \le (f(y+h)-f(y))/h$$

whenever $h > 0$ and $x+h, y+h$ are in I. Thus f'_+ is a nondecreasing function. Similarly, f'_- is a nondecreasing function.

3. It is also clear from (14) that
$$f'_-(x) \le f'_+(x)$$
for all $x \in I$.

4. f is continuous on I. To see this, observe that since both one-sided derivatives exist at every point the function must be continuous on both sides, hence continuous.

We summarize the preceding discussion as a theorem.

Theorem 7.34: *Let f be convex on an open interval I. Then*

(i) *f has finite left and right derivatives at each point of I. Each of these one-sided derivatives is a nondecreasing function of x on I, and*
$$f'_-(x) \le f'_+(x) \text{ for all } x \in I. \tag{15}$$

(ii) *f is continuous on I.*

Note. If f is convex on a *closed* interval $[a,b]$, some of the results do not apply at the endpoints a and b. (See Exercise 7.10.8.) Note, too, that the corresponding results are valid for concave functions on I, the one-sided derivatives now being nonincreasing functions of x and the inequality in (15) being reversed.

We can now obtain the characterizations of convex functions familiar from elementary calculus.

Corollary 7.35: *Let f be defined on an open interval I.*

(i) *If f is differentiable on I, then f is convex on I if and only if f' is nondecreasing on I.*

(ii) *If f is twice differentiable on I, then f is convex on I if and only if $f'' \ge 0$ on I.*

We leave the verification of Corollary 7.35 as Exercise 7.10.9.

Exercises

7.10.1 Show that a function f is convex on an interval I if and only if the determinant

$$\begin{vmatrix} 1 & x_1 & f(x_1) \\ 1 & x_2 & f(x_2) \\ 1 & x_3 & f(x_3) \end{vmatrix}$$

is nonnegative for any choices of $x_1 < x_2 < x_3$ in the interval I.

7.10.2 If f and g are convex on an interval I, show that any linear combination $\alpha f + \beta g$ is also convex provided α and β are nonnegative.

7.10.3 If f and g are convex functions, can you conclude that the composition $g \circ f$ is also convex?

See Note 207

7.10.4 Let f be convex on an open interval (a, b). Show that then there are only two possibilities. Either (i) f is nonincreasing or nondecreasing on the entire interval (a, b) or else (ii) there is a number c so that f is nonincreasing on $(a, c]$ and nondecreasing on $[c, b)$.

7.10.5 Suppose f is convex on an open interval I. Prove that f is differentiable except on a countable set.

See Note 208

7.10.6 Suppose f is convex on an open interval I. Prove that if f is differentiable on I, then f' is continuous on I.

7.10.7 Let f be convex on an open interval that contains the closed interval $[a, b]$. Let

$$M = \max\{f'_+(a), f'_-(b)\}.$$

Show that

$$|f(x) - f(y)| \leq M|x - y|$$

for all x, $y \in [a, b]$.

7.10.8 Theorem 7.34 pertains to functions that are convex on an open interval. Discuss the extent to which the results of the theorem hold when f is convex on a *closed* interval $[a, b]$. In particular, determine whether continuity of f at the endpoints of the interval follows from the definition. Must $f'_+(a)$ and $f'_-(b)$ be finite?

7.10.9 Prove Corollary 7.35.

7.10.10 Let f be convex on an open interval (a,b). Must f be bounded above? Must f be bounded below?

SEE NOTE 209

7.10.11 Let f be convex on an open interval (a,b). Show that f does not have a strict maximum value.

7.10.12 Let f be defined and continuous on an open interval (a,b). Show that f is convex there if and only if there do not exist real numbers α and β such that the function $f(x)+\alpha x+\beta$ has a strict maximum value in (a,b).

7.10.13 ✄ Let $A=\{a_1,a_2,a_3,\dots\}$ be any countable set of real numbers. Let

$$f(x)=\sum_{1}^{\infty}\frac{|x-a_k|}{10^k}.$$

Prove that f is convex on $\mathbb{R}$, differentiable on the set $\mathbb{R}\setminus A$, and nondifferentiable on the set A.

SEE NOTE 210

7.10.14 ✄ **(Inflection Points)** In elementary calculus one studies inflection points. The definitions one finds try to capture the idea that at such a point the sense of concavity changes from strict "up to down" or vice versa. Here are three common definitions that apply to differentiable functions. In each case f is defined on an open interval (a,b) containing the point x_0. The point x_0 is an *inflection point* for f if there exists an open interval $I\subset(a,b)$ such that on I

(Definition A) f' increases on one side of x_0 and decreases on the other side.

(Definition B) f' attains a strict maximum or minimum at x_0.

(Definition C) The tangent line to the graph of f at $(x_0,f(x_0))$ lies below the graph of f on one side of x_0 and above on the other side.

(a) Prove that if f satisfies Definition A at x_0, then it satisfies Definition B at x_0.

(b) Prove that if f satisfies Definition B at x_0, then it satisfies Definition C at x_0.

(c) Give an example of a function satisfying Definition B at x_0, but not satisfying Definition A.

(d) Give an example of an infinitely differentiable function satisfying Definition C at x_0, but not satisfying Definition B.

(e) Which of the three definitions states that the sense of concavity of f is "up" on one side of x_0 and "down" on the other?

SEE NOTE 211

7.10.15 (Jensen's Inequality) Let f be a convex function on an interval I, let $x_1, x_2, \ldots, x_n$ be points of I and let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be positive numbers satisfying

$$\sum_{k=1}^{n} \alpha_k = 1.$$

Show that

$$f\left(\sum_{k=1}^{n} \alpha_k x_k\right) \leq \sum_{k=1}^{n} \alpha_k f(x_k).$$

SEE NOTE 212

7.10.16 Show that the inequality is strict in Jensen's inequality (Exercise 7.10.15) except in the case that f is linear on some interval that contains the points $x_1, x_2, \ldots, x_n$.

7.11 L'Hôpital's Rule

Suppose that f and g are defined in a deleted neighborhood of x_0 and that

$$\lim_{x\to x_0} f(x) = A \text{ and } \lim_{x\to x_0} g(x) = B.$$

According to our usual theory of limits, we then have

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{\lim_{x\to x_0} f(x)}{\lim_{x\to x_0} g(x)} = \frac{A}{B},$$

unless $B = 0$.

But what happens if $B = 0$, which is often the case? A number of possibilities exist: If $B = 0$ and $A \neq 0$, then the limit does not exist. The most interesting case remains: If both A and B are zero, then the limiting behavior depends on the rates at which $f(x)$ and $g(x)$ approach zero.

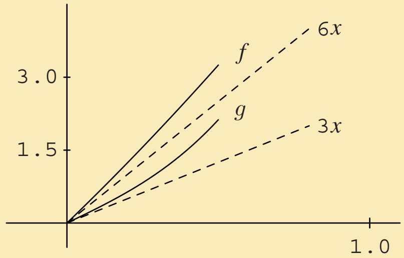

Figure 7.12. Comparison of the rates in Example 7.37.

Example 7.36: Consider

$$\lim_{x\to 0}\frac{6x}{3x}=\lim_{x\to 0}\frac{6}{3}=2.$$

Look at this simple example geometrically. For $x \neq 0$, the height $6x$ is twice that of the height $3x$. The straight line $y = 6x$ approaches zero at twice the rate that the line $y = 3x$ does. ◀

Example 7.37: Now consider the slightly more complicated limit

$$\lim_{x\to 0}\frac{f(x)}{g(x)}=\lim_{x\to 0}\frac{6x+x^2}{3x+5x^3}.$$

If we divide the numerator and denominator by $x \neq 0$, we see that the limit is the same as

$$\lim_{x\to 0}\frac{6+x}{3+5x^2}.$$

This last limit can be calculated by our usual elementary methods as equaling $6/3 = 2$. Here, for $x \neq 0$ near zero, the height $f(x) = 6x + x^2$ is approximately $6x$, while the height of $g(x) = 3x + 5x^3$ is approximately $3x$, that is, the desired ratio is approximately 2. Again, the numerator approaches zero at about twice the rate that the denominator does.

We can be more precise by calculating these rates exactly. Let

$$f(x) = 6x + x^2 \text{ and } \; g(x) = 3x + 5x^3.$$

Then

$$\begin{array}{ll} f'(x) = 6 + 2x, & f'(0) = 6 \\ g'(x) = 3 + 5x^2, & g'(0) = 3. \end{array}$$

This makes precise our statement that the numerator approaches zero twice as fast as the denominator does. (See Figure 7.12 where there is an illustration showing the graphs of the functions f and g compared to the lines $y = 6x$ and $y = 3x$.) ◀

Let us try to generalize from these two examples. Suppose f and g are differentiable in a neighborhood of $x = a$ and that $f(a) = g(a) = 0$. Consider the following calculations and what conditions on f and g are required to make them valid.

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{\left(\frac{f(x)-f(a)}{x-a}\right)}{\left(\frac{g(x)-g(a)}{x-a}\right)} \xrightarrow{x\to a} \frac{f'(a)}{g'(a)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}. \tag{16}$$

If these calculations are valid, they show that under these assumptions ($f(a) = g(a) = 0$ and both $f'(a)$ and $g'(a)$ exist) we should be able to claim that

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

You should check the various conditions that must be met to justify the calculations: $g(x)$ cannot equal zero at any point of the neighborhood in question (other than a); nor can $g(x) = g(a)$, (for $x \neq a$); $f(a)$ and $g(a)$ must equal zero (for the first equality), and f'/g' must be continuous at $x = a$ (for the last equality).

The calculations (16) provide a simple proof of a rudimentary form for a method of computing limits known as L'Hôpital's rule. We say "rudimentary" because some of the conditions we assumed are not needed for the conclusion

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

7.11.1 L'Hôpital's Rule: $\frac{0}{0}$ Form

Enrich.

Our first theorem provides a version of the rule identical with our introductory remarks but under weaker assumptions.

Theorem 7.38 (L'Hôpital's Rule: $\frac{0}{0}$ Form) *Suppose that the functions f and g are differentiable in a deleted neighborhood N of $x=a$. Suppose*

(i) $\lim_{x\to a} f(x)=0$,

(ii) $\lim_{x\to a} g(x)=0$,

(iii) *For every $x\in N$, $g'(x)\neq 0$, and*

(iv) $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ *exists.*

Then $\displaystyle\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$.

Proof. Our hypotheses do not require f and g to be defined at $x=a$. But we can in any case define (or redefine) f and g at $x=a$ by $f(a)=g(a)=0$. Because of assumptions (i) and (ii), this results in continuous functions defined on the full neighborhood $N\cup\{a\}$ of the point $x=a$. We can now apply Cauchy's form of the mean value theorem (7.21).

Suppose $x\in N$ and $a<x$. By Theorem 7.21 there exists $c=c_x$ in (a,x) such that

$$[f(x)-f(a)]g'(c_x)=[g(x)-g(a)]f'(c_x). \tag{17}$$

Since $f(a) = g(a) = 0$, (17) becomes

$$f(x)g'(c_x) = g(x)f'(c_x). \tag{18}$$

Equation (18) is valid for $x > a$ in N. We would like to express (18) in the form

$$\frac{f(x)}{g(x)} = \frac{f'(c_x)}{g'(c_x)}. \tag{19}$$

To justify (19) we show that $g(x)$ is never zero in $N \cap \{x : x > a\}$. (That $g'(c_x)$ is never zero in N is our hypothesis (iii).) If for some $x \in N$, $x > a$, we have $g(x) = 0$, then by Rolle's theorem there would exist a point $t \in (a, x)$ such that $g'(t) = 0$, contradicting hypothesis (ii). Thus equation (19) is valid for all $N \cap \{x : x > a\}$. A similar argument shows that if $x \in N$, $x < a$, then there exists $c_x \in (x, a)$ such that (19) holds.

Now as $x \to a$, c_x also approaches a, since c_x is between a and x. Thus

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(c_x)}{g'(c_x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

since the last limit exists by hypothesis (iv). ■

Note. Observe that we did not require f to be defined at $x = a$, nor did we require that f'/g' be continuous at $x = a$. It is also important to observe that L'Hôpital's rule does *not* imply that, under hypotheses (i), (ii), and (iii) of Theorem 7.38, if $\lim_{x\to a} f(x)/g(x)$ exists, then $\lim_{x\to a} f'(x)/g'(x)$ must also exist. Exercise 7.11.5 provides an example to illustrate this.

Example 7.39: Let us use L'Hôpital's rule to evaluate

$$\lim_{x\to 0} \ln(1 + x)/x.$$

Let $f(x) = \ln(1 + x)$, $g(x) = x$. Then

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0,\ f'(x) = \frac{1}{1 + x}, \quad \text{and} \quad g'(x) = 1.$$

Thus

$$\lim_{x\to 0} \frac{\ln(1+x)}{x} = \lim_{x\to 0} \frac{1}{1+x} = 1.$$

◀

We refer to this theorem as the "$\frac{0}{0}$ form" for obvious reasons. There is also a version of the form $\frac{\infty}{\infty}$ (see Theorem 7.42). In addition, other modifications are possible. The point a can be replaced with $a = \infty$ or $a = -\infty$, (Theorem 7.41), and the results are valid for one-sided limits. (Our proof of Theorem 7.38 actually established that fact since we considered the case $x > a$ and $x < a$ separately.) Various other "indeterminate forms," ones for which the limit depends on the rates at which component parts approach their separate limits, can be manipulated to make use of L'Hôpital's rule possible.

Here is an example in which the forms "1^∞" and "$1^{-\infty}$" come into play. Observe that the function whose limit we wish to calculate is of the form $f(x)^{g(x)}$ where $f(x) \to 1$ as $x \to a$ but $g(x) \to \infty$ as $x \to a+$ and $g(x) \to -\infty$ as $x \to a-$.

Example 7.40: Evaluate $\lim_{x\to 0}(1+x)^{2/x}$. This expression is of the form 1^∞ (when $x > 0$). To calculate $\lim_{x\to 0}(1+x)^{2/x}$, write

$$y = (1+x)^{2/x}, z = \ln y = \frac{2}{x}\ln(1+x).$$

Now the numerator and denominator of the function z satisfy the hypotheses of L'Hôpital's rule. Thus

$$\lim_{x\to 0} z = \lim_{x\to 0} \frac{2\ln(1+x)}{x} = \lim_{x\to 0} \frac{2}{1+x} = 2.$$

Since $\lim_{x\to 0} z = 2$, $\lim_{x\to 0} y = e^2$. ◀

7.11.2 L'Hôpital's Rule as $x \to \infty$

✂ *Enrich.*

We proved Theorem 7.38 under the assumption that $a \in \mathbb{R}$, but the theorem is valid when $a = -\infty$ or $a = +\infty$. In this case we are, of course, dealing with one-sided limits. As before, the relation

$$\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = L$$

implies something about relative rates of growth of the functions $f(x)$ and $g(x)$ as $x \to \infty$. We can base a proof of the versions of L'Hôpital's rule that have $a = \infty$ (or $-\infty$) on Theorem 7.38 by a simple transformation.

Theorem 7.41: *Let f, g be differentiable on some interval $(-\infty, b)$. Suppose*

(i) $\lim_{x\to-\infty} f(x) = 0$,

(ii) $\lim_{x\to-\infty} g(x) = 0$,

(iii) *For every $x \in (-\infty, b)$, $g'(x) \neq 0$, and*

(iv) $\lim_{x\to-\infty} \frac{f'(x)}{g'(x)}$ *exists.*

Then

$$\lim_{x\to-\infty} \frac{f(x)}{g(x)} = \lim_{x\to-\infty} \frac{f'(x)}{g'(x)}.$$

A similar result holds when we replace ∞ by $-\infty$ in the hypotheses.

Proof. Let $x = -1/t$. Then, as $t \to 0+$, $x \to -\infty$ and vice-versa. Define functions F and G by

$$F(t) = f\left(-\frac{1}{t}\right) \text{ and } G(t) = g\left(-\frac{1}{t}\right).$$

Both functions F and G are defined on some interval $(0, \delta)$. We verify easily that

$$\lim_{t \to 0+} F(t) = \lim_{t \to 0+} G(t) = 0$$

and that

$$\lim_{t \to 0+} \frac{F'(t)}{G'(t)} = \lim_{x \to -\infty} \frac{f'(x)}{g'(x)}. \tag{20}$$

Using Theorem 7.38, we infer

$$\lim_{t \to 0+} \frac{F'(t)}{G'(t)} = \lim_{t \to 0+} \frac{F(t)}{G(t)} = \lim_{t \to 0+} \frac{f(-\frac{1}{t})}{g(-\frac{1}{t})} = \lim_{x \to -\infty} \frac{f(x)}{g(x)}. \tag{21}$$

The result follows from (20) and (21) ■

7.11.3 L'Hôpital's Rule: $\frac{\infty}{\infty}$ Form

When $f(x) \to \infty$ and $g(x) \to \infty$ as $x \to a$ we obtain the indeterminate form $\frac{\infty}{\infty}$. L'Hôpital's theorem then takes the form given in Theorem 7.42. Note, however, that we don't require $f(x) \to \infty$ in our hypotheses, or even that $f(x)$ approaches any limit.

Theorem 7.42: *Let f and g be differentiable on a deleted neighborhood N of $x = a$. Suppose that*

(i) $\lim_{x \to a} g(x) = \infty$.

(ii) *For every $x \in N$ $g'(x) \neq 0$.*

(iii) $\lim_{x \to a} f'(x)/g'(x)$ *exists.*

Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

The analogous statements are valid if $a = \pm\infty$ or if $\lim_{x \to a} g(x) = -\infty$.

Proof. We prove the main part of Theorem 7.42 under the assumption that

$$\lim_{x\to a} f'(x)/g'(x)$$

is finite. The case that the limit is infinite as well as variants are left as Exercises 7.11.6 and 7.11.7. It suffices to consider the case of right-hand limits, the proof for left-hand limits being similar. Let

$$L = \lim_{x\to a+} f'(x)/g'(x).$$

We will show that if $p < L < q$, then there exists $\delta > 0$ such that

$$p < f(x)/g(x) < q$$

for $x \in (a, a+\delta)$. Since p and q are arbitrary (subject to the restriction $p < L < q$), we can then conclude

$$\lim_{x\to a+} f(x)/g(x) = L$$

as required.

Choose $r \in (L, q)$. By (iii) and the definition of L there exists δ_1 such that $f'(x)/g'(x) < r$ whenever $x \in (a, a+\delta_1)$. If $a < x < y < a+\delta_1$, then we infer from Theorem 7.21, Cauchy's form of the mean value theorem, and our assumption (ii) that there exists $c \in (x, y)$ such that

$$\frac{f(x)-f(y)}{g(x)-g(y)} = \frac{f'(c)}{g'(c)} < r. \tag{22}$$

Fix y in (22). Since $\lim_{x\to a+} g(x) = \infty$, there exists $\delta_2 > 0$ such that $a+\delta_2 < y$ and such that $g(x) > g(y)$ and $g(x) > 0$ if $a < x < a+\delta_2$. We then have

$$(g(x)-g(y))/g(x) > 0$$

for $x \in (a, a+\delta_2)$, so we can multiply both sides of the inequality (22) by $(g(x)-g(y))/g(x)$, obtaining

$$\frac{f(x)}{g(x)} < r - r\frac{g(y)}{g(x)} + \frac{f(y)}{g(x)} \text{ for } x \in (a, a+\delta_2). \tag{23}$$

Now let $x \to a+$. Then $g(x) \to \infty$ as $x \to a+$ by assumption (i). Since r, $g(y)$, and $f(y)$ are constants, the second and third terms on the right side of (23) approach zero. It now follows from the inequality $r < q$ that there exists $\delta_3 \in (0, \delta_2)$ such that

$$\frac{f(x)}{g(x)} < q \quad \text{whenever} \quad a < x < a + \delta_3. \tag{24}$$

In a similar fashion we find a $\delta_4 > 0$ such that

$$\frac{f(x)}{g(x)} > p \quad \text{whenever} \quad a < x < a + \delta_4.$$

If we let $\delta = \min(\delta_3, \delta_4)$, we have shown that

$$p < \frac{f(x)}{g(x)} < q \quad \text{whenever} \quad x \in (a, a + \delta).$$

Since p and q were arbitrary numbers satisfying $p < L < q$, our conclusion

$$\lim_{x \to a+} \frac{f(x)}{g(x)} = L = \lim_{x \to a+} \frac{f'(x)}{g'(x)}$$

follows. ■

Exercises

7.11.1 Consider the function $f(x) = (3^x - 2^x)/x$ defined everywhere except at $x = 0$.

(a) What value should be assigned to $f(0)$ in order that f be everywhere continuous?

(b) Does $f'(0)$ exist if this value is assigned to $f(0)$?

(c) Would it be correct to calculate $f'(0)$ by computing instead $f'(x)$ by the usual rules of the calculus and finding $\lim_{x \to 0} f'(x)$.

See Note 213

7.11.2 Suppose that f and g are defined in a deleted neighborhood of x_0 and that

$$\lim_{x\to x_0} f(x) = A \neq 0 \text{ and } \lim_{x\to x_0} g(x) = 0.$$

Show that

$$\lim_{x\to x_0} \left|\frac{f(x)}{g(x)}\right| = \infty.$$

SEE NOTE 214

7.11.3 Discuss the limiting behavior as $x \to 0$ for each of the following functions.

(a) $\dfrac{1}{x}$ (b) $\dfrac{1}{x^2}$

(c) $\dfrac{1}{\sin x}$ (d) $\dfrac{1}{x \sin x^{-1}}$

7.11.4 Evaluate each of the following limits.

(a) $\displaystyle\lim_{x\to 0} \frac{e^x - \cos x}{x}$

(b) $\displaystyle\lim_{t\to 0} \frac{\sin t - t}{t^3}$

(c) $\displaystyle\lim_{u\to 1} \frac{u^5 + 5u - 6}{2u^5 + 8u - 10}$

7.11.5 Let $f(x) = x^2 \sin x^{-1}$, $g(x) = x$. Show that

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = 0$$

but that

$$\lim_{x\to 0} \frac{f'(x)}{g'(x)}$$

does not exist.

7.11.6 The proof we provided for Theorem 7.42 required that $\lim_{x\to a} f'(x)/g'(x)$ be finite. Prove that the result holds if this limit is infinite.

7.11.7 Prove the part of Theorem 7.42 dealing with $a = \pm\infty$ or $\lim_{x\to a} g(x) = -\infty$.

7.11.8 Evaluate the following limits.

(a) $\displaystyle\lim_{x\to\infty} \frac{x^3}{e^x}$

(b) $\displaystyle\lim_{x\to\infty} \frac{\ln x}{x}$

(c) $\displaystyle\lim_{x\to 0+} x\ln x$

(d) $\displaystyle\lim_{x\to 0+} x^x$

7.11.9 This exercise gives information about the relative rates of increase of certain types of functions. Prove that for each positive number p,

$$\lim_{x\to\infty} \frac{\ln x}{x^p} = \lim_{x\to\infty} \frac{x^p}{e^x} = 0.$$

7.11.10 Give an example of functions f and g defined on $\mathbb{R}$ such that

$$\lim_{x\to\infty} g(x) = \infty,\ \limsup_{x\to\infty} f(x) = \infty,\ \liminf_{x\to\infty} f(x) = -\infty$$

and Theorem 7.42 applies.

SEE NOTE 215

7.12 Taylor Polynomials

Enrich.

Suppose f is continuous on an open interval I and $c \in I$. The constant function $g(x) = f(c)$ approximates f closely when x is sufficiently close to the point c, but may or may not provide a good approximation elsewhere. If f is differentiable on I, then we see from the mean value theorem (Theorem 7.20) that for each $x \in I$ $(x \neq c)$ there exists z between x and c such that

$$f(x) = f(c) + f'(z)(x - c).$$

The expression $R_0(x) = f'(z)(x-c) = f(x) - f(c)$ provides the size of the error obtained in approximating the function f by a constant function $P_0(x) = f(c)$. We can think of this as approximation by a zero-degree polynomial.

We do not expect a constant function to be a good approximation to a given continuous function in general. But our acquaintance with Taylor series (as presented in elementary calculus courses) suggests that if a function is sufficiently differentiable, it can be approximated well by polynomials of sufficiently high degree.

Suppose we wish to approximate f by a polynomial P_n of degree n. In order for the polynomial P_n to have a chance to approximate f well in a neighborhood of a point c, we should require

$$P_n(c) = f(c), P_n'(c) = f'(c), \ldots, P_n^{(n)}(c) = f^{(n)}(c).$$

In that case we at least guarantee that P_n "starts out" with the correct value, the correct rate of change, etc. to give it a chance to approximate f well in some neighborhood I of c. The test however is this. Write

$$f(x) = P_n(x) + R_n(x).$$

Is it true that the "error" or "remainder" $R_n(x)$ is small when $x \in I$?

In order to answer this sort of question, it would be useful to have workable forms for this error term $R_n(x)$. We present two forms for the remainder. The first is due to Joseph-Louis Lagrange (1736–1813), who obtained Theorem 7.43 in 1797. He used integration methods to prove the theorem. We provide a popular and more modern proof based on the mean value theorem.

Theorem 7.43 (Lagrange) *Let f possess at least $n+1$ derivatives on an open interval I and let $c \in I$. Let*

$$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

and let $R_n(x) = f(x) - P_n(x)$. Then for each $x \in I$ there exists z between x and c ($z = c$ if $x = c$) such that

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}.$$

Proof. Fix $x \in I$. Then there is a number M (depending on x, of course) such that

$$f(x) = P_n(x) + M(x-c)^{n+1}.$$

We wish to show that $M = (f^{(n+1)}(z))/(n+1)!$ for some z between x and c.

Consider the function g defined on I by

$$\begin{aligned} g(t) &= f(t) - P_n(t) - M(t-c)^{n+1} \\ &= R_n(t) - M(t-c)^{n+1}. \end{aligned}$$

Now P_n is a polynomial of degree at most n, so $P_n^{(n+1)}(t) = 0$ for all $t \in I$. Thus

$$g^{(n+1)}(t) = f^{(n+1)}(t) - (n+1)!M \quad \text{for all } t \in I. \tag{25}$$

Also, since $f^{(k)}(c) = P_n^{(k)}(c)$ for $k = 1, 2, \ldots, n$, we readily see that

$$g^{(k)}(c) = 0 \quad \text{for } k = 0, 1, 2, \ldots, n. \tag{26}$$

Suppose now that $x > c$, the case $x < c$ having a similar proof, and the case $x = c$ being obvious. We have chosen M in such a way that $g(x) = 0$ and, by (26), we see that $g(c) = 0$. Thus g satisfies the hypotheses of Rolle's theorem on the interval $[c, x]$. Therefore there exists a point $z_1 \in (c, x)$ such that $g'(z_1) = 0$.

Now apply Rolle's theorem to g' on the interval $[c, z_1]$, obtaining a point $z_2 \in (c, z_1)$ such that $g''(z_2) = 0$.

Continuing in this way we use (26) and Rolle's theorem repeatedly to obtain a point $z_n \in (c, z_{n-1})$ such that $g^{(n)}(z_n) = 0$. Finally, we apply Rolle's theorem to the function $g^{(n)}$ on the interval $[c, z_n]$. We obtain a point $z \in (c, z_n)$ such that $g^{(n+1)}(z) = 0$. From (25) we deduce

$$f^{(n+1)}(z) = (n+1)!M,$$

completing the proof. ■

Note. The function P_n is called the *nth Taylor polynomial for f*. You will recognize P_n as the nth partial sum of the Taylor series studied in elementary calculus. (See also Chapter 16.) The function R_n is called the *remainder* or *error* function between f and P_n. If P_n is to be a good approximation to f, then R_n must be small in absolute value.

Observe that $P_n(c) = f(c)$ and that

$$P_n^{(k)}(c) = f^{(k)}(c) \quad \text{for} \quad k = 0, 1, 2, \ldots, n.$$

Observe also that the mean value theorem is the special case of Theorem 7.43 obtained by taking $n = 0$: on the interval $[c, x]$ there is a point z with

$$f(x) - f(c) = f'(z)(x - c).$$

Lagrange's result expresses the error term R_n in a particular way. It provides a sense of the error in approximating f by P_n. Note that we do not get an exact statement of the error term since it is given in terms of the value $f^{(n+1)}(z)$ at *some* point z. But if we know a little bit about the function $f^{(n+1)}$ on the interval in question, we might be able to say that this error is not very large.

Example 7.44: Suppose we wish to approximate the function $f(x) = \sin x$ on the interval $[-a, a]$ by a Taylor polynomial of degree 3, with $c = 0$. Here

$$f'(x) = \cos x\ ,\ \ f''(x) = -\sin x\ ,\ \ f'''(x) = -\cos x \ \text{ and } f^{(4)}(x) = \sin x.$$

Thus

$$\begin{aligned} P_3(x) &= \cos(0)x - \frac{\sin(0)}{2!}x^2 - \frac{\cos(0)}{3!}x^3 = x - \frac{x^3}{3!} \quad \text{and} \\ R_3(x) &= \frac{\sin z}{4!}x^4 \quad \text{for some } z \text{ in } [-a, a]. \end{aligned}$$

The exact error depends on which z makes this all true. But since $|\sin z| \le 1$ for all z, we get immediately that

$$|R_3(x)| \le a^4/4! = a^4/24,$$

so P_3 approximates f to within $a^4/24$ on the interval $[-a, a]$. For a small, the approximation should be sufficient for the purposes at hand. For large a, a higher-degree polynomial can produce the desired accuracy, since

$$|R_n(x)| \le \frac{|x^{n+1}|}{(n+1)!}.$$

◄

Various other forms for the error term R_n are useful. The integral form is one of them. We state this form without proof. We assume that you are familiar with the integral as studied in calculus courses.

Theorem 7.45 (Integral Form of Remainder) *Suppose that the function f possesses at least $n+1$ derivatives on an open interval I and that $f^{(n+1)}$ is Riemann integrable on every closed interval contained in I. Let $c \in I$. Then*

$$R_n(x) = \frac{1}{n!}\int_c^x f^{(n+1)}(t)(x-t)^n\,dt \quad \textit{for all } x \in I.$$

We shall see this form of the remainder again in Chapter 16 when we study Taylor series.

Exercises

7.12.1 Exhibit the Taylor polynomial about $x = 0$ of degree n for the function $f(x) = e^x$. Find n so that $|R_n(x)| \le .0001$ for all $x \in [0, 2]$.

7.12.2 Show that if f is a polynomial of degree n, then it is its own Taylor polynomial of degree n with $c = 0$.

7.12.3 Calculate the Taylor polynomial of degree 5 with $c = 1$ for the functions $f(x) = x^5$ and $g(x) = \ln x$.

7.12.4 Let $f(x) = \frac{1}{x+2}$, $c = -1$, and $n = 2$. Show that

$$\frac{1}{x+2} = 1 - (x+1) + (x+1)^2 + R_3$$

where, for some z between x and -1,

$$R_3 = -\frac{(x+1)^3}{(2+z)^4}.$$

7.12.5 Let $f(x) = \ln(1+x)$, $c = 0$, and $(x > -1)$. Show that

$$f(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots + (-1)^{n-1}\frac{x^n}{n} + R_n$$

where

$$R_n = \frac{(-1)^n}{n+1}\left(\frac{x}{1+z}\right)^{n+1}$$

for some z between 0 and x. Estimate R_n on the interval $[0, 1/10]$.

7.12.6 Just because a function possesses derivatives of all orders on an interval I does not guarantee that some Taylor polynomial approximates f in a neighborhood of some point of I. Let

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

(a) Show that f has derivatives of all orders and that $f^{(k)}(0) = 0$ for each $k = 0, 1, 2, \ldots$.

(b) Write down the polynomial P_n with $c = 0$.

(c) Write down Lagrange's form for the remainder of order n. Observe its magnitude and take the time to understand why P_n is not a good approximation for f on any interval I, no matter how large n is.

7.13 Challenging Problems for Chapter 7

7.13.1 **(Straddled derivatives)** Let $f:\mathbb{R}\to\mathbb{R}$ and let $x_0\in\mathbb{R}$. Prove that f is differentiable at x_0 if and only if

$$\lim_{u\to x_0-,\, v\to x_0+}\frac{f(v)-f(u)}{v-u}$$

exists (finite), and, in this case, $f'(x_0)$ equals this limit.

7.13.2 **(Unstraddled Derivatives)** Let $f:\mathbb{R}\to\mathbb{R}$ and let $x_0\in\mathbb{R}$. We say f is strongly differentiable at x_0 if

$$\lim_{u\to x_0,\, v\to x_0,\, u\neq v}\frac{f(v)-f(u)}{v-u}$$

exists.

(a) Show that a differentiable function need not be strongly differentiable everywhere.

(b) Show that a strongly differentiable function must be differentiable.

(c) If f is strongly differentiable at a point x_0 and differentiable in a neighborhood of x_0, show that f' must be continuous there.

7.13.3 Let p be a polynomial of the nth degree that is everywhere nonnegative. Show that

$$p(x)+p'(x)+p''(x)+\cdots+p^{(n)}(x)\geq 0$$

for all x.

See Note 216

7.13.4 Suppose that f is continuous on $[0,1]$, differentiable on $(0,1)$, and $f(0)=0$ and $f(1)=1$. For every integer n show that there must exist n distinct points $\xi_1, \xi_2, \ldots, \xi_n$ in that interval so that

$$\sum_{k=1}^{n}\frac{1}{f'(\xi_k)}=n.$$

7.13.5 Show that there exists precisely one real number α with the property that for every function f differentiable on $[0,1]$ and satisfying $f(0)=0$ and $f(1)=1$ there exists a number ξ in $(0,1)$ (which depends, in general, on f) so that

$$f'(\xi)=\alpha\xi.$$

7.13.6 Let f be a continuous function. Show that the set of points where f is differentiable but not strongly differentiable (as defined in Exercise 7.13.2) is of the first category.

7.13.7 Let f be a continuous function on an open interval I. Show that f is convex on I if and only if

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}.$$

SEE NOTE 217

7.13.8 **(Wronskians)** The Wronskian of two differentiable functions f and g is the determinant

$$W(f,g) = \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix}.$$

Prove that if $W(f,g)$ does not vanish on an interval I and $f(x_1) = f(x_2) = 0$ for points $x_1 < x_2$ in I, then there exists $x_3 \in (x_1, x_2)$ such that $g(x_3) = 0$. [The functions $f(x) = \sin x, g(x) = \cos x$ furnish an example.]

SEE NOTE 218

7.13.9 ✂ Let f be a continuous function on an open interval I. Show that f is convex if and only if

$$\limsup_{h\to 0} \frac{f(x+h)+f(x-h)-2f(x)}{h^2} \geq 0$$

for every $x \in I$.

SEE NOTE 219

7.13.10 ✂ Let f be continuous on an interval (a,b).

(a) Prove that the four Dini derivates of f and the difference quotient $\frac{f(y)-f(x)}{y-x}$ $(x \neq y \in (a,b))$ have the same bounds.

(b) Prove that if one of the Dini derivates is continuous at a point x_0, then f is differentiable at x_0.

(c) Show by example that the statements in the first two parts can fail for discontinuous functions.

7.13.11 ✂ **(Denjoy-Young-Saks Theorem)** The theorem with this name is a far-reaching theorem relating the four Dini derivates D^+f, D_+f, D^-f and D_-f. It was proved independently by an English mathematician, Grace Chisolm Young (1868–1944), and a French mathematician, Arnaud Denjoy (1884–1974), for continuous functions in 1916 and 1915 respectively. Young then extended the result to a larger class of functions called measurable functions. Finally, the Polish mathematician Stanislaw Saks (1897–1942) proved the theorem for all real-valued functions in 1924. Here is their theorem.

> **Theorem (Denjoy-Young-Saks)** Let f be an arbitrary finite function defined on $[a, b]$. Then except for a set of measure zero every point $x \in [a, b]$ is in one of four sets:
>
> (1) A_1 on which f has a finite derivative.
>
> (2) A_2 on which $D^+f = D_-f$ (finite), $D^-f = \infty$ and $D_+f = -\infty$.
>
> (3) A_3 on which $D^-f = D_+f$ (finite), $D^+f = \infty$ and $D_-f = -\infty$.
>
> (4) A_4 on which $D^-f = D^+f = \infty$ and $D_-f = D_+f = -\infty$.

(a) Sketch a picture illustrating points in the sets A_2, A_3 and A_4. To which set does $x = 0$ belong when $f(x) = \sqrt{|x|}\sin x^{-1}$, $f(0) = 0$?

(b) Use the Denjoy-Young-Saks theorem to prove that an increasing function f has a finite derivative except on a set of measure zero.

(c) Use the Denjoy-Young-Saks theorem to show that if all derived numbers of f are finite except on a set of measure zero, then f is differentiable except on a set of measure zero.

(d) Use the Denjoy-Young-Saks theorem to show that, for every finite function f, the set $\{x : f'(x) = \infty\}$ has measure zero.

7.13.12 Let f be a continuous function on an interval $[a, b]$ with a second derivative at all points in (a, b). Let $a < x < b$. Show that there exists a point $\xi \in (a, b)$ so that

$$\frac{\frac{f(x)-f(a)}{x-a} - \frac{f(b)-f(a)}{b-a}}{x-b} = \tfrac{1}{2}f''(\xi).$$

SEE NOTE 220

7.13.13 Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function with $f(0) = 0$ and suppose that $|f'(x)| \le |f(x)|$ for all $x \in \mathbb{R}$. Show that f is identically zero.

SEE NOTE 221

7.13.14 Let $f:\mathbb{R}\to\mathbb{R}$ have a third derivative that exists at all points. Suppose that

$$\lim_{x\to\infty} f(x)$$

exists and that

$$\lim_{x\to\infty} f'''(x) = 0.$$

Show that

$$\lim_{x\to\infty} f'(x) = \lim_{x\to\infty} f''(x) = 0.$$

See Note 222

7.13.15 Let f be defined on an interval I of length at least 2 and suppose that f'' exists there. If $|f(x)| \leq 1$ and $|f''(x)| \leq 1$ for all $x \in I$ show that $|f'(x)| \leq 2$ on the interval.

See Note 223

7.13.16 Let $f:\mathbb{R}\to\mathbb{R}$ be infinitely differentiable and suppose that

$$f\left(\frac{1}{n}\right) = \frac{n^2}{n^2+1}$$

for all $n = 1, 2, 3, \dots$. Determine the values of

$$f'(0),\ f''(0),\ f'''(0),\ f^{(4)}(0), \dots.$$

See Note 224

7.13.17 Let $f:\mathbb{R}\to\mathbb{R}$ have a third derivative that exists at all points. Show that there must exist at least one point ξ for which

$$f(\xi)f'(\xi)f''(\xi)f'''(\xi) \geq 0.$$

See Note 225

Notes

[161]Exercise 7.2.1. Write $x = x_0 + h$.

[162]Exercise 7.2.6. Write

$$f(x+h) - f(x-h)$$
$$= [f(x+h) - f(x)] + [f(x) - f(x-h)].$$

[163]Exercise 7.2.7. Use

$$1 - \cos x = 2\sin^2 x/2.$$

When you take the square root be sure to use the absolute value.

[164]Exercise 7.2.12. Just use the definition of the derivative. Give a counterexample with $f(0) = 0$ and $f'(0) > 0$ but so that f is not increasing in any interval containing 0.

[165]Exercise 7.2.13. Even for polynomials, $p(x)$ increasing does not imply that $p'(x) > 0$ for all x. For example, take $p(x) = x^3$. That has only one point where the derivative is not positive. Can you do any better?

[166]Exercise 7.2.14. Actually the assumptions are different. Here we assume $f'(x_0)$ does exist, whereas in the trapping principle we had to assume more inequalities to deduce that it exists.

[167]Exercise 7.2.15. Review Exercise 5.10.3 first.

[168]Exercise 7.2.16. Advanced (very advanced) methods would allow you to find a function continuous on $[0, 1]$ that is differentiable at *no* point of that interval. For the purpose of this exercise just try to find one that is not differentiable at $1/2$, $1/3$, $1/4$, ... (Novices constructing examples often feel they need to give a simple formula for functions. Here, for example, you can define the function on $[1/2, 1]$, then on $[1/4, 1/2]$, then on $[1/8, 1/4]$, and so on ... and then finally at 0.)

[169]Exercise 7.2.18. Find two examples of functions, one continuous and one discontinuous at 0, with an infinite derivative there.

[170]Exercise 7.2.19. Imitate the proof of Theorem 7.6. Find a counterexample to the question.

[171]Exercise 7.3.5. Use Theorem 7.7 (the product rule) and for the induction step consider

$$\frac{d}{dx}x^n = \frac{d}{dx}[x][x^{n-1}].$$

[172]Exercise 7.3.10. This formula is known as Leibniz's rule (which should indicate its age since Leibniz, one of the founders of the calculus, was born in 1646). It extends both Exercises 7.3.8 and 7.3.9. The formula is

$$(fg)^{(n)}(x_0)$$

$$= \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} f^{(k)}(x_0) g^{(n-k)}(x_0).$$

[173]Exercise 7.3.11. Consider a sequence $x_n \to x_0$ with $x_n \neq x_0$ and $f(x_n) = f(x_0)$.

[174]Exercise 7.3.12. Let

$$f(x) = x^2 \sin x^{-1}$$

$(f(0) = 0)$ and take $x_0 = 0$. Utilize the fact that 0 is a limit point of the set $\{x : f(x) = 0\}$.

[175]Exercise 7.3.17. If $I(x)$ is the inverse function then $I(\sin x) = x$. The chain rule gives derivative as $I'(\sin x) = 1/\cos x$. This needs some work. Use

$$\cos x = \sqrt{1 - \sin^2 x}$$

and obtain

$$I'(\sin x) = \frac{1}{\sqrt{1 - \sin^2 x}}.$$

Now replace the $\sin x$ by some other variable. Caution: While doing this exercise make sure that you know how the arcsin function $\sin^{-1} x$ is actually defined. It is not the inverse of the function $\sin x$ since that function has no inverse.

[176]Exercise 7.3.19. Draw a good picture. The graph of $y = g(x)$ is the reflection in the line $y = x$ of the graph of $y = f(x)$. What is the slope of the reflected tangent line?

[177]Exercise 7.3.21. Use the idea in the example. If $f(x) = x^{1/m}$, then $[f(x)]^m = x$ and use the chain rule. If

$$F(x) = x^{n/m},$$

then

$$[F(x)]^m = x^n$$

and use the chain rule.

[178]Exercise 7.3.22. Once you know that

$$\frac{d}{dx}e^x = e^x$$

you can determine that

$$\frac{d}{dx}\ln x = 1/x$$

using inverse functions. Then consider $x^p = e^{p(\ln x)}$.

[179]Exercise 7.3.23. The formula you should obtain is

$$a_k = \frac{p^{(k)}(0)}{k!}$$

for $k = 0, 1, 2, \dots$.

[180]Exercise 7.3.24. If you succeed, then you have proved the binomial theorem using derivatives. Of course, you need to compute $p(0)$, $p'(0)$, $p''(0)$, $p'''(0)$, ...to do this.

[181]Exercise 7.5.6. Define sets A_n consisting of all x for which $f(t) < f(x)$ for all $0 < |x - t| < \frac{1}{n}$ and observe that $\bigcup_{n=1}^{\infty} A_n$ is the set in question.

[182]Exercise 7.5.7. Modify the hint in Exercise 7.5.6.

[183]Exercise 7.6.3. Use Rolle's theorem to show that if x_1 and x_2 are distinct solutions of $p(x) = 0$, then between them is a solution of $p'(x) = 0$.

[184]Exercise 7.6.5. Use Rolle's theorem twice. See Exercise 7.6.7 for another variant on the same theme.

[185]Exercise 7.6.6. Since f is continuous we already know (look it up) that f maps $[a, b]$ to some closed bounded interval $[c, d]$. Use Rolle's theorem to show that there cannot be two values in $[a, b]$ mapping to the same point.

[186]Exercise 7.6.7. cf. Exercise 7.6.5.

[187]Exercise 7.6.8. First show directly from the definition that the Lipshitz condition will imply a bounded derivative. Then use the mean value theorem to get the converse, that is, apply the mean value theorem to f on the interval $[x, y]$ for any $a \leq x < y \leq b$.

[188]Exercise 7.6.9. Note that an increasing function f would allow only positive numbers in S.

[189]Exercise 7.6.12. Apply the mean value theorem to f on the interval $[x, x+a]$ to obtain a point ξ in $[x, x+a]$ with

$$f(x+a) - f(x) = af'(\xi).$$

[190]Exercise 7.6.13. Use the mean value theorem to compute

$$\lim_{x \to a+} \frac{f(x) - f(a)}{x - a}.$$

[191]Exercise 7.6.14. This is just a variant on Exercise 7.6.13. Show that under these assumptions f' is continuous at x_0.

[192]Exercise 7.6.15. Use the mean value theorem to relate

$$\sum_{i=1}^{\infty} (f(i+1) - f(i))$$

to

$$\sum_{i=1}^{\infty} f'(i).$$

Note that f is increasing and treat the former series as a telescoping series.

[193]Exercise 7.6.16. The proof of the mean value theorem was obtained by applying Rolle's theorem to the function

$$g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a).$$

For this mean value theorem apply Rolle's theorem twice to a function of the form

$$h(x) = f(x) - f(a) - f'(a)(x - a) - \alpha(x - a)^2$$

for an appropriate number α.

[194]Exercise 7.6.18. Write

$$f(x + h) + f(x - h) - 2f(x) =$$
$$[f(x + h) - f(x)] + [f(x - h) - f(x)]$$

and apply the mean value theorem to each term.

[195]Exercise 7.6.21. Let $\phi(x)$ be

$$\begin{vmatrix} f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \\ f(x) & g(x) & h(x) \end{vmatrix}$$

and imitate the proof of Theorem 7.21.

[196]Exercise 7.7.1. Interpret as a monotonicity statement about the function

$$f(x) = (1 - x)e^x.$$

[197]Exercise 7.7.3. We do not assume differentiability at b. For example, this would apply to the function $f(x) = |x|$, with $b = 0$.

[198]Exercise 7.7.5. Interpret this as a monotonicity property for the function $F(x) = f(x)/x$. We need to show that F' is positive. Show that this is true if $f'(x) > f(x)/x$ for all x. But how can we show this? Apply the mean value theorem to f on the interval $[0, x]$ (and don't forget to use the hypothesis that f' is an increasing function).

[199]Exercise 7.7.6. If not, there is an interval $[a, b]$ with $f(a) = f(b) = 0$ and neither f nor g vanish on (a, b). Show that $f(x)/g(x)$ is monotone (increasing or decreasing) on $[a, b]$.

[200]Exercise 7.8.7. Let $\varepsilon > 0$ and consider $f(x) + \varepsilon x$.

[201]Exercise 7.8.9. Figure out a way to express $\mathbb{R}$ as a countable union of disjoint dense sets A_n and then let $f(x) = n$ for all $x \in A_n$. For an example subtract an appropriate linear function F from f such that $f - F$ is not an increasing function, and apply Theorem 7.30.

[202]Exercise 7.8.10. In connection with this exercise we should make this remark. If $A = \{a_k\}$ is any countable set, then the function defined by the series

$$\sum_{k=0}^{\infty} \frac{-|x - a_k|}{2^k}$$

has $D^+f(x) < D_-f(x)$ for all $x \in A$. This can be verified using the results in Chapter 14 on uniform convergence.

[203]Exercise 7.9.1. For the third part use the function $F(x) = x^2 \sin x^{-1}$, $F(0) = 0$ to show that there exists a differentiable function f such that $f'(x) = \cos x^{-1}$, $f(0) = 0$. Consider $g(x) = f(x) - x^3$ on an appropriate interval.

[204]Exercise 7.9.3. If either FG' or GF' were a derivative, so would the other be since

$$(FG)' = FG' + GF'.$$

In that case $FG' - GF'$ is also a derivative. But now show that this is impossible [because of (c)].

[205]Exercise 7.9.4. Use $fg' = (fg)' - f'g$. You need to know the fundamental theorem of calculus to continue.

[206]Exercise 7.9.5. If f' is continuous, then it is easy to check that E_α is closed. In the opposite direction suppose that every E_α is closed and f' is not continuous. Then show that there must be a number β and a sequence of points $\{x_n\}$ converging to a point z and yet $f'(x_n) \geq \beta$ and $f'(z) < \beta$. Apply the Darboux property of the derivative to show that this cannot happen if E_β is closed. Deduce that f' is continuous.

[207]Exercise 7.10.3. If f is convex on an interval I and g is convex *and also nondecreasing* on the interval $f(I)$, then you should be able to prove that $g \circ f$ is also convex. Show also that if the monotonicity assumption on g is dropped this might not be true.

[208]Exercise 7.10.5. Show that at every point of continuity of f'_+ the function is differentiable. How many discontinuities does the (nondecreasing) function f'_+ have?

[209]Exercise 7.10.10. Give an example of a convex function on the interval $(0,1)$ that is not bounded above; that answers the first question. For the second question use Exercise 7.10.4 to show that f must be bounded below.

[210]Exercise 7.10.13. The methods of Chapter 14 would help here. There we learn in general how to check for the differentiability of functions defined by series. For now just use the definitions and compute carefully.

[211]Exercise 7.10.14. For (d) let

$$f(x) = \begin{cases} e^{-1/x^2}(\sin 1/x)^2, & \text{for } x > 0 \\ 0, & \text{for } x = 0, \\ -e^{-1/x^2}(\sin 1/x)^2, & \text{for } x < 0 \end{cases}.$$

The three definitions in the exercise are not equivalent even for infinitely differentiable functions. They are, however, equivalent for *analytic* functions; that is, functions represented by power series (a topic we cover in Chapter 16). Since the scope of elementary calculus is more or less limited to functions that are analytic on the intervals on which the functions are concave up or down, we might argue that on that level, the definition to take is the one that is simplest to develop. We should mention, however, that there are differentiable functions that are not concave-up or concave-down on any interval!

[212]Exercise 7.10.15. Order the terms so that

$$x_1 \leq x_2 \leq \cdots \leq x_n.$$

And write

$$p = \sum_{k=1}^{n} \alpha_k x_k.$$

Choose a number M between $f'_-(p)$ and $f'_+(p)$. Check that

$$x_1 \leq p \leq x_n.$$

Check that

$$f(x_k) \geq M(x_k - p) + f(p)$$

for $k = 1, 2, \ldots, n$. Now use these inequalities to obtain Jensen's inequality.

[213]Exercise 7.11.1. Use L'Hôpital's rule to find that $f(0)$ should be $\ln(3/2)$. Use the definition of the derivative and L'Hôpital's rule twice to compute

$$f'(0) = [(\ln 3)^2 - (\ln 2)^2]/2.$$

Exercise 7.6.13 shows that the technique in (c) part does in fact compute the derivative provided only that you can show that this limit exists.

[214]Exercise 7.11.2. Treat the cases $A > 0$ and $A < 0$ separately.

[215]Exercise 7.11.10. We must have $\lim_{x\to\infty} f'(x) = 0$ in this case. (Why?)

[216]Exercise 7.13.3. Consider the function

$$H(x) = p(x) + p'(x) + p''(x) + \cdots + p^{(n)}(x)$$

and note, in particular, the relation between H, H' and p.

[217]Exercise 7.13.7. Such functions are called *midpoint convex*. By the definition of convexity we need to show that if $x_1, x_2 \in I$ and $\alpha \in [0,1]$, then the inequality

$$f(\alpha x_1 + (1-\alpha)x_2) \leq \alpha f(x_1) + (1-\alpha)f(x_2)$$

is satisfied. Use the midpoint convexity condition to show that this is true whenever α is a fraction of the form $p/2^q$ for integers p and q. Now use continuity to show that it holds for all $\alpha \in [0,1]$. Without continuity this argument fails and, indeed, there exist discontinuous midpoint convex functions that fail to be convex. [For an extensive account of what is known about such conditions, see B. S. Thomson, *Symmetric Properties of Real Functions*, Marcel Dekker, (New York, 1994).]

[218]Exercise 7.13.8. If g does not vanish on (x_1, x_2), then Rolle's theorem applied to the quotient f/g provides a contradiction. Incidentally, Josef de Wronski (1778–1853), whose name was attached firmly to this concept in 1882 in a multivolume *History of Determinants*, was a rather curious figure whom you are unlikely to encounter in any other context. One biographer writes about him:

For many years Wronski's work was dismissed as rubbish. However, a closer examination of the work in more recent times shows that, although some is wrong and he has an incredibly high opinion of himself and his ideas, there is also some mathematical insights of great depth and brilliance hidden within the papers.

[219]Exercise 7.13.9. Consider the function

$$H(x) = f(x) + cx^2 + ax + b$$

for $c > 0$ and various choices of lines $y = ax + b$ and make use of Exercise 7.10.14.

[220]Exercise 7.13.12. This is from the 1939 Putnam Mathematical Competition.

[221]Exercise 7.13.13. This is from the 1946 Putnam Mathematical Competition.

[222]Exercise 7.13.14. This is from the 1958 Putnam Mathematical Competition.

[223]Exercise 7.13.15. This is from the 1962 Putnam Mathematical Competition.

[224]Exercise 7.13.16. This is from the 1992 Putnam Mathematical Competition.

[225]Exercise 7.13.17. This is from the 1998 Putnam Mathematical Competition.

Chapter 8

THE CALCULUS INTEGRAL

Dripped Chapter[1]

In this chapter we study the integral

$$\int_a^b f(x)\,dx$$

of a function f defined on a compact interval $[a, b]$. We can restrict attention throughout to *continuous* functions or, more generally, functions with at most a *finite number of discontinuities*.

This puts some considerable limitations on how far the theory can go and would be inadequate for most serious applications of integrals. We consider this the "calculus version of integration theory" because, for all practical purposes, this is all that a freshman course in calculus usually manages to impart about integration theory. There is normally an attempt at presenting Riemann's integration theory (which applies also to some badly discontinuous functions) but the average student ends up with the narrow interpretation

[1] Note to the instructor: For a *truly elementary* course this solitary ***dripped*** chapter could be used. The integration theory would then revert to just the calculus (or Newton) integral on the real line. The argument for this is that the Riemann integration theory does not do much, in any case, to prepare the student for advanced courses. Material for an analysis course with minimal ambitions as regards integration theory can be tailored to avoid the Riemann integral altogether except as an historical fact.

For a less elementary course this ***dripped*** chapter is to be considered motivation for the genuine integration theory on the real line that follows in subsequent chapters.

below anyway. The later ***dripped*** chapters will develop the correct integration that is used for all modern applications. This chapter alone, however, is a useful (if minimal) introduction to the integral.

♠ INTEGRATION THEORY FOR THIS CHAPTER:
If f is a function defined for all but finitely many points on an interval $[a,b]$ then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

means only that there must exist a continuous function F on that interval so that $F'(x) = f(x)$ for all but finitely many points x for which $a < x < b$.

Such a function F is called a *primitive* for f on the interval $[a,b]$, or in many calculus classes an *indefinite integral* of f.

This working definition ♠ is our basis for defining the calculus integral. It would not be inappropriate to use this exclusively in a freshman calculus class, since this applies to exactly the class of functions that are discussed in such courses anyway.

We will discover in the course of this chapter that:

1. *All* continuous functions possess an integral in this sense.

2. *Some* discontinuous functions possess an integral in this sense.

For this chapter let us work with this definition. Later, in subsequent chapters, we will enlarge the scope of the theory to handle many more discontinuous functions. One advantage arises immediately: if integration theory is thus simply based on a statement about derivatives, then we can prove all we need to prove by just using properties of derivatives.

Many of the integration exercises of this chapter can be handled by what we already know about derivatives.

8.1 The calculus integral of continuous functions

The following describes the reality of integration theory for most calculus classes:

- f is a continuous function on a compact interval $[a, b]$.
- This means there must exist a continuous function $F : [a, b] \to \mathbb{R}$ with the property that
$$F'(x) = f(x) \text{ for all } a < x < b.$$
- Then the value of the integral is determined by computing
$$\int_a^b f(x)\,dx = F(b) - F(a).$$

We are insisting here, for the calculus student, on four things:

1. The integration theory will be restricted *initially at least* to continuous functions.
2. We require an assurance that all continuous functions do possess antiderivatives.
3. We acknowledge that we do not possess methods that will find explicit formulas for antiderivatives of *all* continuous functions.
4. We accept, *as calculus students*, that the only meaning assigned to the integral expression $\int_a^b f(x)\,dx$ is this statement about antiderivatives.

There are two theorems essential to proceeding with this approach.

8.1.1 Uniqueness of the integral

We assign the integral the meaning

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

by agreeing to use some function F that serves as a primitive for f, i.e., F is continuous on $[a,b]$ and $F'(x) = f(x)$ for every $a < x < b$ with perhaps finitely many exceptions. Such primitive functions are not unique so we must be sure that the value of the integral does not depend on which primitive we choose.

Theorem 8.1: *Suppose that $f : [a,b] \to \mathbb{R}$ and that there are functions F, $G : [a,b] \to \mathbb{R}$ for f, so that F and G are continuous on $[a,b]$ and*

$$F'(x) = G'(x) = f(x)$$

for every $a < x < b$. Then

$$F(b) - F(a) = G(b) - F(a).$$

More generally our working definition of an integral ♠ requires the following variant.

Theorem 8.2: *Suppose that $f : [a,b] \to \mathbb{R}$ and that there are functions F, $G : [a,b] \to \mathbb{R}$ for f, so that F and G are continuous on $[a,b]$ and*

$$F'(x) = G'(x) = f(x)$$

for all but finitely many points x in (a,b). Then

$$F(b) - F(a) = G(b) - F(a).$$

Both of these theorems follow from the mean-value theorem and the reader is asked in Exercise 8.1.1 and 8.1.2 to supply the details.

8.1.2 Existence of antiderivatives

Suppose that $f : [a,b] \to \mathbb{R}$ is continuous. Then the integral $\int_a^b f(x)\,dx$ must exist. To be assured of this we need only prove that every such function has a primitive. This is supplied by the following existence theorem. This is important for several reasons:

1. We wish to know that our integration theory applies to a large class of functions [all continuous functions].

2. We wish to discuss $\int_a^b f(x)\,dx$ even in situations where we are unable to exhibit a primitive function for f explicitly.

The theorem actually proves a little more than we need to handle continuous functions.

Theorem 8.3: *Suppose that $f : (a,b) \to \mathbb{R}$ is a bounded function. Then there exists a continuous function $F : [a,b] \to \mathbb{R}$ so that $F'(x) = f(x)$ for every $a < x < b$ at which f is continuous.*

Proof. It will be enough to assume that $f : (0,1) \to \mathbb{R}$ and that f is nonnegative. (Exercises 8.1.3 and 8.1.4 ask for the justifications for this assumption.) Let F_0 denote the function on $[0,1]$ that has $F_0(0) = 0$ and has constant slope equal to

$$c_{01} = \sup\{f(t) : 0 < t < 1\}.$$

Subdivide $[0,1]$ into $[0,\frac{1}{2}]$ and $[\frac{1}{2},1]$ and let F_1 denote the continuous, piecewise linear function on $[0,1]$ that has $F_0(0) = 0$ and has constant slope equal to

$$c_{11} = \sup\{f(t) : 0 < t \leq \tfrac{1}{2}\}$$

on $[0,\frac{1}{2}]$ and constant slope equal to

$$c_{12} = \sup\{f(t) : \tfrac{1}{2} \leq t < 1\}$$

on $[0, \frac{1}{2}]$. This construction is continued. For example, at the next stage, Subdivide $[0, 1]$ further into $[0, \frac{1}{4}]$, $[\frac{1}{4}, \frac{1}{2}]$, $[\frac{1}{2}, \frac{3}{4}]$, and $[\frac{3}{4}, 1]$. Let F_2 denote the continuous, piecewise linear function on $[0, 1]$ that has $F_0(0) = 0$ and has constant slope equal to $c_{11} = \sup\{f(t) : 0 < t \leq \frac{1}{4}\}$ on $[0, \frac{1}{4}]$, constant slope equal to $c_{12} = \sup\{f(t) : \frac{1}{4} \leq t \leq \frac{1}{2}\}$ on $[\frac{1}{4}, \frac{1}{2}]$, constant slope equal to $c_{13} = \sup\{f(t) : \frac{1}{2} \leq t \leq \frac{3}{4}\}$ on $[\frac{1}{2}, \frac{3}{4}]$, and constant slope equal to $c_{14} = \sup\{f(t) : \frac{3}{4} \leq t < 1\}$ on $[\frac{3}{4}, 1]$.

In this way we construct a sequence of such functions $\{F_n\}$. Note that each F_n is continuous and nondecreasing. Moreover a look at the geometry reveals that

$$F_n(x) \geq F_{n+1}(x)$$

for all $0 \leq x \leq 1$ and all $n = 0, 1, 2, \ldots$. In particular $\{F_n(x)\}$ is a nonincreasing sequence of nonnegative numbers and consequently

$$F(x) = \lim_{n\to\infty} F_n(x)$$

exists for all $0 \leq x \leq 1$. We prove that $F'(x) = f(x)$ at all points x in $(0, 1)$ at which the function f is continuous.

Fix a point x in $(0, 1)$ at which f is assumed to be continuous and let $\varepsilon > 0$. Choose $\delta > 0$ so that the oscillation

$$\omega f([x - 2\delta, x + 2\delta])$$

of f on the interval $[x - 2\delta, x + 2\delta]$ does not exceed ε. Let h be fixed so that $0 < h < \delta$. Choose an integer N sufficiently large that

$$|F_N(x) - F(x)| < \varepsilon h \quad \text{and} \quad |F_N(x + h) - F(x + h)| < \varepsilon h.$$

From the geometry of our construction notice that the inequality

$$|F_N(x + h) - F_N(x) - f(x)h| \leq h\omega f([x - 2h, x + 2h]),$$

must hold for large enough N. (Simply observe that the graph of F_N will be composed of line segments, each of whose slopes differ from $f(x)$ by no more than $\omega f([x - 2h, x + 2h])$.)

Putting these inequalities together we find that

$$|F(x+h)-F(x)-f(x)h| \leq |F_N(x+h)-F_n(x)-f(x)h|+|F_N(x)-F(x)|+|F_N(x+h)-F(x+h)| < 3\varepsilon h.$$

This shows that the right-hand derivative of F at x must be exactly $f(x)$. A similar argument will handle the left-hand derivative and we have verified the statement in the theorem about the derivative.

We leave it as an entertainment for the reader (Exercise 8.1.5) to check that the function F defined here is continuous at every point of $[0,1]$. ■

Corollary 8.4: *Suppose that $f:(a,b)\to\mathbb{R}$ is bounded and has a finite number of discontinuity points. Then the calculus integral*

$$\int_a^b f(x)\,dx$$

must exist, i.e., f has a continuous primitive on $[a,b]$.

Exercises

8.1.1 Show that there cannot be two numbers that would be assigned to the symbol $\int_a^b f(x)\,dx$ for a continuous function f (i.e., that the value is independent of which antiderivative one happens to find).

SEE NOTE 226

8.1.2 Show that there cannot be two numbers that would be assigned to the symbol $\int_a^b f(x)\,dx$ for *any* function f (i.e., show that if F, $G:[a,b]\to\mathbb{R}$ are continuous functions for which $F'(x)=G'(x)$ for all but finitely many points x in (a,b), then $F(b)-F(a)=G(b)-G(a)$.).

SEE NOTE 227

8.1.3 Suppose that $f:[a,b]\to\mathbb{R}$ and set $g(t)=f(a+t(b-a))$ for all $0\leq t\leq 1$. If G is a primitive for g on $[0,1]$ show how to find a primitive for f on $[a,b]$.

8.1.4 Suppose that $f:[a,b]\to\mathbb{R}$ is a bounded function and that $K=\inf\{f(x): a<x<b\}$. Set $g(t)=f(t)-K$ for all $a\leq t\leq b$. Show that g is nonnegative and bounded. Suppose that G is a primitive for g on $[a,b]$; show how to find a primitive for f on $[a,b]$.

8.1.5 In the proof of Theorem 8.3 we did not trouble ourselves to check that the function F is continuous at all points of the interval $[0,1]$. Give the necessary details to show this.

SEE NOTE 228

8.1.6 If f is constant and $f(x) = \alpha$ for all x in $[a,b]$ show that

$$\int_a^b f(x)\,dx = \alpha(b-a).$$

8.1.7 If f is continuous and $f(x) \geq 0$ for all x in $[a,b]$ show that

$$\int_a^b f(x)\,dx \geq 0.$$

8.1.8 If f is continuous and $m \leq f(x) \leq M$ for all x in $[a,b]$ show that

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a).$$

8.1.9 Calculate $\int_0^1 x^p\,dx$ (for whatever values of p you can manage).

8.1.10 (Additive Property) Let f be continuous on $[a,c]$ and suppose that $a < b < c$. Then

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx.$$

8.1.11 (Linear Property) Let f and g be continuous on $[a,b]$. Then, for all r, $s \in \mathbb{R}$,

$$\int_a^b [rf(x) + sg(x)]\,dx = r\int_a^b f(x)\,dx + s\int_a^b g(x)\,dx.$$

8.1.12 (Monotone Property) Let f and g be continuous on $[a,b]$. Then, if $f(x) \leq g(x)$ for all $a \leq x \leq b$,

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$

8.1.13 If f is continuous on an interval $[a,b]$ and

$$\|f\|_\infty = \max\{|f(x)| : x \in [a,b]|\}$$

show that

$$\left|\int_a^b f(x)\,dx\right| \leq \|f\|_\infty (b-a).$$

8.1.14 If f is continuous on an interval $[a,b]$ show that

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx.$$

8.1.15 (Mean-Value Theorem for Integrals) If f is continuous show that there is a point ξ in (a,b) so that

$$\int_a^b f(x)\,dx = f(\xi)(b-a).$$

8.1.16 If f is continuous and $m \leq f(x) \leq M$ for all x in $[a,b]$ show that

$$m\int_a^b g(x)\,dx \leq \int_a^b f(x)g(x)\,dx \leq M\int_a^b g(x)\,dx$$

for any continuous, nonnegative function g.

8.1.17 If f is continuous and nonnegative on an interval $[a,b]$ and

$$\int_a^b f(x)\,dx = 0$$

show that f is identically equal to zero there.

8.1.18 (Second Mean-Value Theorem for Integrals) If f and g are continuous on an interval $[a,b]$ and g is nonnegative, show that there is a number $\xi \in (a,b)$ such that

$$\int_a^b f(x)g(x)\,dx = f(\xi)\int_a^b g(x)\,dx.$$

8.1.19 If f is continuous on an interval $[a,b]$ and

$$\int_a^b f(x)g(x)\,dx = 0$$

for every continuous function g on $[a,b]$ show that f is identically equal to zero there.

8.1.20 (Integration by parts) Suppose that f, g, f' and g' are continuous on $[a,b]$. Establish the formula

$$\int_a^b f(x)g'(x)\,dx = [f(b)g(b) - f(a)g(a)] - \int_a^b f'(x)g(x)\,dx.$$

8.1.21 (Integration by substitution) State conditions on f and g so that the formula

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(s)\,ds$$

is valid.

8.1.22 State conditions on f, g and h so that this version of an integration by substitution formula

$$\int_a^b f(g(h(x)))g'(h(x))h'(x)\,dx = \int_{g(h(a))}^{g(h(b))} f(s)\,ds$$

is valid.

8.1.23 (Cauchy-Schwarz inequality) If f and g are continuous on an interval $[a,b]$ show that

$$\left(\int_a^b f(x)g(x)\,dx\right)^2 \leq \left(\int_a^b [f(x)]^2\,dx\right)\left(\int_a^b [g(x)]^2\,dx\right).$$

SEE NOTE 229

8.1.3 The "improper" calculus integral

The calculus integral, *as we have defined it*, applies to all functions that have a primitive. We know that if $f : [a,b] \to \mathbb{R}$ is continuous then it must have a primitive. Thus the integral handles all continuous functions.

Consider the integral

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx.$$

The function

$$f(x) = \frac{1}{\sqrt{x}}$$

integrated here is not continuous on all of $[0,1]$. (Nor is it defined at every point of the interval $[0,1]$ but that is not a concern in our theory). We know that bounded functions with only a finite number of discontinuity points can be integrated. But this function is *unbounded*.

To be assured that it has an integral we would have to *find* a primitive since, for the moment, we have no theory assuring us that one must exist.

But there is no difficulty here since ordinary calculus methods suffice. Simply compute

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = 2\sqrt{1} - 2\sqrt{0} = 2.$$

The justification is that the function $F(x) = 2\sqrt{x}$ is continuous on $[0,1]$ and satisfies

$$F'(x) = \frac{1}{\sqrt{x}} \quad (0 < x < 1).$$

Calculus students are normally encouraged instead to use the following procedure:

- $$\int_t^1 \frac{1}{\sqrt{x}}\,dx = 2\sqrt{1} - 2\sqrt{t} \quad (0 < t < 1),$$
  which is valid because $f(x) = \frac{1}{\sqrt{x}}$ is continuous on $[t,1]$.

- $\lim_{t\to 0+}\left(2\sqrt{1} - 2\sqrt{t}\right) = 2.$

- Claim that
$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0+}\int_t^1 \frac{1}{\sqrt{x}}\,dx = 2$$
but now interpreted as an *improper calculus integral.*

We see that our definition of an integral includes this procedure without the fuss of taking a limit (although we do have to check the continuity of the primitive function, which is the same thing). This is true in general. Thus improper integrals can be easily avoided in the calculus by using the calculus integral in its place.[2]

Theorem 8.5: *Let $f:(a,b)\to\mathbb{R}$ be a continuous function. Suppose that*
$$\lim_{h\to 0+}\lim_{k\to 0+}\int_{a+h}^{b-k} f(x)\,dx$$
exists, where the integral is taken as a calculus integral of a continuous function on the interval $[a+h,b-k]$. Then the integral exists and
$$\int_a^b f(x)\,dx = \lim_{h\to 0+}\lim_{k\to 0+}\int_{a+h}^{b-k} f(x)\,dx.$$

Exercises

8.1.1 Calculate $\int_0^1 x^p\,dx$ (for whatever values of p you can manage).

[2]Often calculus courses work with the improper Riemann integral, even if only in theory. That integral is included in the integration theory of the subsequent chapters. Thus the "improper" procedure can be avoided in general, and simply reinterpreted as a check for continuity of a proposed indefinite integral.

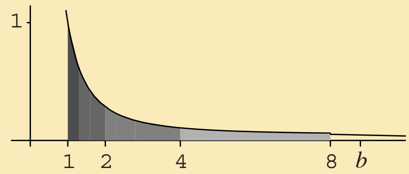

Figure 8.1. Computation of $\int_1^\infty x^{-2}\,dx = 1$.

8.2 The infinite integral

How should we interpret the integral

$$\int_1^\infty \frac{dx}{x^2}?$$

The method suggested by Cauchy is quite compelling. Consider this example:

$$\int_1^\infty \frac{dx}{x^2} = \lim_{X\to\infty}\int_1^X \frac{dx}{x^2} = \lim_{X\to\infty}\left(1-\frac{1}{X}\right) = 1.$$

In Figure 8.1 we show graphically how to interpret the area that is represented by $\int_1^\infty x^{-2}\,dx$. Note that

$$\int_1^2 x^{-2}\,dx = 1/2,\ \int_2^4 x^{-2}\,dx = 1/4,\ \int_4^8 x^{-2}\,dx = 1/8$$

and so we would expect

$$\int_1^\infty x^{-2}\,dx = 1/2 + 1/4 + 1/8 + \cdots = 1$$

as indeed this method does give. (See Figure 8.1.)

We give a formal definition valid just for an infinite interval of the form $[a,\infty)$. The case $(-\infty,b]$ is similar. The case $(-\infty, +\infty)$ is best split up into the sum of two integrals, from $(-\infty, a]$ and $[a, \infty)$, each of which can be handled in this fashion. (See Exercise 8.2.2.)

Definition 8.6: Let f be a continuous function on an interval $[a, \infty)$. Then we define

$$\int_a^\infty f(x)\,dx$$

to be

$$\lim_{X\to\infty}\int_a^X f(x)\,dx$$

if this limit exists, and in this case the integral is said to be *convergent*. If both integrals

$$\int_a^\infty f(x)\,dx \quad\text{and}\quad \int_a^\infty |f(x)|\,dx$$

converge the integral is said to be *absolutely convergent*.

The role of the extra condition of absolute convergence is much like its role in the study of infinite series. Note that the integral $\int_1^\infty x^{-2}\,dx$ is convergent and also absolutely convergent merely because the integrand is nonnegative.

Exercises

8.2.1 Formulate a definition of the integral $\int_{-\infty}^b f(x)\,dx$ for a function continuous on $(-\infty, b]$. Supply examples of convergent and divergent integrals of this type.

8.2.2 Formulate a definition of the integral $\int_{-\infty}^\infty f(x)\,dx$ for a function continuous on $(-\infty, \infty)$. Supply examples of convergent and divergent integrals of this type.

See Note 230

8.2.3 For what values of p is the integral $\int_1^\infty x^{-p}\,dx$ convergent?

8.2.4 Show that
$$\int_0^\infty x^n e^{-x}\,dx = n!.$$

8.2.5 Let f be a continuous function on $[1,\infty)$ such that $\lim_{x\to\infty} f(x) = \alpha$. Show that if the integral $\int_1^\infty f(x)\,dx$ converges, then α must be 0.

8.2.6 Let f be a continuous function on $[1,\infty)$ such that the integral $\int_1^\infty f(x)\,dx$ converges. Can you conclude that $\lim_{x\to\infty} f(x) = 0$?

8.2.7 Let f be a continuous, decreasing function on $[1,\infty)$. Show that the integral $\int_1^\infty f(x)\,dx$ converges if and only if the series $\sum_{n=1}^\infty f(n)$ converges.

8.2.8 Give an example of a function f continuous on $[1,\infty)$ so that the integral $\int_1^\infty f(x)\,dx$ converges but the series $\sum_{n=1}^\infty f(n)$ diverges.

8.2.9 Give an example of a function f continuous on $[1,\infty)$ so that the integral $\int_1^\infty f(x)\,dx$ diverges but the series $\sum_{n=1}^\infty f(n)$ converges.

8.2.10 Show that
$$\int_0^\infty \frac{\sin x}{x}\,dx$$
is convergent but not absolutely convergent.

See Note 231

8.2.11 (Cauchy Criterion for Convergence) Let $f : [a,\infty) \to \mathbb{R}$ be a continuous function. Show that the integral $\int_a^\infty f(x)\,dx$ converges if and only if for every $\varepsilon > 0$ there is a number M so that, for all $M < c < d$,
$$\left| \int_c^d f(x)\,dx \right| < \varepsilon.$$

8.2.12 (Cauchy Criterion for Absolute Convergence) Let $f : [a,\infty) \to \mathbb{R}$ be a continuous function. Show that the integral $\int_a^\infty f(x)\,dx$ converges absolutely if and only if for every $\varepsilon > 0$ there is a number M so that, for all $M < c < d$,
$$\int_c^d |f(x)|\,dx < \varepsilon.$$

8.2.13 As a project determine which of the properties of the integral in the previous set of exercises (which apply only to continuous functions on a finite interval) can be extended to integrals on an infinite interval $[a, \infty)$. Give proofs.

8.3 Cauchy's analysis of the integral

To evaluate an integral

$$\int_a^b f(x)\,dx$$

we must first find an antiderivative F and then compute $F(b) - F(a)$. Any antiderivative will do, but if we cannot compute one how should we then determine the integral? To this point the only thing we know about this integration method is expressed by the antiderivative. We would like to determine the value of the integral more directly from the values of f on the interval $[a, b]$.

The first clue is the mean-value theorem for integrals which is merely a rewriting of the usual mean-value theorem of the calculus.

8.3.1 First mean-value theorem for integrals

Theorem 8.7: *Let f be a bounded function that is continuous on an interval (a, b). Then there is at least one point $a < \xi < b$ for which*

$$\int_a^b f(x)\,dx = f(\xi)(b-a).$$

We already know that there is a primitive function F for f on $[a, b]$, i.e., a continuous function for which $F(x) = f(x)$ at every point of the open interval (a, b). Thus this follows directly from the usual mean-value theorem.

This shows that *some* value of f can be used to compute the integral, but there is no method for finding that particular value $f(\xi)$ that works without first finding $F(b) - F(a)$. But this defeats our reason for asking the question in the first place.

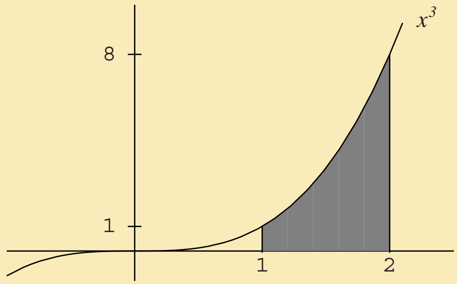

Figure 8.2. Region bounded by $x = 1$, $x = 2$, $y = x^3$, and $y = 0$.

8.3.2 The method of exhaustion

The development of the integral by Newton has been described here by the calculus integral that dominates the chapter.

Cauchy's development of the integral as a limit of Riemann sums came much later, but has its origins, most likely, in the geometry of the ancient Greeks, who had long ago described a method for computing areas of regions enclosed by curves. This *method of exhaustion* involves computing the areas of simpler figures (squares, triangles, rectangles) that approximate the area of the region.

Consider the example

$$\int_1^2 x^3\,dx$$

interpreted as an area. The region is that bounded on the left and right by the lines $x = 1$ and $x = 2$, below by the line $y = 0$, and above by the curve $y = x^3$. (See Figure 8.2.)

Using the method of exhaustion, we may place this figure inside a collection of rectangles by dividing the interval $[1, 2]$ into n equal sized subintervals each of length $1/n$. This means selecting the points

$$1,\ 1 + 1/n,\ 1 + 2/n,\ \ldots,\ 1 + (n-1)/n$$

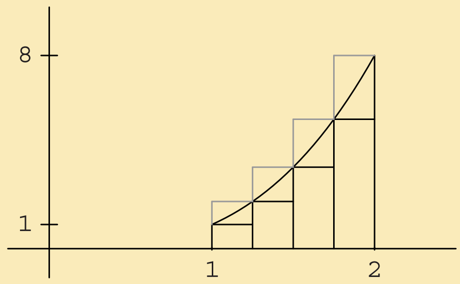

Figure 8.3. Method of exhaustion ($n = 4$).

and constructing rectangles with the height of the rectangles determined by right hand endpoints. The total area of these rectangles exceeds the true area and is precisely

$$\sum_{k=1}^{n}(1+k/n)^3(1/n).$$

The method of exhaustion requires a lower estimate as well and the true area of the region must be greater than

$$\sum_{k=1}^{n}(1+(k-1)/n)^3(1/n).$$

(See Figure 8.3 for an illustration with $n = 4$.)

The method of exhaustion requires us to show that as n increases *both* approximations, the upper one and the lower one, approach the same number. Cauchy saw that, because of the continuity of the function $f(x) = x^3$, these limits would be the same. More than that, any choice of points ξ_k from the interval

$[1 + (k-1)/n, 1 + (k)/n]$ would have the property that

$$\lim_{n\to\infty} \sum_{k=1}^{n} {\xi_k}^3 (1/n)$$

would exist.

This procedure, borrowed heavily from the Greeks, leads to the same methods that we will describe in the next section (arising from mean-value considerations). Since this will work for any continuous function, it offered to Cauchy a way to *define* the calculus integral

$$\int_a^b f(x)\,dx$$

for any continuous function f without any reference whatsoever to notions of derivatives or antiderivatives. The key ingredients here are first dividing the interval $[a, b]$ by a finite sequence of points

$$a = x_0 < x_1 < x_2 < x_3 < \cdots < x_{n-1} < x_n = b,$$

thus forming a collection of nonoverlapping subintervals with associated points called *a partition* of $[a, b]$

$$([x_0, x_1], \xi_1),\ ([x_1, x_2], \xi_2),\ \ldots,\ ([x_{n-1}, x_n], \xi_n)$$

(it is not important that the intervals have equal size, just that they get small). Then we form the sums

$$\sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1}) \tag{1}$$

with respect to this partition. It is an unfortunate trick of fate that the sums (1) that originated with Cauchy are called *Riemann sums* because of Riemann's later (much later) use of them in defining his integral.

8.3.3 Riemann sums

We show how a calculus integral

$$\int_a^b f(x)\,dx$$

of a continuous function f can be interpreted using Riemann sums.

We subdivide the interval $[a,b]$ into a sequence of subintervals $[a_{k-1},a_k]$ $(k=1,2,\dots,n)$ thus:

$$a = a_0 < a_1 < a_2 < \dots a_{n-1} < a_n = b.$$

Then in each subinterval the first mean-value theorem for integrals will provide a point

$$a_{k-1} \le \xi_k \le a_k$$

for which

$$\int_{a_{k-1}}^{a_k} f(x)\,dx = f(\xi_k)(a_{k-1}-a_k).$$

But then we must have, from elementary properties of the calculus integral, that

$$\int_a^b f(x)\,dx = \sum_{k=1}^n \int_{a_{k-1}}^{a_k} f(x)\,dx$$

$$= \sum_{k=1}^n f(\xi_k)(a_{k-1}-a_k).$$

We call the collection

$$\pi = \{([a_{k-1},a_k],\xi_k) : k = 1,2,\dots,n\}$$

a *partition* of the interval $[a,b]$. The corresponding sum

$$\sum_{k=1}^n f(\xi_k)(a_{k-1}-a_k)$$

is called a *Riemann sum* over the partition π.

We have proved the following weak theorem which demonstrates (perhaps in a peculiar manner) that every calculus integral of a continuous function can be obtained exactly as the value of *some* Riemann sum.

Theorem 8.8: *Let f be a bounded continuous function on an interval (a, b). Subdivide the interval $[a, b]$ into any sequence of subintervals $[a_{k-1}, a_k]$ ($k = 1, 2, \ldots, n$) thus:*

$$a = a_0 < a_1 < a_2 < \ldots a_{n-1} < a_n = b.$$

Then there is a choice of partition

$$\pi = \{([a_{k-1}, a_k], \xi_k) : k = 1, 2, \ldots, n\}$$

so that

$$\int_a^b f(x)\,dx = \sum_{k=1}^{n} f(\xi_k)(a_{k-1} - a_k).$$

8.3.4 The integral of continuous functions as a limit of Riemann sums

We have seen that every calculus integral of a continuous function can be written exactly as a Riemann sum, although there is no method available for choosing the right partition. The key idea, due to Cauchy, is that for *continuous* functions the choice of the points ξ_k in the partitions is not so critical. Other points that are close by will alter the Riemann sum by only a small amount. That means any integral of a continuous function can be obtained by an *approximation* using Riemann sums.

Theorem 8.9: *Let f be a continuous function on an interval $[a,b]$ and let $\varepsilon > 0$. Then there is a positive number δ with the following property. Subdivide the interval $[a,b]$ into any sequence of subintervals $[a_{k-1}, a_k]$ $(k = 1, 2, \ldots, n)$ thus:*

$$a = a_0 < a_1 < a_2 < \ldots a_{n-1} < a_n = b$$

so that each $a_k - a_{k-1} < \delta$. Then for any choice of partition

$$\pi = \{([a_{k-1}, a_k], \xi_k) : k = 1, 2, \ldots, n\}$$

we have

$$\left| \int_a^b f(x)\,dx - \sum_{k=1}^{n} f(\xi_k)(a_{k-1} - a_k) \right| < \varepsilon.$$

Proof. If f is continuous, then it is uniformly continuous (Theorem 5.48). Choose $\delta > 0$ small enough that

$$|f(x) - f(y)| < \frac{\varepsilon}{b-a}$$

whenever x and y are points of $[a,b]$ with $|x - y| < \delta$. Then

$$\left| \sum_{k=1}^{n} f(\xi'_k)(a_{k-1} - a_k) - \sum_{k=1}^{n} f(\xi_k)(a_{k-1} - a_k) \right| =$$

$$\left| \sum_{k=1}^{n} [f(\xi'_k) - f(\xi_k)](a_{k-1} - a_k) \right| < \sum_{k=1}^{n} \frac{\varepsilon(a_{k-1} - a_k)}{b-a} = \varepsilon$$

for any choices of ξ_k and ξ'_k from $[a_{k-1}, a_k]$. The proof is completed by making the selection of the ξ'_k so that the Riemann sum is exactly the integral:

$$\int_a^b f(x)\,dx = \sum_{k=1}^{n} f(\xi'_k)(a_{k-1} - a_k)$$

which is possible because of Theorem 8.8. ■

A special case of this theorem allows us to compute an integral as a limit of a sequence. In practice this is seldom the best way to compute it, but it is interesting and useful in some parts of the theory. More sophisticated numerical methods do much the same thing.

Corollary 8.10: *Let f be a continuous function on an interval $[a,b]$. Then*

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{k=0}^{n-1} f\left(a + \frac{k}{n}(b-a)\right).$$

8.3.5 The calculus integral as a limit of Riemann sums

We have seen that every calculus integral of a continuous function can be written as a limit of Riemann sums. The same is true even for discontinuous functions but some caution is needed for the kind of limit. Functions integrated by this method may be badly discontinuous and a single positive number δ is not refined enough to describe how small our partitions must be.

Theorem 8.11: *Let $f : [a,b] \to \mathbb{R}$ possess a calculus integral, in fact suppose that f has a primitive $F : [a,b] \to \mathbb{R}$ for which $F'(x) = f(x)$ for* every *point $x \in (a,b)$. Then for every $\varepsilon > 0$ and every x in $[a,b]$ there is a positive number $\delta(x)$ with the following property: For any choice of partition of the interval $[a,b]$*

$$\pi = \{([a_{k-1}, a_k], \xi_k) : k = 1, 2, \dots, n\}$$

we have

$$\left| \int_a^b f(x)\,dx - \sum_{k=1}^{n} f(\xi_k)(a_{k-1} - a_k) \right| < \varepsilon$$

provided that each $a_k - a_{k-1} < \delta(\xi_k)$.

Proof. We use $\int_a^b f(x)\,dx = F(b) - F(a)$. Let $\varepsilon > 0$ and choose $0 < \eta(b-a+2) < \varepsilon$. Define $\delta(x)$ so that

1. if $x = a$ and $[a, v] \subset [a, b]$ with $v - a < \delta(a)$ then
$$|f(a)|(v-u) + |F(v) - F(a)| < \eta.$$

2. if $x = b$ and $[u, b] \subset [a, b]$ with $b - u < \delta(b)$ then
$$|f(b)|(b-u) + |F(b) - F(u)| < \eta.$$

3. if $a < x < b$ and $[u, v] \subset [a, b]$ then
$$|F(v) - F(u) - f(x)(v-u)| \leq \eta(v-u).$$

This uses the differentiation formula $F'(x) = f(x)$ at the interior points of $[a, b]$ and the continuity of F at the endpoints.

Now suppose we have a partition of the interval $[a, b]$
$$\pi = \{([a_{k-1}, a_k], \xi_k) : k = 1, 2, \dots, n\}$$
for which $a_k - a_{k-1} < \delta(\xi_k)$. We check
$$\left|\int_a^b f(x)\,dx - \sum_{k=1}^n f(\xi_k)(a_{k-1} - a_k)\right| =$$
$$\left|F(b) - F(a) - \sum_{k=1}^n f(\xi_k)(a_{k-1} - a_k)\right| =$$
$$\left|\sum_{k=1}^n [F(a_k) - F(a_{k-1})] - \sum_{k=1}^n f(\xi_k)(a_{k-1} - a_k)\right| \leq$$
$$\sum_{k=1}^n \left|[F(a_k) - F(a_{k-1})] - f(\xi_k)(a_{k-1} - a_k)\right|.$$

Consider the n-terms of this sum: if $\xi_1 = a$ or $\xi_n = b$ then we can estimate the corresponding term by the way in which we defined $\delta(a)$ and $\delta(b)$. For example, if $\xi_1(a) = a$, then the first term is

$$|[F(a_1) - F(a)] - f(a)(a_1 - a)| \leq |F(a_1) - F(a)| + |f(a)|(a_1 - a) < \eta.$$

All remaining terms have $a < \xi_k < b$ and we can estimate the sum of these by the way in which we have defined $\delta(\xi_k)$. In particular, we should arrive at the estimate

$$\left| \int_a^b f(x)\,dx - \sum_{k=1}^{n} f(\xi_k)(a_{k-1} - a_k) \right| \leq \eta(b-a) + 2\eta < \varepsilon$$

as required. ■

The theorem just proved used a primitive function that was differentiable *everywhere* inside the interval. With a slight adjustment we can show that all calculus integrals have the same property.

Corollary 8.12: *Suppose that $f : [a, b] \to \mathbb{R}$ has a calculus integral. Then for every $\varepsilon > 0$ and every x in $[a, b]$ there is a positive number $\delta(x)$ with the following property: For any choice of partition of the interval $[a, b]$*

$$\pi = \{([a_{k-1}, a_k], \xi_k) : k = 1, 2, \dots, n\}$$

we have

$$\left| \int_a^b f(x)\,dx - \sum_{k=1}^{n} f(\xi_k)(a_{k-1} - a_k) \right| < \varepsilon$$

provided that each $a_k - a_{k-1} < \delta(\xi_k)$.

Exercises

8.3.1 To complete the computations in this section, show that

$$\lim_{n\to\infty} \sum_{k=1}^{n} (1 + (k)/n)^3 (1/n) = 15/4.$$

SEE NOTE 232

8.3.2 If f is constant and $f(x) = \alpha$ for all x in $[a, b]$ use limits of Riemann sums to show that

$$\int_a^b f(x)\,dx = \alpha(b-a).$$

8.3.3 If f is continuous and $f(x) \geq 0$ for all x in $[a, b]$ use limits of Riemann sums to show that

$$\int_a^b f(x)\,dx \geq 0.$$

8.3.4 If f is continuous and $m \leq f(x) \leq M$ for all x in $[a, b]$ show use limits of Riemann sums that

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a).$$

8.3.5 Calculate $\int_0^1 x^p\,dx$ (for whatever values of p you can manage) using limits of Riemann sums by partitioning $[0, 1]$ into subintervals of equal length.

8.3.6 Calculate $\int_a^b x^p\,dx$ (for whatever values of p you can manage) using limits of Riemann sums by partitioning $[a, b]$ into subintervals $[a, aq]$, $[aq, aq^2]$, ..., $[aq^{n-1}, b]$ where $aq^n = b$. (Note that the subintervals are not of equal length, but that the lengths form a geometric progression.)

8.3.7 Use the method of the preceding exercise to show that

$$\int_1^2 \frac{dx}{x^2} = \frac{1}{2}$$

and check it by the usual calculus method.

8.3.8 Compute the Riemann sums for the integral $\int_a^b x^{-2}\,dx$ $(a > 0)$ taken over a partition

$$[x_0, x_1],\ [x_1, x_2],\ \ldots,\ [x_{n-1}, x_n]$$

of the interval $[a, b]$ and with associated points $\xi_i = \sqrt{x_i x_{i-1}}$. What can you conclude from this?

SEE NOTE 233

8.3.9 Compute the Riemann sums for the integral $\int_a^b x^{-1/2}\,dx$ $(a > 0)$ taken over a partition

$$[x_0, x_1],\ [x_1, x_2],\ \ldots,\ [x_{n-1}, x_n]$$

of the interval $[a, b]$ and with associated points

$$\xi_i = \left(\frac{\sqrt{x_i} + \sqrt{x_{i-1}}}{2}\right)^2.$$

What can you conclude from this?

8.3.10 Show that

$$\lim_{n\to\infty} n\left\{\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \frac{1}{(n+3)^2} + \cdots + \frac{1}{(2n)^2}\right\} = \frac{1}{2}.$$

8.3.11 Calculate

$$\lim_{n\to\infty} \frac{e^{1/n} + e^{2/n} + \cdots + e^{(n-1)/n} + e^{n/n}}{n}$$

by expressing this limit as a definite integral of some continuous function and then using calculus methods.

8.3.12 Express

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right)$$

as a definite integral where f is continuous on $[0, 1]$.

8.3.13 For a bounded function f and any partition π

$$[x_0, x_1],\ [x_1, x_2],\ \ldots,\ [x_{n-1}, x_n]$$

of the interval $[a, b]$ write

$$M(f, \pi) = \sum_{k=1}^{n} \sup\{f(x) : x \in [x_{k-1}, x_k]\}(x_k - x_{k-1})$$

and

$$m(f, \pi) = \sum_{k=1}^{n} \inf\{f(x) : x \in [x_{k-1}, x_k]\}(x_k - x_{k-1})$$

These are called the *upper sums* and *lower sums* for the partition for the function f.

(a) Show that if π_2 contains all of the points of the partition π_1, then
$$m(f,\pi_1) \leq m(f,\pi_2) \leq M(f,\pi_2) \leq M(f,\pi_1).$$

(b) Show that if π_1 and π_2 are arbitrary partitions and f is any bounded function, then
$$m(f,\pi_1) \leq M(f,\pi_2).$$

(c) Show that if π is any arbitrary partition and f is any bounded function on $[a,b]$ then
$$c(b-a) \leq m(f,\pi) \leq M(f,\pi) \leq C(b-a)$$
where $C = \sup f$ and $c = \inf f$.

(d) Show that with any choice of associated points the Riemann sum over a partition π is in the interval $[m(f,\pi), M(f,\pi)]$.

(e) Show that, if f is continuous, every value in the interval between $m(f,\pi)$ and $M(f,\pi)$ is equal to some particular Riemann sum over the partition π with an appropriate choice of associated points ξ_k.

(f) Show that if f is not continuous the preceding assertion may be false.

(g) Prove Corollary 8.12. (You will need to accommodate a finite number of points at which the primitive may not have a derivative.)

8.4 Extensions of the integral

The calculus integral is sufficiently broad to handle all of the problems that arise in most elementary analysis courses. But for advanced applications the theory has severe limitations. The difficulty is in the second of the statements defining the calculus integral:

1. F is continuous on $[a,b]$,

2. $F'(x) = f(x)$ for all but finitely many points in (a,b),

3. $\int_a^b f(x)\,dx = F(b) - F(a)$.

The phrase "finitely many points" needs to be relaxed, to allow certain infinite sets of points x where the derivative $F'(x) = f(x)$ is allowed to fail. But how should one do this?

8.4.1 Riemann's integral

The method of Cauchy was taken by Riemann as a definition of an integral

$$\int_a^b f(x)\,dx.$$

In effect Riemann converted Theorem 8.9 into a definition of an integral which would then be guaranteed to integrate at least all continuous functions.

He dropped the assumption that the function f must be continuous (which then guarantees that the limit of the Riemann sums must exist). He then defined a function to be *integrable* should the limit of the Riemann sums happen to exist. This defines a class of functions that is evidently larger than the class of continuous functions and for which an integration theory can be developed.

But the Riemann integral and the calculus integral have an uneasy relationship. Should a function possess an integral in both senses then the same value would be assigned. But there are functions that possess a calculus integral and yet are not Riemann integrable, and similarly there are Riemann integrable functions that do not have a calculus integral in our sense.

Integration theory was much impeded in the 19th century by a broad acceptance of Riemann's definition. It should, however, be dropped from modern curricula as it is far too weak and constraining a theory for most purposes. We shall not mention it again except in its historical context.

8.4.2 Lebesgue's integral

At the beginning of the twentieth-century Lebesgue developed an approach to integration theory that completely changed the perspective. He demonstrated that the problem of integrating bounded functions on an interval $[a,b]$ is equivalent to the problem of developing a measure theory for subsets of $[a,b]$. The connection is the statement

$$m(E) = \int_a^b \chi_E(x)\,dx \tag{2}$$

where $\chi_E(x)$ is the characteristic function of a set $E \subset [a,b]$ and $m(E)$ is the "measure" of that set.

One direction is clear. If an integral is defined for a broad class of bounded functions, then a measure is defined by (2) for a large class of sets.

The converse direction is less transparent, but not difficult to follow. If a measure m is available then an integral can be defined, first, for all linear combinations of characteristic functions:

$$\int_a^b \left(\sum_{i=1}^n c_i \chi_{E_i}(x) \right) dx = \sum_{i=1}^n c_i m(E_i).$$

From that an integral can then be defined for all bounded functions that can be uniformly approximated by linear combinations of characteristic functions.

Thus Lebesgue's program was to develop the integral in a completely different manner than had been done before. Rather than taking a calculus approach (inverting a derivative) or to take Riemann's approach (integral as a limit of Riemann sums) his program, in broad outline, looks like this:

1. Construct a measure theory for a large class of subsets of an interval $[a, b]$.

2. Develop properties of the measure.

3. Use that measure to define an integral.

4. Develop properties of the integral.

5. Show that that measure-theoretic integral includes the Riemann integral and includes the calculus integral of bounded functions.

In this text we do not follow Lebesgue's program, but we will reproduce all of the theory of the program. It has been considered for a long time that the development of the measure theory as a first step in integration theory is a necessary step, but one that is difficult to learn. For that reason many instructors have avoided teaching a correct version of integration theory on the real line altogether.

8.4.3 The Henstock-Kurzweil extension

Half a century after Lebesgue's integration theory was introduced it was noticed that Riemann's methods suffered, in fact, from only a minor defect. The ε, δ definition of the Riemann integral is clearly a uniform definition, insisting on using partitions that are uniformly small (i.e., the same δ at all points). A pointwise definition is more general, but not any more difficult. Simply adjust the Riemann definition (for all ε there is a $\delta \dots$) to allow a pointwise treatment (for all ε and all $x \in [a,b]$ there is a $\delta(x) \dots$).

The difference is technically very minor. In place of a single positive δ as used in expressing the calculus integral as a limit of Riemann sums (Section 8.3.4) we shall use a positive function $\delta(x)$ as was used in expressing the calculus integral as a limit of Riemann sums (Section 8.3.5). In effect we turn Corollary 8.12 into an appropriate definition of an integral more general than the calculus integral.

This is the approach to integration theory taken in the rest of the text. We do not, however, express these notions in the ε, $\delta(x)$ language but use instead the language of covering relations.

8.4.4 An extended calculus integral

The following describes a more general version[3] of the integration theory ♠ developed in this chapter:

- f is a real-valued function defined at all but countably many points of a compact interval $[a,b]$.
- f is the derivative of some continuous function in this sense: there exists a continuous function $F : [a,b] \to \mathbb{R}$ with the property that $F'(x) = f(x)$ for all $a < x < b$ with at most a countable number of exceptions.
- Then the value of the integral is determined by computing

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

[3]This version of the calculus (Newton's) integral is featured in the textbook **Mathematical Analysis I** by Elias Zakon, available from the Trillia Group at the website http://www.trillia.com/zakon-analysisI.html.

This version of the integral is not ideal but it does remove the defect of the calculus integral in that finite exceptional sets are now replaced by infinite (but countable) exceptional sets. It may be hard to imagine that one needs more, but countable exceptional sets are still too limited for advanced applications. In later chapters we will go beyond this integral and develop the full theory of the ordinary integral on the real line. Even so this variant has its uses in teaching the methods of the calculus.

To use this integral we require the following lemma that shows that the value of the integral $F(b)-F(a)$ does not depend on the particular choice of indefinite integral F.

Lemma 8.13: *Let f, F, $G : [a,b] \to \mathbb{R}$ be functions and suppose that F and G are continuous. Let $F'(x) = f(x)$ for all x in (a,b) with countably many exceptions and let $G'(x) = f(x)$ for all x in (a,b) with countably many exceptions. Then*

$$F(b)-F(a) = G(b)-G(a).$$

Proof. We suppose that $F'(x) = f(x)$ for $x \in (a,b)$ except possibly at points of the sequence $\{c_j\}$. We suppose that $G'(x) = f(x)$ for $x \in (a,b)$ except possibly at points of the sequence $\{d_j\}$. Write $H(x) = F(x) - G(x)$. Then H is continuous on $[a,b]$ and $H'(x) = 0$ for $x \in (a,b)$ except possibly at points of the sequences $\{c_j\}$ and $\{d_j\}$.

Let $\varepsilon > 0$. Let $\mathcal{C}_1$ be the collection of all intervals $[u,v] \subset [a,b]$, with the property that either $u = a$ or else $v = b$ and

$$|H(v) - H(u)| < \varepsilon/2.$$

Let $\mathcal{C}_2$ be the collection of all intervals $[u,v] \subset [a,b]$, with the property that for some $j = 1, 2, 3, \dots$, $[u,v]$ contains a point c_j or d_j and

$$|H(v) - H(u)| < \varepsilon 2^{-j-1}.$$

Let $\mathcal{C}_3$ be the collection of all intervals $[u,v] \subset [a,b]$, with the property that

$$\left| \frac{H(v) - H(u)}{v - u} \right| < \varepsilon.$$

Note that the collection $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$ satisfies the hypotheses of Cousin's lemma (Lemma 4.26) on the interval $[a, b]$. At any point $x = a$, $x = b$, $x = c_i$, or $x = d_i$ this is because H is continuous (Exercise 8.4.1). At any other point x this is because $H'(x) = 0$ (Exercise 8.4.2).

Consequently, by Cousin's lemma we may select a partition

$$a = x_0 < x_1 < \cdots < x_n = b$$

of $[a, b]$ such that $[x_{i-1}, x_i] \in \mathcal{C}$ for $i = 1, \dots, n$. We simply decide in each case whether $[x_{i-1}, x_i]$ is in $\mathcal{C}_1$, $\mathcal{C}_2$, or $\mathcal{C}_3$. Then we have the estimate

$$|H(b) - H(a)| = \left| \sum_{i}^{n} H(x_i) - H(x_{i-1}) \right|$$

$$\leq \sum_{i}^{n} |H(x_i) - H(x_{i-1})| \leq \varepsilon + 2 \sum_{j=1}^{\infty} \varepsilon 2^{-j-1} + \varepsilon(b - a) = \varepsilon[1 + 1 + (b - a)].$$

Since $\varepsilon > 0$ is arbitrary it follows that $H(b) - H(a) = 0$ and consequently that $F(b) - F(a) = G(b) - G(a)$.
■

Exercises

8.4.1 In the proof of Lemma 8.13 check that there is a $\delta > 0$ so that $\mathcal{C}_1$ contains all intervals $[a, v]$ and all interval $[u, b]$ for which $0 < v - a < \delta$ and $0 < b - u < \delta$.

8.4.2 In the proof of Lemma 8.13 check that for each $x \in (a, b)$ at which $H'(x) = 0$ there is a $\delta > 0$ so that $\mathcal{C}_3$ contains all intervals $[u, v] \subset [a, b]$ for which $0 < v - u < \delta$.

8.4.3 Give an example of a function that possesses an integral in the sense just given but does not have a calculus integral according to our definition ♠.

8.5 Challenging Problems for Chapter 8

8.5.1 Here is an argument claiming that Riemann and Cauchy's method will handle all derivatives, not just continuous derivatives. We take a completely naive approach and start with the definition of the derivative itself. If $F' = f$ everywhere, then, at each point ξ and for every $\varepsilon > 0$, there is a $\delta > 0$ so that

$$|F(x'') - F(x') - f(\xi)(x'' - x')| < \varepsilon(x'' - x') \tag{3}$$

for $x' \leq \xi \leq x''$ and $0 < x'' - x' < \delta$.

A careless student might argue that one can recover $F(b) - F(a)$ as a limit of Riemann sums for f as follows. Let

$$a = x_0 < x_1 < x_2 \ldots x_n = b$$

be a partition of $[a, b]$, and let $\xi_i \in [x_{i-1}, x_i]$. Then

$$F(b) - F(a) = \sum_{i=1}^{n} (F(x_{i-1}) - F(x_i)) = \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) + R$$

where

$$R = \sum_{i=1}^{n} (F(x_i) - F(x_{i-1}) - f(\xi_i)(x_i - x_{i-1})) .$$

Thus $F(b) - F(a)$ has been given as a Riemann sum for f plus some error term R. But it appears now that, if the partition is arranged to be smaller than the number δ so that (3) may be used, we have

$$\begin{aligned} |R| \ &\leq \ \sum_{i=1}^{n} \Big| F(x_i) - F(x_{i-1}) - f(\xi_i)(x_i - x_{i-1}) \Big| \\ &< \ \sum_{i=1}^{n} \varepsilon(x_i - x_{i-1}) = \varepsilon(b - a). \end{aligned}$$

Evidently, then, *if there are no mistakes here* it follows that f is Riemann integrable and that $\int_a^b f(t)\, dt = F(b) - F(a)$.

Spot the error in this careless student argument.

SEE NOTE 234

8.5.2 Let $f : [0,1] \to \mathbb{R}$ be a differentiable function such that $|f'(x)| \leq M$ for all $x \in (0,1)$. Show that

$$\left| \int_0^1 f(x)\,dx - \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) \right| \leq \frac{M}{n}.$$

SEE NOTE 235

Notes

[226]Exercise 8.1.1. If F and G are both primitives of f apply the mean-value theorem to $H = F - G$.

[227]Exercise 8.1.2. If F and G are both primitives of f with exceptional points $a < c_1 < c_2 < \cdots < c_n < b$, then apply the mean-value theorem to $H = F - G$ on each interval $[a, c_1]$, $[c_1, c_2]$, $\dots$, $[c_n, b]$.

[228]Let M be an upper bound for the function f of that proof. Check, first, that

$$0 \leq F_n(y) - F_n(x) \leq M(y - x)$$

for all $x < y$ in $[0,1]$. Deduce that F is continuous, in fact uniformly continuous, on $[0,1]$.

[229]Exercise 8.1.22. This is called the *Cauchy-Schwarz inequality* and is the analog for integrals of that inequality in Exercise 3.5.13. It can be proved the same way and does not involve any deep properties of integrals.

[230]Exercise 8.2.2. Define

$$\int_{-\infty}^{\infty} f(x)\,dx$$

to be the sum of

$$\int_{-\infty}^{a} f(x)\,dx$$

and

$$\int_{a}^{\infty} f(x)\,dx.$$

Be sure to prove that this definition would not depend on the choice of a.

[231]Exercise 8.2.10. Compare with Exercise 3.4.26. Note, too, that it may seem to require special handling at the left-hand endpoint but it does not.

[232]Exercise 8.3.1. You will need to find a formula for

$$\sum_{k=1}^{n} k^3.$$

[233]Exercise 8.3.8. Be sure, first, to check that these associated points are legitimate. Show that each of these sums has the same value (think of telescoping sums!). What, then, would be the limit of the Riemann sums?

[234]Exercise 8.5.1. The error is that the choice of δ depends on the point ξ considered and so is not a constant. This is an error you have doubtless made in other contexts: A local condition that holds for *each* point x is misinterpreted as holding uniformly for *all* x.

[235]Exercise 8.5.2. This is from the 1947 Putnam Mathematical Competition.

Chapter 9

COVERING RELATIONS

Dripped Chapter[1]

The language of integration theory, as presented here, depends on an understanding of and facility with partitions and Riemann sums. A partition is a special case of a covering relation. This chapter defines all of the terminology and examines all of the techniques needed to carry on to the integral.

9.1 Partitions and subpartitions

Construct a subdivision of a compact interval $[a, b]$,

$$a = a_0 < a_1 < a_2 < \cdots < a_{k-1} < a_k = b$$

and select points ξ_1, ξ_2, ..., ξ_k so that each point ξ_i belongs to the corresponding interval $[a_{i-1}, a_i]$. Then the collection

$$\pi = \{([a_{i-1}, a_i], \xi_i) : i = 1, 2, \ldots, k\}$$

[1]Note to the instructor: For a *modest* course in integration theory on the real line this ***dripped*** chapter together with Chapters 10, 11 and 12 could be used. The more difficult proofs (eg. for the Vitali theorem and the Lebesgue differentiation theorem) would be skipped, although the statements are quite accessible. You might also allude, without proof, to the monotone convergence theorem and the fundamental theorem of the calculus that are given in detail in Chapters 13 and 17, but skipping over other dripped material.

is called a *partition* of $[a,b]$. Any subset of a partition is called a *subpartition*.

We consider this a special kind of *covering relation*.

9.2 Covering relations

Families of pairs $([u,v],w)$, where $[u,v]$ is a compact interval and w a point in that interval, are called *covering relations*. Every partition and every subpartition is a covering relation.

All covering relations are just subsets of one big covering relation:

$$\{([u,v],w) : u,v,w \in \mathbb{R},\ u < v \text{ and } u \le w \le v\ \}.$$

We shall most frequently use the Greek symbol β to denote a covering relation. We have already used the Greek symbol π to denote those covering relations which are partitions.

9.2.1 Prunings

Given a number of covering relations arising in a problem we often have to combine them or "prune out" certain subsets of them. We use the following techniques quite frequently:

Definition 9.1: If β is a covering relation and E a set of real numbers then we write:

- $\beta[E] = \{([u,v],w) \in \beta : w \in E\}$.
- $\beta(E) = \{([u,v],w) \in \beta : [u,v] \subset E\}$.

to indicate these subsets of the covering relation β from which we have removed inconvenient members.

9.2.2 Full covers

A full cover is one that, in very loose language, contains all sufficiently small intervals at a point.

Definition 9.2: Let E be a set of real numbers. A covering relation β is said to be a *full cover* of E if for each $x \in E$ there is a positive number δ so that β contains every pair $([u,v],w)$ for which $v-u<\delta$.

By a *full cover* without reference to any set we mean a full cover of all of $\mathbb{R}$.

Full covers arise naturally as ways to describe continuity, differentiation, integration, and numerous other processes of analysis. The student should attempt many (perhaps all) of the exercises in order to gain a facility in covering arguments.

9.2.3 Fine covers

A fine cover[2] is one that, in very loose language, contains arbitrarily small intervals at a point.

Definition 9.3: Let E be a set of real numbers. A covering relation β is said to be a *fine cover* of E if for each $x \in E$ and any positive number δ the covering relation β contains at least one pair $([u,v],w)$ for which $v-u<\delta$.

By a *fine cover* without reference to any set we mean a fine cover of all of $\mathbb{R}$. Fine covers arise in the same way that full covers arise. In a sense the fine cover comes from a negation of a full cover. For example (as you will see in the Exercises) full covers could be used to describe continuity conditions and fine covers would then twist this to describe the situation where continuity fails.

9.2.4 Uniformly full covers

A uniformly full cover is one that, in very loose language, contains all sufficiently small intervals at a point, where the smallness required is considered the same for all points

Definition 9.4: Let E be a set of real numbers. A covering relation If β is said to be a *uniformly full cover* of E if there is a positive number δ so that β contains every pair $([u,v],w)$ for which $v-u<\delta$.

[2]Known also as a *Vitali cover*.

We are not so much interested in uniformly full covers. To verify that a covering relation is full just requires us to test what happens at each point. To verify that a covering relation is *uniformly* full requires more: we have to find a positive number δ that works at every point. The exclusive use of uniformly full covers would lead to a restrictive theory; the Riemann integral (which is banished from this textbook) is based on uniformly full covers. Our integration theory uses full covers and, as a consequence, is much more general and is easier.[3]

Exercises

9.2.1 We have defined previously a Cousin cover. What is the difference between that and a full cover?

9.2.2 Suppose that β is a full cover of a set E and that G is an open set containing E. Show that $\beta(G)$ is also a full cover of E.

9.2.3 Suppose that β is a fine cover of a set E and that G is an open set containing E. Show that $\beta(G)$ is also a fine cover of E. [This is described as "pruning the fine cover" by the open set G.]

9.2.4 Suppose that β is a uniformly full cover of a set E and that G is an open set containing E. Show that $\beta(G)$ is not necessarily a uniformly full cover of E. Would it be a full cover?

9.2.5 Suppose that β_1 and β_2 are both full covers of a set E. Show that $\beta_1 \cap \beta_2$ is also a full cover of E.

9.2.6 Suppose that β_1 and β_2 are both fine covers of a set E. Show that $\beta_1 \cap \beta_2$ need not be a fine cover of E.

9.2.7 Suppose that β_1 is a full cover of a set E and β_2 is a fine cover. Show that $\beta_1 \cap \beta_2$ is also a fine cover of E. Need it be a full cover?

9.2.8 Suppose that β_1 and β_2 are full covers of sets E_1 and E_2 respectively. Show that $\beta_1 \cup \beta_2$ is a full cover of $E_1 \cup E_2$.

9.2.9 Suppose that β_1 and β_2 are fine covers of sets E_1 and E_2 respectively. Show that $\beta_1 \cup \beta_2$ is a fine cover of $E_1 \cup E_2$.

[3] It is easier since the requirement in Riemann integration to always check that the covers used are not merely full, but uniformly full, imposes unnecessary burdens on many proofs.

9.2.10 Let $F : \mathbb{R} \to \mathbb{R}$. Define

$$\beta = \{([u, v], w) : |F(u) - F(v)| < \varepsilon\}.$$

Show that β is full at a point x_0 for all $\varepsilon > 0$ if and only if F is continuous at that point.

9.2.11 Let $F : \mathbb{R} \to \mathbb{R}$, $c \in \mathbb{R}$ and define

$$\beta = \{([u, v], w) : |F(u) - F(v) - c(v - u)| < \varepsilon(v - u)\}.$$

Show that β is full at a point x_0 for all $\varepsilon > 0$ if and only if $F'(x_0) = c$.

9.2.12 Let $F : \mathbb{R} \to \mathbb{R}$ and define

$$\beta = \{([u, v], w) : |F(u) - F(v)| \geq \varepsilon\}.$$

Show that β is fine at a point x_0 for some value of $\varepsilon > 0$ if and only if F is not continuous at that point.

9.2.13 Let $F : \mathbb{R} \to \mathbb{R}$, $c \in \mathbb{R}$ and define

$$\beta = \{([u, v], w) : |F(u) - F(v) - c(v - u)| \geq \varepsilon(v - u)\}.$$

Show that β is fine at a point x_0 for some value of $\varepsilon > 0$ if and only if $F'(x_0) = c$ is false.

9.2.14(Heine-Borel) Let $\mathcal{G}$ be a family of open sets so that every point in a compact set K is contained in at least one member of the family. Show that the covering relation

$$\beta = \{(I, x) : x \in I \text{ and } I \subset G \text{ for some } G \in \mathcal{G}\}.$$

is a full cover of K (cf. the Heine-Borel Theorem).

9.2.15(Bolzano-Weierstrass) Let E be an infinite set that contains no points of accumulation. Show that

$$\beta = \{(I, x) : x \in I \text{ and } I \cap E \text{ is finite}\}.$$

must be a full cover (cf. the Bolzano-Weierstrass Theorem).

9.2.16 Let $\{x_n\}$ be a sequence of real numbers and let

$$\beta = \{(I, x) : x \in I \text{ and } I \text{ contains only finitely many of the } x_n\}.$$

If β is a fine cover of a set E what (if anything) can you conclude?

9.2.17 Let $\{x_n\}$ be a sequence of real numbers and let

$$\beta = \{(I, x) : x \in I \text{ and } I \text{ contains only finitely many of the } x_n\}.$$

If β is a fine cover of a set E what (if anything) can you conclude?

9.2.18 Let $\{x_n\}$ be a sequence of real numbers and let

$$\beta = \{(I, x) : x \in I \text{ and } I \text{ contains infinitely many of the } x_n\}.$$

If β is a fine cover of a set E what (if anything) can you conclude?

9.2.19 Let $\{x_n\}$ be a sequence of real numbers and let

$$\beta = \{(I, x) : x \in I \text{ and } I \text{ contains infinitely many of the } x_n\}.$$

If β is a fine cover of a set E what (if anything) can you conclude?

9.3 Cousin covering lemma

We have elsewhere discussed the Cousin covering lemma, but repeat it here for convenience and to stress the role that it plays in covering arguments in analysis and in integration theory.

Lemma 9.5 (Cousin covering lemma) *Let β be a full cover. Then β contains a partition of every compact interval.*

Proof. Note, first, that if β fails to contain a partition of some interval $[a, b]$ then it must fail to contain a partition of much smaller subintervals. For example if $a < c < b$, if π_1 is a partition of $[a, c]$ and π_2 is a partition of $[c, b]$, then $\pi_1 \cup \pi_2$ is certainly a partition of $[a, b]$.

We use this feature repeatedly. Suppose that β fails to contain a partition of $[a, b]$. Choose a subinterval $[a_1, b_1]$ with length less than 1/2 the length of $[a, b]$ so that β fails to contain a partition of $[a_1, b_1]$. Continue inductively, selecting a nested sequence of compact intervals $[a_n, b_n]$ with lengths shrinking to zero so that β fails to contain a partition of each $[a_n, b_n]$.

By the nested interval property there is point z belonging to each of the intervals. As β is a full cover, there must exist a $\delta > 0$ so that β contains (I, z) for any compact subinterval I of $[a, b]$ with length smaller than δ. In particular β contains all $([a_n, b_n], z)$ for n large enough to assure us that $b_n - a_n < \delta$. The set $\pi = \{([a_n, b_n], z)\}\}$ containing a single element is itself a partition of $[a_n, b_n]$ that is contained in β. That contradicts our assumptions. Consequently β must contain a partition of $[a, b]$. Since $[a, b]$ was arbitrary, β must contain a partition of any compact interval. ■

9.4 Riemann sums

The integral is defined as a limit of Riemann sums. In fact we wish to define upper and lower integrals first, so the upper integral is a limsup of Riemann sums and the lower integral is a liminf of Riemann sums. The notation for Riemann sums can assume any of the following forms (1), (2), (3),or (4), depending on which is convenient:

Take an interval $[a,b]$ and subdivide as follows:

$$a = x_0 < x_1 < x_2 < x_3 < \cdots < x_{n-1} < x_n = b.$$

Then form a partition of $[a,b]$:

$$\pi = ([x_0,x_1],\xi_1),\ ([x_1,x_2],\xi_2),\ \ldots,\ ([x_{n-1},x_n],\xi_n)$$

Sums of the following form are called *Riemann sums* with respect to this partition:

$$\sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1}). \tag{1}$$

These can also be written as

$$\sum_{([u,v],w)\in\pi} f(w)(v-u) \tag{2}$$

or

$$\sum_{([u,v],w)\in\pi} f(w)\mathcal{L}([u,v]) \tag{3}$$

or

$$\sum_{(I,w)\in\pi} f(w)\mathcal{L}(I). \tag{4}$$

Here we are using $\mathcal{L}$ as a *length function*:

$$\mathcal{L}([u,v]) = v - u$$

is simply the length of the interval $[u, v]$. We can in this way also conveniently assign a length to the intersection of two compact intervals.. For example,

$$\mathcal{L}([u_1, v_1] \cap [u_2, v_2])$$

would be the length of the interval $[u_1, v_1] \cap [u_2, v_2]$ (if it is an interval) and would have length zero if the two intervals do not overlap.

Exercises

9.4.1 Let $F : [a, b] \to \mathbb{R}$ and let π be a partition of $[a, b]$. Verify the computations

$$\sum_{([u,v],w)\in\pi} (v - u) = b - a$$

and

$$\sum_{([u,v],w)\in\pi} (F(v) - F(u)) = F(b) - F(a).$$

9.4.2 Let $F : [a, b] \to \mathbb{R}$ and let π be a partition of $[a, b]$. Show that

$$\sum_{([u,v],w)\in\pi} |F(v) - F(u)| \geq |F(b) - F(a)|.$$

9.4.3 Let $F : [a, b] \to \mathbb{R}$ be a Lipschitz function with Lipschitz constant M and let π be a partition of the interval $[a, b]$. Show that

$$\sum_{([u,v],w)\in\pi} |F(v) - F(u)| \leq M(b - a)|.$$

9.4.4 Let F, $f : [a, b] \to \mathbb{R}$ and let π be a partition of $[a, b]$ and suppose that

$$F(v) - F(u) \geq f(w)(v - u)$$

for all $([u, v], w) \in \pi$. Show that

$$\sum_{([u,v],w)\in\pi} f(w)(v - u)) \leq F(b) - F(a).$$

9.4.5 Let $F : [a,b] \to \mathbb{R}$ be a function with the property that

$$\sum_{([u,v],w)\in\pi} |F(v) - F(u)| = 0.$$

for every partition π of the interval $[a,b]$. What can you conclude?

9.4.6 Let $F : [0,1] \to \mathbb{R}$ be a function with the property that it is monotonic on each of the intervals $[0,1/3]$, $[1/3,2/3]$, and $[2/3,1]$. What is the largest possible value of

$$\sum_{([u,v],w)\in\pi} |F(v) - F(u)|$$

for arbitrary partitions π of the interval $[a,b]$.

9.4.7 Describe the difference between the two sums

$$\sum_{([u,v],w)\in\pi} f(w)(v-u)$$

and

$$\sum_{(I,w)\in\pi([c,d])} f(w)(v-u)$$

where $[c,d]$ is an interval.

9.4.8 Describe the difference between the two sums

$$\sum_{([u,v],w)\in\pi} f(w)(v-u)$$

and

$$\sum_{([u,v],w)\in\pi[E]} f(w)(v-u).$$

where E is a set.

9.4.9 How could you interpret the expression

$$\sum_{([u,v],w)\in\pi_1\cup\pi_2} f(w)(v-u)?$$

9.4.10 How could you interpret the expression

$$\sum_{(([u_1,v_1],w_1)\in\pi_1} \sum_{([u_2,v_2],w_2)\in\pi_2} f(w_1)\mathcal{L}([u_1,v_1]\cap[u_2,v_2])?$$

if π_1 and π_2 are both partitions of the same interval $[a,b]$?

9.4.11 Show that

$$\sum_{(([u_1,v_1],w_1)\in\pi_1} f(w_1)\mathcal{L}([u_1,v_1]) - \sum_{([u_2,v_2],w_2)\in\pi_2} f(w_2)\mathcal{L}([u_2,v_2]) =$$

$$\sum_{(([u_1,v_1],w_1)\in\pi_1} \sum_{([u_2,v_2],w_2)\in\pi_2} [f(w_1) - f(w_2)\mathcal{L}([u_1,v_1]\cap[u_2,v_2])$$

if π_1 and π_2 are both partitions of the same interval $[a,b]$?

9.4.12 Let $f : [a,b] \to \mathbb{R}$ be a continuous function. What could you require of two partitions π_1 and π_2 of the interval $[a,b]$ in order to conclude that

$$\left| \sum_{(([u_1,v_1],w_1)\in\pi_1} f(w_1)(v_1-u_1) - \sum_{([u_2,v_2],w_2)\in\pi_2} f(w_2)(v_2-u_2) \right| < \varepsilon?$$

Notes

[226]Exercise 8.1.1. If F and G are both primitives of f apply the mean-value theorem to $H = F - G$.

[227]Exercise 8.1.2. If F and G are both primitives of f with exceptional points $a < c_1 < c_2 < \cdots < c_n < b$, then apply the mean-value theorem to $H = F - G$ on each interval $[a,c_1]$, $[c_1,c_2]$, ..., $[c_n,b]$.

[228]Let M be an upper bound for the function f of that proof. Check, first, that

$$0 \le F_n(y) - F_n(x) \le M(y-x)$$

for all $x < y$ in $[0,1]$. Deduce that F is continuous, in fact uniformly continuous, on $[0,1]$.

[229]Exercise 8.1.22. This is called the *Cauchy-Schwarz inequality* and is the analog for integrals of that inequality in Exercise 3.5.13. It can be proved the same way and does not involve any deep properties of integrals.

[230]Exercise 8.2.2. Define

$$\int_{-\infty}^{\infty} f(x)\,dx$$

to be the sum of

$$\int_{-\infty}^{a} f(x)\,dx$$

and

$$\int_{a}^{\infty} f(x)\,dx.$$

Be sure to prove that this definition would not depend on the choice of a.

[231]Exercise 8.2.10. Compare with Exercise 3.4.26. Note, too, that it may seem to require special handling at the left-hand endpoint but it does not.

[232]Exercise 8.3.1. You will need to find a formula for

$$\sum_{k=1}^{n} k^3.$$

[233]Exercise 8.3.8. Be sure, first, to check that these associated points are legitimate. Show that each of these sums has the same value (think of telescoping sums!). What, then, would be the limit of the Riemann sums?

[234]Exercise 8.5.1. The error is that the choice of δ depends on the point ξ considered and so is not a constant. This is an error you have doubtless made in other contexts: A local condition that holds for *each* point x is misinterpreted as holding uniformly for *all* x.

[235]Exercise 8.5.2. This is from the 1947 Putnam Mathematical Competition.

Chapter 10

THE INTEGRAL

Dripped Chapter[1]

This chapter introduces the natural integral on the real line. It includes the calculus integral and all variants of the Newton integral. It includes the Riemann integral that has been in the past (most unfortunately) the integral of choice for elementary real analysis courses. It includes, as well, the integral of Henri Lebesgue that is covered in graduate courses (but presented normally in graduate school as a special case of the general theory of measure.)

The usual development of Lebesgue's integral starts with measure theory as the primary (indeed only) tool. Fine covering arguments (i.e., Vitali coverings) enter rather later. Full covering arguments and Riemann sums enter not at all. But this latter tool, the full covering arguments, allows a different way to present the integral and, most importantly, added a new and useful tool to its study. Measure theory still remains a vital and essential part of integration methods, but it will be introduced slowly over the next few chapters.

[1] Note to the instructor: For a *modest* course in integration theory this ***dripped*** chapter, along with Chapter 8 [for motivation] and Chapter 9 [for the language of covering relations] could be used alone.

10.1 Upper and lower integrals

Definition 10.1: For a function $f : [a,b] \to \mathbb{R}$ we define an upper integral by

$$\overline{\int_a^b} f(x)\,dx = \inf_{\beta} \sup_{\pi\subset\beta} \sum_{([u,v],w)\in\pi} f(w)(v-u)$$

where the supremum is taken over all partitions π of $[a,b]$ contained in β, and the infimum over all full covers β.

Similarly we define a lower integral, either by writing

$$\underline{\int_a^b} f(x)\,dx = -\overline{\int_a^b}[-f(x)]\,dx,$$

or directly as

$$\underline{\int_a^b} f(x)\,dx = \sup_{\beta} \inf_{\pi\subset\beta} \sum_{([u,v],w)\in\pi} f(w)(v-u)$$

where, again, π is a partition of $[a,b]$ and β is a full cover.

Exercises

10.1.1 Check that

$$\underline{\int_a^b} f(x)\,dx = -\overline{\int_a^b}[-f(x)]\,dx.$$

10.1.2 Let $f : \mathbb{R} \to \mathbb{R}$. Show that

$$\underline{\int_a^b} f(x)\,dx \leq \overline{\int_a^b} f(x)\,dx.$$

See Note 236

10.1.3 Show that a function f can be altered at a finite number of points without altering the values of the upper and lower integrals.

10.1.4 Show that a function f can be altered at a countable number of points without altering the values of the upper and lower integrals.

10.1.5 Let $f : \mathbb{R} \to \mathbb{R}$ and suppose that $a < b < c$. Show that

$$\overline{\int_a^c} f(x)\,dx = \overline{\int_a^b} f(x)\,dx + \overline{\int_b^c} f(x)\,dx,$$

assuming the sum makes sense.

SEE NOTE 237

10.1.6 Let f, $g : \mathbb{R} \to \mathbb{R}$. What rule should hold for the upper and lower integrals

$$\overline{\int_a^b} [f(x) + g(x)]\,dx \quad \text{and} \quad \underline{\int_a^b} [f(x) + g(x)]\,dx?$$

10.1.7 Define a partition π to be *endpointed* if only elements of the form $([u, w], w)$ or $([w, v], w)$ appear and there is no element $([u, v], w) \in \pi$ for which $u < w < v$. Show that a restriction in the definition of integrals to use endpointed partitions only would not change the theory at all.

SEE NOTE 238

10.1.1 The integral and integrable functions

If the upper and lower integrals are identical we write the common value as

$$\int_a^b f(x)\,dx$$

allowing finite or infinite values. We say in this case that the integral is *determined.* When the integral is not determined then

$$\underline{\int_a^b} f(x)\,dx < \overline{\int_a^b} f(x)\,dx$$

and there is no integral.

If the integral is determined and this value is also finite then we say f is *integrable* and

$$\int_a^b f(x)\,dx$$

is called simply the *integral*, now assuming a finite value.

Definition 10.2: Let $f : \mathbb{R} \to \mathbb{R}$ and suppose that $a < b$. Then

1. f is *integrable* on $[a, b]$ if the integral $\int_a^b f(x)\,dx$ is determined and is finite.
2. f is *absolutely integrable*[a] on $[a, b]$ if both f and $|f|$ are integrable on $[a, b]$.
3. f is *nonabsolutely integrable* on $[a, b]$ if f is integrable on $[a, b]$, but $|f|$ is not.[b]

[a] Absolutely integrable functions are said to be *Lebesgue integrable* although Lebesgue's original definition was very different. He did not use Riemann sums, although he did, in the end, check that his integral could, nonetheless, be obtained from Riemann sums.

[b] As we will see later, whenever f is nonabsolutely integrable on $[a, b]$ the integral $\int_a^b |f(x)|\,dx = \infty$.

Exercises

10.1.1 Let $f : [a, b] \to \mathbb{R}$ show that a sufficient condition for f to be integrable on $[a, b]$ with $c = \int_a^b f(x)\,dx$ is that for every $\varepsilon > 0$ there is a full cover so that

$$\left| c - \sum_{([u,v],w)\in\pi} f(w)(v-u) \right| < \varepsilon$$

for every partition π of $[a, b]$ contained in β.

SEE NOTE 239

10.1.2 Let $f:[a,b]\to\mathbb{R}$ be an integrable function and let π be any partition of $[a,b]$. Show that

$$\left|\int_a^b f(x)\,dx - \sum_{([u,v],w)\in\pi} f(w)(v-u)\right| \le \sum_{([u,v],w)\in\pi} \omega f([u,v])\mathcal{L}([u,v]).$$

[Here $\omega f(I)$ denotes the *oscillation* of the function f on the interval I, defined as $\sup_{s,t\in I}|f(s)-f(t)|$.]

10.1.3 Show that an integrable function f can be altered at a finite number of points without altering the value of the integral.

10.1.4 Show that an integrable function f can be altered at a countable number of points without altering the value of the integrals.

10.1.5 Define a function to be *uniformly integrable* [i.e., Riemann integrable] if in the definition one uses the uniformly full covers from Section 9.2.4, rather than the more general full covers. Show that a function that is integrable in this narrow sense must be bounded.

10.1.6 Continuing Exercise 10.1.6, show that a function f that is uniformly integrable on an interval $[a,b]$ must satisfy the following restrictive property: for every $\varepsilon>0$ there must exist a partition π for which

$$\sum_{([u,v],w)\in\pi} \omega f([u,v])(v-u) < \varepsilon.$$

10.1.7 Continuing the preceding two exercises (if you have the patience to work this hard on the Riemann integral), show that a function f is uniformly integrable on an interval $[a,b]$ if and only if it is bounded and satisfies the following property: for every $\varepsilon>0$ there must exist a partition π for which

$$\sum_{([u,v],w)\in\pi} \omega f([u,v])(v-u) < \varepsilon.$$

10.2 Integrability criteria

In this section we extend our understanding of the integral by introducing a number of integrability criteria. The theoretical development of the integral depends these integrability criteria.

10.2.1 First Cauchy criterion

Theorem 10.3: *A necessary and sufficient condition in order for a function $f : [a, b] \to \mathbb{R}$ to be integrable on a compact interval $[a, b]$ is that there is a number c so that for all $\varepsilon > 0$ a full cover β can be found so that*

$$\left| \sum_{([u,v],w)\in\pi} f(w)(v-u) - c \right| < \varepsilon$$

for all partitions π of $[a, b]$ contained in β.

Proof. In Exercise 10.1.1 we checked that this condition is sufficient. On the other hand, if we know that f is integrable with $c = \int_a^b f(x)\,dx$ then, using the definition of the upper integral, for any $\varepsilon > 0$ we choose a full cover β_1 so that

$$\sum_{([u,v],w)\in\pi} f(w)(v-u) < c + \varepsilon$$

for all partitions π of $[a, b]$ contained in β_1. Similarly, using the definition of the lower integral, we choose a full cover β_2 so that

$$\sum_{([u,v],w)\in\pi} f(w)(v-u) > c - \varepsilon$$

for all partitions π of $[a, b]$ contained in β_2. Take $\beta = \beta_1 \cap \beta_2$. This is a full cover with the property stated. ∎

10.2.2 Second Cauchy criterion

Theorem 10.4: *A necessary and sufficient condition in order for a function $f : [a,b] \to \mathbb{R}$ to be integrable on a compact interval $[a,b]$ is that, for all $\varepsilon > 0$, a full cover β can be found so that*

$$\left| \sum_{(I,w)\in\pi} \sum_{(I',w')\in\pi'} [f(w) - f(w')]\mathcal{L}(I \cap I') \right| < \varepsilon \tag{1}$$

for all partitions π, π' of $[a,b]$ contained in β.

Proof. Start by checking that when π and π' are both partitions of the same interval $[a,b]$ then, for any subinterval I of $[a,b]$

$$\mathcal{L}(I) = \sum_{(I',w')\in\pi'} \mathcal{L}(I \cap I')$$

from which it is easy to see that

$$\sum_{(I,w)\in\pi} f(w)\mathcal{L}(I) = \sum_{(I,w)\in\pi} \sum_{(I',w')\in\pi'} f(w)\mathcal{L}(I \cap I').$$

This allows the difference that would normally appear in a Cauchy type criterion

$$\left| \sum_{(I,w)\in\pi} f(w)\mathcal{L}(I) - \sum_{(I',w')\in\pi'} f(w')\mathcal{L}(I') \right|$$

to assume the simple form given in (1). In particular that statement can be rewritten as

$$\left| \sum_{(I,w)\in\pi} f(w)\mathcal{L}(I) - \sum_{(I',w')\in\pi'} f(w)\mathcal{L}(I) \right| < \varepsilon. \tag{2}$$

The condition is necessary. For if f is integrable then the first Cauchy criterion supplies a full cover β so that

$$\left|\sum_{(I,w)\in\pi} f(w)\mathcal{L}(I) - c\right| < \varepsilon/2$$

for all partitions π of $[a,b]$ contained in β. Any two Riemann sums would both be this close to c and hence within ε of each other.

Suppose the condition holds. We can see from (2) that the upper and lower integrals must be finite. We wish to show that they are equal.

Using the definition of the upper integral, there is at least one partition π of $[a,b]$ contained in β with

$$\sum_{(I,w)\in\pi} f(w)\mathcal{L}(I) > \overline{\int_a^b} f(x)\,dx - \varepsilon$$

Using the definition of the lower integral, there is at least one partition π' of $[a,b]$ contained in β with

$$\sum_{(I,w)\in\pi'} f(w)\mathcal{L}(I) < \underline{\int_a^b} f(x)\,dx + \varepsilon.$$

Together with (2) these show that

$$\overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx < 2\varepsilon.$$

Since ε is an arbitrary positive number the upper and lower integrals are equal.

■

10.2.3 Integrability on subintervals

Lemma 10.5 (Integrability on subintervals) *If $f : [a, b] \to \mathbb{R}$ is integrable then it is also integrable on any compact subinterval of $[a, b]$.*

Proof. Let $\varepsilon > 0$. Suppose that f is integrable on $[a, b]$ and $[c, d]$ is a compact subinterval. Take any full cover β so that the second Cauchy criterion is satisfied for β.

Observe that for every pair of partitions π_1, and $\pi_2 \subset \beta$ of the subinterval $[c, d]$, there is a subpartition π from β so that $\pi_1 \cup \pi$ and $\pi_1 \cup \pi$ are partitions of the full interval $[a, b]$. In particular then

$$\left| \sum_{(I,w)\in\pi_1} f(w)\mathcal{L}(I) - \sum_{(I,w)\in\pi_2} f(w)\mathcal{L}(I) \right| =$$

$$\left| \sum_{(I,w)\in\pi\cup\pi_1} f(w)\mathcal{L}(I) - \sum_{(I,w)\in\pi\cup\pi_2} f(w)\mathcal{L}(I) \right| < \varepsilon$$

The integrability of f on $[c, d]$ follows now from the second Cauchy criterion. ■

10.2.4 The indefinite integral

Theorem 10.6 (The indefinite integral) *If $f : [a, b] \to \mathbb{R}$ is integrable then there is a function $F : [a, b] \to \mathbb{R}$, called an indefinite integral for f, so that*

$$\int_c^d f(x)\,dx = F(d) - F(c)$$

for every compact subinterval $[c, d]$ of $[a, b]$.

Proof. Theorem 10.5 supplies the existence of the integral on the subintervals. Check that the integral is an additive interval function. Then

$$F(t) = \int_a^t f(x)\,dx \quad (a \leq t \leq b)$$

will have this property. ■

10.2.5 Absolutely integrable functions

Using normal inequality techniques we easily observe that the expression (1) that we use for the second Cauchy criterion must be smaller than a quite similar expression:

$$\left|\sum_{(I,w)\in\pi}\ \sum_{(I',w')\in\pi'} [f(w) - f(w')]\mathcal{L}(I\cap I')\right| \leq$$

$$\sum_{(I,w)\in\pi}\ \sum_{(I',w')\in\pi'} \left|f(w) - f(w')\right|\mathcal{L}(I\cap I').$$

It takes a sharp (and young) eye to spot the difference, but the larger side of this inequality may be strictly larger. This leads to a stronger integrability criterion than that in the second Cauchy criterion. This is the motivation for the criterion, named after E. J. McShane.

We prove that McShane's criterion is a sufficient condition for absolute integrability [i.e., Lebesgue integrability]. In a more advanced course we would prove the converse, namely that this criterion is both a necessary and sufficient condition for absolute integrability.

Definition 10.7: [McShane's criterion] A function $f : [a, b] \to \mathbb{R}$ is said to *satisfy McShane's criterion* on $[a, b]$ provided that for all $\varepsilon > 0$ a full cover β can be found so that

$$\sum_{(I,w)\in\pi} \sum_{(I',w')\in\pi'} \left|f(w) - f(w')\right| \mathcal{L}(I \cap I') < \varepsilon$$

for all partitions π, π' of $[a, b]$ contained in β.

Theorem 10.8: *If f satisfies McShane's criterion on $[a, b]$ then f is absolutely integrable, i.e., both f and $|f|$ are integrable there and*

$$-\int_a^b f(x)\,dx \le \int_a^b |f(x)|\,dx \le \int_a^b f(x)\,dx.$$

Proof. It is immediate that if f satisfies McShane's criterion it also satisfies Cauchy's second criterion. Thus the function f is integrable. We then observe that, since

$$\left||f(x)| - |f(x')|\right| \le \left|f(x) - f(x')\right|,$$

it is clear that whenever f satisfies McShane's criterion so too does $|f|$. Thus $|f|$ too is integrable on $[a, b]$. The inequalities of the theorem simply follow from the inequalities $-|f(x)| \le f(x) \le |f(x)|$ which hold for all x. ∎

Exercises

10.2.1 Show that if f satisfies McShane's criterion on $[a, b]$ then it is satisfies McShane's criterion on any subinterval $[c, d]$.

10.2.2 Suppose that f and g both satisfy McShane's criterion on $[a, b]$. Show that so too does any linear combination $rf + sg$.

10.2.3 Suppose that each of the functions $f_1, f_2, \ldots, f_n : [a,b] \to \mathbb{R}$ satisfies McShane's criterion on a compact interval $[a,b]$ and that a function $L : \mathbb{R}^n \to \mathbb{R}$ is given satisfying

$$|L(x_1, x_2, \ldots, x_n) - L(y_1, y_2, \ldots, y_n)| \leq M \sum_{i=1}^{n} |x_i - y_i|$$

for some number M and all $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$. Show that the function $g(x) = L(f_1(x), f_2(x), \ldots, f_n(x))$ satisfies McShane's criterion on $[a,b]$.

10.2.4 Show that there is an integrable function on the interval $[0,1]$ that does not satisfy McShane's criterion.

SEE NOTE 240

10.2.6 Henstock's zero variation criterion

Henstock's criterion[2] gives a necessary and sufficient condition for a specified function F to be the indefinite integral of a function f.

Theorem 10.9: *A necessary and sufficient condition for a function $f : [a,b] \to \mathbb{R}$ to be integrable on a compact interval $[a,b]$ and for F to be its indefinite integral is that for every $\varepsilon > 0$ there exists a full cover β such that*

$$\sum_{([u,v],w)\in\pi} |F(v) - F(u) - f(w)(v-u)| < \varepsilon, \tag{3}$$

for every subpartition π of $[a,b]$ contained in β.

[2]Henstock claimed that he took this idea from Stanisław Saks in a similar setting. So some authors call this the *Henstock-Saks Lemma*. It is important to notice the structure of the lemma: it states that a certain function

$$h(([u,v],w)) = F(v) - F(u) - f(w)(v-u)$$

has *zero variation* in the sense that we will soon define in Chapter 12. The criterion can also be (better) described as asserting, in the language of Chapter 18, that

$$\int_a^b |dF(x) - f(x)dx| = 0.$$

Proof. Suppose that this criterion holds. Then (3) immediately shows that

$$\left| F(b) - F(a) - \sum_{([u,v],w)\in\pi} f(w)(v-u) \right|$$

$$= \left| \sum_{([u,v],w)\in\pi} [F(v) - F(u) - f(w)(v-u) \right|$$

$$\leq \sum_{([u,v],w)\in\pi} |F(v) - F(u) - f(w)(v-u)| < \varepsilon.$$

It follows that $F(b) - F(a) = \int_a^b f(x)\,dx$ by the first Cauchy criteria. The same argument will work on any subinterval to check that F is an indefinite integral for f.

Conversely let us suppose that F is an indefinite integral for f on $[a,b]$ and $\varepsilon > 0$. By the Cauchy criterion there is a full cover β such that

$$\left| F(b) - F(a) - \sum_{([u,v],w)\in\pi} f(w)(v-u) \right| < \varepsilon/4 \tag{4}$$

for every partition π of $[a,b]$ contained in β and it will be our goal to establish (3) from this.

Fix π and let $\pi' \subset \pi$ be any nonempty subset. Since β is full and contains partitions of any compact interval, we will find a useful way to supplement the subpartition π' so as to form a useful partition of $[a,b]$: we write

$$\pi \setminus \pi' = \{([u_1,v_1],w_1), ([u_2,v_2],w_2), \dots ([u_k,v_k],w_k)\}.$$

Our hypothesis requires F to be an indefinite integral for f on each $[u_i, v_i]$ $(i = 1, 2, \dots, k)$ and so for each $i = 1, 2, \dots, k$ we are able to select a partition $\pi_i \subset \beta$ of the interval $[u_i, v_i]$ in such a way that

$$\left| F(v_i) - F(u_i) - \sum_{([u,v],w)\in\pi_i} f(w)(v-u) \right| < \varepsilon/(4k). \tag{5}$$

Thus if we augment π' to form

$$\pi'' = \pi \cup \pi_1 \cup \pi_2 \cup \dots \cup \pi_k$$

we obtain a partition of $[a, b]$ contained in β and thus also satisfying an inequality of the form (4). Computing with these ideas, we see

$$\sum_{([u,v],x)\in\pi'} [F(v) - F(u)] = F(b) - F(a) - \sum_{i=1}^{k} [F(v_i) - F(u_i)]$$

and

$$\sum_{([u,v],w)\in\pi'} f(w)(v-u) = \sum_{([u,v],w)\in\pi''} f(w)(v-u) - \sum_{i=1}^{k} \left(\sum_{([u,v],w)\in\pi_i} f(w)(v-u) \right).$$

Putting these together with the estimates (4) and (5) we obtain

$$\left| \sum_{([u,v],x)\in\pi'} [[F(v) - F(u)] - f(x)(v-u)] \right| \le \left| F(b) - F(a) - \sum_{([u,v],x)\in\pi''} f(x)(v-u) \right|$$

$$+ \sum_{i=1}^{k} \left| [F(v_i) - F(u_i)] - \sum_{([u,v],x)\in\pi_i} f(x)(v-u) \right| < \varepsilon/4 + k(\varepsilon/(4k) = \varepsilon/2.$$

Let us emphasize what we now see: if π' is *any* subset of π we have obtained this inequality:

$$\left| \sum_{([u,v],w)\in\pi'} [F(v) - F(u) - f(x)(v-u)] \right| < \varepsilon/2.$$

To complete the proof let

$$\pi^+ = \{([u,v],w) \in \pi : F(v) - F(u) - f(w)(v-u) \geq 0\}$$

and

$$\pi^- = \{([u,v],w) \in \pi : F(v) - F(u) - f(w)(v-u) < 0\}.$$

Then

$$\sum_{([u,v],w)\in\pi^+} |F(v) - F(u) - f(w)(v-u)|$$
$$= \sum_{([u,v],w)\in\pi^+} [F(v) - F(u) - f(w)(v-u)] < \varepsilon/2$$

and

$$\sum_{([u,v],w)\in\pi^-} |F(v) - F(u) - f(w)(v-u)|$$
$$= \sum_{([u,v],w)\in\pi^-} -[F(v) - F(u) - f(w)(v-u)] < \varepsilon/2.$$

Adding the two inequalities proves (3). ∎

For those readers willing to pursue the integration theory as far as the Stieltjes versions of Chapter 18, the Henstock criterion will assume the simple form

$$\int_a^b |dF(x) - f(x)\,dG(x)| = 0$$

that replaces the more clumsy formulation of Theorem 10.9. This notation makes working with the criterion rather easier.

10.3 Continuous functions are absolutely integrable

We started our integration theory with the calculus integral that was confined to continuous functions. We see now that all continuous functions are integrable.

Theorem 10.10: *If $f : [a, b] \to \mathbb{R}$ is continuous then f is absolutely integrable on $[a, b]$.*

Proof. Let $\varepsilon > 0$ and define β to be the collection of all pairs $([x, y], z)$ subject only to the condition that if $a \leq z \leq b$ and $[x, y] \subset [a, b]$ then $\omega f([x, y]) < \varepsilon/2(b - a)$. Check, using the continuity of f in the interval $[a, b]$, that β is a full cover. Verify that if $([x, y], w)$ and $([x', y'], w')$ both belong to β with $[x, y]$ and $[x'y']$ subintervals of $[a, b]$ then, either $[x, y]$ and $[x'y']$ have no points in common or else $|f(w) - f(w')| < \varepsilon/(b - a)$.

Complete the proof by checking that

$$\sum_{([x,y],w)\in\pi} \sum_{([x',y'],w')\in\pi'} \left|f(w) - f(w')\right| \mathcal{L}([x, y] \cap [x', y']) < \varepsilon$$

for any pair of partitions π and π' of $[a, b]$. Thus f satisfies McShane's criterion on $[a, b]$. It follows that f is absolutely integrable there. ■

10.4 Elementary properties of the integral

All of our elementary properties of the integral are anticipated by the calculus integral which shares all the same properties. Our interest here is that these same properties now hold under very general and weak hypotheses.

10.4.1 Integration and order

Theorem 10.11: *Suppose that f, $g : [a,b] \to \mathbb{R}$ are both integrable and that $f(x) \leq g(x)$ for each x in that interval. Then*

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$

Proof. This follows easily from the inequality

$$\sum_{([u,v],w)\in\pi} f(w)(v-u) \leq \sum_{([u,v],w)\in\pi} g(w)(v-u)$$

which would be true for any partition π of the interval $[a,b]$. ■

10.4.2 Integration of linear combinations

Theorem 10.12: *Suppose that f, $g : [a,b] \to \mathbb{R}$ are both integrable. Then so too is any linear combination $rf + sg$ and*

$$\int_a^b [rf(x) + sg(x)]\,dx = r\left(\int_a^b f(x)\,dx\right) + s\left(\int_a^b g(x)\,dx\right).$$

Proof. Use Exercise 10.1.1 and some simple algebra. ■

10.4.3 The integral as an additive interval function

Theorem 10.13: *If $f : [a,c] \to \mathbb{R}$ is integrable on each of the intervals $[a,b]$, $[b,c]$, and $[a,c]$ then*

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

Proof. This follows from Exercise 10.1.5. ■

10.4.4 Change of variable

Let $\phi : [a,b] \to \mathbb{R}$ be a strictly increasing differentiable function. We would expect from elementary formulas of the calculus that

$$\int_{\phi(a)}^{\phi(b)} f(x)\,dx = \int_a^b f(\phi(t))\phi'(t)\,dt.$$

If f is itself everywhere a derivative then this could be justified. If f is assumed only to be integrable then a different proof, using ϕ to map full covers and partitions, is needed.

Theorem 10.14 (Change of variable) *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a strictly increasing, differentiable function. If $f : \mathbb{R} \to \mathbb{R}$ is integrable on $[\phi(a), \phi(b)]$ then*

$$\int_{\phi(a)}^{\phi(b)} f(x)\,dx = \int_a^b f(\phi(t))\phi'(t)\,dt.$$

Proof. Let $\varepsilon > 0$ and define β to be the collection of all pairs $([x,y],z)$ subject only to the conditions that

$$\left|\frac{\phi(y)-\phi(x)}{y-x} - \phi'(z)\right| < \frac{\varepsilon}{2(b-a)|(1+|f(\phi(z)|)}.$$

Since ϕ is everywhere differentiable this is a full cover. Note that we can write $\phi(y)-\phi(x)$ also as $\mathcal{L}(J)$ where $J = \phi([x,y])$ is just the compact interval that ϕ maps $[x,y]$ onto.

Write

$$\beta_1' = \{(\phi([x,y]), \phi(x)) : ([x,y],z) \in \beta_1\}$$

and check that β_1' is also a full cover. Observe that elements $(J,x) = (\phi([x,y]), \phi(z))$ of β_1' must satisfy

$$|f(\phi(x))\mathcal{L}(\phi([x,y])) - f(\phi(x))\phi'(x)\mathcal{L}([x,y])| < \varepsilon\mathcal{L}([x,y])/2(b-a).$$

The expression $f(\phi(t))\mathcal{L}(\phi([x,y]))$ here is better viewed as $f(x)\mathcal{L}(J)$.

Choose a full cover β_2' so that

$$\left|\int_{\phi(a)}^{\phi(b)} f(x)\,dx - \sum_{(J,x)\in\pi'} f(x)\mathcal{L}(J)\right| < \varepsilon/2$$

for all partitions $\pi' \subset \beta_2'$ of the interval $[\phi(a), \phi(b)]$. Write β_2 for the collection of all (I, x) for which $(I, x) = (\phi(J), \phi(t))$ for some $(J, t) \in \beta_2'$. This is a full cover of $[a, b]$.

Write $\beta = \beta_1 \cap \beta_2$. Check that β is a full cover of $[a, b]$ and check that

$$\left|\int_{\phi(a)}^{\phi(b)} f(x)\,dx - \sum_{(I,x)\in\pi} f(\phi(x))\phi'(x)\mathcal{L}(I)\right| < \varepsilon$$

for all partitions $\pi \subset \beta$ of the interval $[a, b]$. An appeal to the first Cauchy criterion then completes the proof. ■

Exercises

10.4.1 Show that an improper version of the integral (cf. Section 8.1.3) is not needed.[3] That is, prove the following: Let $f : [a, b] \to \mathbb{R}$ be integrable on every interval $[c, d] \subset (a, b)$ and suppose that

$$\lim_{h\to 0+} \lim_{k\to 0+} \int_{a+h}^{b-k} f(x)\,dx$$

exists. Show that f is integrable on $[a, b]$ and that

$$\int_a^b f(x)\,dx = \lim_{h\to 0+} \lim_{k\to 0+} \int_{a+h}^{b-k} f(x)\,dx.$$

See Note 241

[3]It might appear initially that this criterion could be useful. In fact other techniques will usually prove integrability much more easily in any concrete situation. It is, even so, interesting that the so-called "improper extension" does not produce anything new for this theory of integration. (It does for the Riemann integral and for the Lebesgue integral.

10.4.2 Show that the assertion in Exercise 10.4.1 would be false if both occurrences of the word "integrable" were replaced by "absolutely integrable."

10.5 The fundamental theorem of the calculus

The fundamental theorem of the calculus is the statement that connects integration and differentiation. Loosely it states that the two processes are mutually inverse (although this needs some considerable work to make it precise). The full version of the fundamental theorem of the calculus will appear in Chapter 13. Our version in this section is quite a bit weaker, but can be attained with minimal tools. As such it is worth presenting here for students who will not learn the most general statements.

10.5.1 Derivative of the integral of continuous functions

Theorem 10.15: *Let $f : [a,b] \to \mathbb{R}$ be an integrable function on the interval $[a,b]$. Let*

$$F(t) = \int_a^t f(x)\,dx \qquad (a \leq t \leq b).$$

Assume that $x_0 \in [a,b]$ is a point of continuity of f. Then

1. *If $a < x_0 < b$ then $F'(x_0) = f(x_0)$.*
2. *If $a = x_0$ then the right hand derivative $F'_+(x_0) = f(x_0)$.*
3. *If $x_0 = b$ then the left hand derivative $F'_-(x_0) = f(x_0)$.*

Proof. Let x_0 be a point of continuity of f and let $\varepsilon > 0$. Then there is a $\delta > 0$ so that $|f(x) - f(x_0)| < \varepsilon$ if $|x - x_0| < \delta$ and $x \in [a,b]$. Let $[u,v] \subset [a,b]$ be any interval that contains x_0 and has length less than δ. Simply compute

$$\left| \int_u^v f(x)\,dx - f(x_0)(v-u) \right| = \left| \int_u^v f(x)\,dx - \int_u^v f(x_0)\,dx \right|$$

$$\leq \int_u^v |f(x) - f(x_0)| \, dx \leq \varepsilon(v-u).$$

From this the conclusions of the theorem are easy to check. ■

10.5.2 Relation to the calculus integral

The integral as defined here includes the calculus integral. A precise statement of this is given in the following theorem:

Theorem 10.16: *Let f be a continuous function on an interval $[a,b]$. Then f is absolutely integrable on $[a,b]$ and*

$$\frac{d}{dt}\int_a^t f(x)\,dx = f(t)$$

at every point t of (a,b).

Proof. We know the individual pieces of this theorem already. Theorem 10.10 supplies us with the fact that a continuous function f must be integrable, even absolutely integrable. And Theorem 10.15 supplies us with the fact the derivative of the indefinite integral at each point x in (a,b) is exactly $f(x)$ because f is continuous at each such point. ■

10.5.3 Integral of the derivative

We started our integration theory with the calculus integral that was defined in a way to invert derivatives of continuous functions. We prove now that all derivatives (not just continuous derivatives) are integrable and that the value of the integral is exactly given by the usual calculus formula. In particular our integral includes Newton's version of the integral.

Theorem 10.17: *Let F, $f : [a, b] \to \mathbb{R}$ and suppose that F is a continuous function that is differentiable at every point x, $a < x < b$. If $f(x) = F'(x)$ for each of these values then f is integrable on $[a, b]$ and, moreover,*

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Proof. Let $\varepsilon > 0$ and define β to be the collection of all pairs $([u, v], w)$ subject only to the conditions that

1. if $w = a$, and $[u, v] \subset [a, b]$ then $u = a$ and
$$|f(a)|(v - a) + |F(v) - F(a)| < \varepsilon.$$
2. if $w = b$ and $[u, v] \subset [a, b]$ then $v = b$ and
$$|f(b)|(b - u) + |F(b) - F(u)| < \varepsilon.$$
3. if $a < w < b$ and $[u, v] \subset [a, b]$ then $|F(v) - F(u) - f(w)(v - u)| \leq \varepsilon(v - u)$.

This β is a full cover: just check at each point. Use the continuity of F at the endpoints of the interval and use the differentiation assumption $F'(x) = f(x)$ at points inside the interval. (The points outside $[a, b]$ are irrelevant and our definition of β placed no restriction on them in any case.) Now verify that

$$\sum_{([u,v],w)\in\pi} |F(v) - F(u) - f(w)(v - u)| < (2 + b - a)\varepsilon$$

for all partitions π of $[a, b]$ contained in β. By the Henstock criterion f is integrable and F is its indefinite integral on $[a, b]$. ■

10.5.4 Relation to the Newton integral

The integral as defined here includes not just the calculus integral (of continuous functions) but the Newton integral defined for *all* derivatives (not just continuous ones). This is a consequence of Theorem 10.17.

In fact our integral includes all variants of the Newton integral. The most interesting version allows a countable exceptional set. We have the necessary techniques to check this. A precise statement of this is given in the following theorem:

Theorem 10.18: *Let F be a continuous function on an interval $[a,b]$ that is differentiable on (a,b) with the exception possibly of a countable set. Then F' is integrable on $[a,b]$ and*

$$\int_a^b F'(x)\,dx = F(b) - F(a).$$

Proof. As a technical point we note that $f(x) = F'(x)$ has not been defined at the endpoints a and b, nor is it defined at the countable exceptional set

$$C = \{c_1, c_2, c_3, \dots\}$$

where $F'(x)$ may not exist. But the integral is defined using Riemann sums and such points need to be addressed since they can easily occur inside the sum.

This point is clarified later, in Chapter 11, where we will allow many more points at which a function f to be integrated need not be defined. Our solution here is simple. We assume that $f(a)$, $f(b)$, and $f(c_i)$ are in fact defined, but we show that this has no impact on the statement of the theorem: f is integrable in any case and $\int_a^b f(x)\,dx = F(b) - F(a)$.

Let $\varepsilon > 0$ and choose $0 < \eta < (b-a+1)\varepsilon$. We use the fact that $F'(x) = f(x)$ for $a < x < b$ to define β_1 to be the collection of all pairs $([u,v],w)$ subject only to the conditions that, if $a < w < b$, and w is not one of the points in the sequence c_1, c_2, c_3, ... then $[u,v] \subset (a,b)$ and

$$|F(v) - F(u) - f(w)(v-u)| \le \eta(v-u). \tag{6}$$

This must be a full cover.

We use the continuity of F at the endpoints a, b, and at the exceptional points c_1, c_2, c_3, ... to choose a full cover β_2 so that

$$|F(v) - F(a)| + |f(a)|(v-a) < \eta/2 \tag{7}$$

and

$$|F(b) - F(u)| + |f(b)|(b - u) < \eta/4 \tag{8}$$

whenever $([a, v], a)$ or $([u, b], b)$ is in β_2 and

$$|F(v) - F(u)| + |f(c_i)|(v - u) < \eta 2^{-i-2} \tag{9}$$

whenever $([u, v], c_i)$ is in β_2 $(i = 1, 2, 3, \dots)$.

Set $\beta = \beta_1 \cap \beta_2$. This is also a full cover. Now we verify the Henstock criterion for this full cover β. Let $\pi \subset \beta$ be a partition of the interval $[a, b]$. By the way in which it has been defined the partition π must include pairs $([a, v_0], a)$ and $([u_0, b], b)$. For these pairs we use (7), and (8). The partition π may also include pairs $([u_i, v_i], c_i)$ and for these we take advantage of (9). Finally we use (6) which is valid for every other pair $([u, v], w) \in \pi$. Thus we easily check that

$$\sum_{([u,v],w)\in\pi} |F(v) - F(u) - f(w)(v - u)| < \eta(b - a) + \sum_{i=1}^{\infty} \eta 2^{-i} < \varepsilon.$$

Since this inequality holds for every partition π of $[a, b]$ contained in β the zero variation criterion of Henstock (Theorem 10.9) is satisfied and the proof is complete. ■

Notes

[236]Exercise 10.1.2. Make use in your proof of the fact that the intersection of two full covers, is again a full cover.

[237]Exercise 10.1.5. Infinite values are allowed but we would have to avoid $\infty + (-\infty)$ or $-\infty + \infty$. This is simpler if you first check that a single value $f(b)$ is irrelevant to the computations so that you may assume that $f(b) = 0$. Then ensure that any partition π contained in your choice of β of the interval $[a, b]$, $[a, c]$ or $[b, c]$ would have to contain an element (I, b).

[238]Exercise 10.1.7. Check, first, that full covers do in fact contain endpointed partitions (as well as ordinary partitions). Then note that, if a partition π contains a pair $([u,v],w)$ for which $u < w < v$ that element can be replaced by the two items $([u,w],w)$ and $([w,v],w)$. That does not change the Riemann sums here because, for example,

$$f(w)[v-u] = f(w)[w-u] + f(w)[v-w].$$

Finally check that if β is a full cover there must be a smaller full cover $\beta' \subset \beta$ so that $([u,v],w) \in \beta'$ with $u < w < v$ if and only if both $([u,w],w)$ and $([w,v],w)$ are in β'.

[239]Exercise 10.1.1. Use β to find estimates for the upper and lower integrals. (Later we will show that this condition is, in fact, both necessary and sufficient.)

[240]Exercise 10.2.4. Find a differentiable function F for which $|F'|$ cannot be integrable on $[0,1]$.

[241]Exercise 10.4.1. Use the Henstock zero variation criterion.

Chapter 11

NULL SETS AND NULL FUNCTIONS

Dripped Chapter[1]

The study of integration and differentiation on the real line requires detailed knowledge of the sets and functions that can be ignored in much of the theory. These are the null sets and the null functions.

11.1 Sets of measure zero

A set has measure zero if it is small in the sense of length. In analysis there are a number of ways in which a set might be considered as "small." For example, the Cantor set is not small in the sense of counting: It is uncountable. It is small in another different sense: It is nowhere dense, that is there is no interval at all in which it is dense. Now we turn to another way in which the Cantor set can be considered small: It has "zero length."

[1]Note to the instructor: For a *modest* course in integration theory this ***dripped*** chapter and all later ones can be skipped over. The basic properties of the integral are now evident. Most importantly we have had a rather easier time in this theory than we would have experienced if we had chosen the Riemann integral as our integral of choice.

Even so it should be hard to resist not doing a little more. Sets of measure zero play a key role in integration theory, even in Riemann integration theory (although most elementary courses shun them entirely). The arguments are the same as we have just seen; everything reduces really to a covering argument, whether it is the definition of the integral or the exploration of concepts such as measure zero sets. The proof of the simplified Vitali covering theorem in Section 11.4.2 should be accessible.

Example 11.1: Suppose we wish to measure the "length" of the Cantor set. Since the Cantor set is rather bizarre, we might look instead at the sequence of intervals that have been removed. There is no difficulty in assigning a meaning of length to an interval; the length of (a, b) is $b - a$. What is the total length of the intervals removed in the construction of the Cantor set? From the interval $[0, 1]$ we remove first a middle third interval of length $1/3$, then two middle third intervals of length $1/9$, and so on so that at the nth stage we remove 2^{n-1} intervals each of length 3^{-n}. The sum of the lengths of all intervals so removed is

$$1/3 + 2(1/9) + 4(1/27) + \cdots =$$

$$1/3\,(1 + 2/3 + (2/3)^2 + (2/3)^3 + \dots) = 1.$$

From the interval $[0, 1]$ we appear to have removed all of the length. What is left over, the Cantor set, must have length zero.

This method of computing lengths has some merit but it is not the one we wish to adopt here. Another approach to "measuring" the length of the Cantor set is to consider the length that *remains* at each stage. At the first stage the Cantor set is contained inside the union

$$[0, 1/3] \cup [2/3, 1],$$

which has length $2(1/3)$. At the next stage it is contained inside a union of four intervals, with total length $4(1/9)$. Similarly, at the nth stage the Cantor set is contained inside the union of 2^n intervals each of length 3^{-n}. The sum of the lengths of all these intervals is $(2/3)^n$, and this tends to zero as n gets large. Thus, as before, it seems we should assign zero length to the Cantor set. ◀

We convert the second method of the example into a definition of what it means for a set to be of measure zero. "Measure" is the technical term used to describe the "length" of sets that need not be intervals. In the example we used closed intervals while in our definition below we will use open intervals and open sets. There is no difference (see Exercise 11.1.2). In the example we covered the Cantor set with a finite sequence of intervals while in our definition below we use an infinite sequence. For the Cantor set there is no difference but for other sets (sets that are not bounded or are not closed) there is a difference.

11.1.1 Lebesgue measure of open sets

The property that a set E will be a set of measure zero is actually a statement about the family of open sets containing E. E is measure zero if there are arbitrarily "small" open sets containing E.

For a precise version of this we require a definition for the Lebesgue measure $\mathcal{L}(G)$ of an open set G. Later on, in Chapter 17, we will study Lebesgue's measure in general. The attention here remains only on that measure for open sets.

Definition 11.2: Let G be an open set. Then the Lebesgue measure $\mathcal{L}(G)$ of an open set G is defined to be the total sum of the lengths of all the component intervals of G.

According to this definition $\mathcal{L}(\emptyset) = 0$ (since there are no component intervals). If G has infinitely many component intervals $(\{a_i, b_i)\}$ then the measure is the sum of an infinite series:

$$\mathcal{L}(G) = \sum_{i=1}^{\infty}(b_i - a_i).$$

[An unbounded component interval would have length ∞.]

The only tool we need for working with this concept is given by the subadditivity property.

Lemma 11.3 (Subadditivity) *Let G_1, G_2, G_3, ... be a sequence of open sets. Then the union*

$$G = \bigcup_{i=1}^{\infty} G_i$$

is also an open set and

$$\mathcal{L}(G) \leq \sum_{i=1}^{\infty} \mathcal{L}(G_i).$$

Proof. Certainly G is open since any union of open sets is open. Let

$$T = \sum_{i=1}^{\infty} \mathcal{L}(G_i).$$

Note that T is simply the sum of all the component intervals of all the G_i.

Let $(\{a_j, b_j)\}$ denote the component intervals of G. Take (a_1, b_1) and choose any $[c_1, d_1] \subset (a_1, b_1)$. A compactness argument shows that $[c_1, d_1]$ is contained in finitely many of the component intervals making up the sum T. We conclude that $d_1 - c_1 \leq T$. That would be true for any choice of $[c_1, d_1] \subset (a_1, b_1)$, so that $b_1 - a_1 \leq T$. A similar argument using m components (a_1, b_1), (a_2, b_2), ..., (a_m, b_m) will establish that

$$\sum_{j=1}^{m} (b_j - a_j) \leq T$$

from which

$$\mathcal{L}(G) = \sum_{j=1}^{\infty} (b_j - a_j) \leq T$$

evidently follows. ■

11.1.2 Sets of measure zero

Definition 11.4: Let E be a set of real numbers. Then E is said to have *measure zero* if for every $\varepsilon > 0$ there is an open set G containing E for which $\mathcal{L}(G) < \varepsilon$.

Example 11.5: The empty set has measure zero. (It satisfies the definition easily, with $G = \emptyset$ in fact.) ◀

Example 11.6: Every finite set has measure zero. If

$$E = \{x_1, x_2, \ldots x_N\}$$

and $\varepsilon > 0$, then the sequence of intervals

$$\left(x_i - \frac{\varepsilon}{2N}, x_i + \frac{\varepsilon}{2N}\right) \quad i = 1, 2, 3, \ldots, N$$

covers the set E and the sum of all the lengths is ε. The union of these intervals is an open set G that contains E and has Lebesgue measure $\mathcal{L}(G)$ smaller than ε. ◀

Example 11.7: Every infinite, countable set has measure zero. If

$$E = \{x_1, x_2, \ldots\}$$

and $\varepsilon > 0$, then the sequence of intervals

$$\left(x_i - \frac{\varepsilon}{2^{i+1}}, x_i + \frac{\varepsilon}{2^{i+1}}\right) \quad i = 1, 2, 3, \ldots$$

covers the set E. Let G be the union of these intervals. Since

$$\sum_{k=1}^{\infty} 2\left(\frac{\varepsilon}{2^{k+1}}\right) = \sum_{k=1} \varepsilon 2^{-k} = \varepsilon,$$

we conclude (from Lemma 11.3) that $\mathcal{L}(G) < \varepsilon$. ◀

Example 11.8: The Cantor set has measure zero. Let $\varepsilon > 0$. Choose n so that $(2/3)^n < \varepsilon$. Then the nth stage intervals in the construction of the Cantor set give us 2^n closed intervals each of length $(1/3)^n$. This covers the Cantor set with 2^n closed intervals of total length $(2/3)^n$, which is less than ε. If the closed intervals trouble you (the definition requires open intervals), see Exercise 11.1.2 or argue as follows. Since $(2/3)^n < \varepsilon$ there is a positive number δ so that

$$(2/3)^n + \delta < \varepsilon.$$

Enlarge each of the closed intervals to form a slightly larger open interval, but change the length of each only enough so that the sum of the lengths of all the 2^n closed intervals does not increase by more than δ. The resulting collection of open intervals also covers the Cantor set, and the sum of the length of these intervals is less than ε. ◀

11.1.3 Sequences of measure zero sets

One of the most fundamental of the properties of sets having measure zero is how sequences of such sets combine. We recall that the union of any sequence of countable sets is also countable. We now prove that the union of any sequence of measure zero sets is also a measure zero set.

Theorem 11.9: *Let E_1, E_2, E_3, ... be a sequence of sets of measure zero. Then the set E formed by taking the union of all the sets in the sequence is also of measure zero.*

Proof. Let $\varepsilon > 0$. Choose open sets $G_n \supset E_n$ so that

$$\mathcal{L}(G_n) < 2^{-n}\varepsilon.$$

Then set $G = \bigcup_{n=1}^{\infty} G_n$. Observe, by the subadditivity property (i.e., from Lemma 11.3), that G is an open set containing E for which $\mathcal{L}(G) < \varepsilon$. ■

11.1.4 Compact sets of measure zero

Let us return to the situation for the Cantor set once again. For each $\varepsilon > 0$ we were able to choose a *finite* cover of open intervals with total length less than ε. This is not the case for all sets of measure zero. For example, the set of all rational numbers on the real line is countable and hence also of measure zero. Any finite collection of small intervals must fail to cover that set, in fact cannot come close to covering all rational numbers. For what sets is it possible to select finite coverings of small total length? The answer is that this is possible for *compact* sets of measure zero.

Theorem 11.10: *Let E be a compact set of measure zero. Then for every $\varepsilon > 0$ there is a* finite *collection of open intervals*

$$\{(a_k, b_k) : k = 1, 2, 3, \dots, N\}$$

that covers the set E and so that

$$\sum_{k=1}^{N}(b_k - a_k) < \varepsilon.$$

Proof. Since E has measure zero, there is an open set G containing E for which $\mathcal{L}(G) < \varepsilon$. Let $\{(a_k, b_k)\}$ denote the component intervals of G. By the Heine-Borel theorem there is a finite N so that

$$\{(a_k, b_k) : k = 1, 2, \dots, N\}$$

covers the set E. Since

$$\sum_{k=1}^{N}(b_k - a_k) \le \mathcal{L}(G) < \varepsilon.$$

the proof is complete. ■

Exercises

11.1.1 Show that E is a set of measure zero if and only if there is a finite or infinite sequence

$$(a_1, b_1), (a_2, b_2), (a_3, b_3), (a_4, b_4), \dots$$

of open intervals covering the set E so that

$$\sum_{k=1}^{\infty}(b_k - a_k) \le \varepsilon.$$

11.1.2 Show that E is a set of measure zero if and only if there is a finite or infinite sequence

$$[a_1, b_1], [a_2, b_2], [a_3, b_3], [a_4, b_4], \dots$$

of compact intervals covering the set E so that

$$\sum_{k=1}^{\infty}(b_k - a_k) \le \varepsilon.$$

11.1.3 Show that every subset of a set of measure zero also has measure zero.

11.1.4 If E has measure zero, show that the translated set

$$E + \alpha = \{x + \alpha : x \in E\}$$

also has measure zero.

11.1.5 If E has measure zero, show that the expanded set

$$cE = \{cx : x \in E\}$$

also has measure zero for any $c > 0$.

11.1.6 If E has measure zero, show that the reflected set

$$-E = \{-x : x \in E\}$$

also has measure zero.

11.1.7 Without referring to Theorem 11.9, show that the union of any two sets of measure zero also has measure zero.

11.1.8 If $E_1 \subset E_2$ and E_1 has measure zero but E_2 has not, what can you say about the set $E_2 \setminus E_1$?

11.1.9 Show that any interval (a, b) or $[a, b]$ is not of measure zero.

11.1.10 Give an example of a set that is not of measure zero and does not contain any interval $[a, b]$.

11.1.11 A careless student claims that if a set E has measure zero, then it is true that the closure $\overline{E}$ must also have measure zero. After all, if E is contained in $\bigcup_{i=1}^{\infty}(a_i, b_i)$ with small total length then $\overline{E}$ is contained in $\bigcup_{i=1}^{\infty}[a_i, b_i]$, also with small total length. Is this correct?

11.1.12 If a set E has measure zero what can you say about interior points of that set?

11.1.13 If a set E has measure zero what can you say about boundary points of that set?

11.1.14 Suppose that a set E has the property that $E \cap [a, b]$ has measure zero for every compact interval $[a, b]$. Must E also have measure zero?

11.1.15 Show that the set of real numbers in the interval $[0, 1]$ that do not have a 7 in their infinite decimal expansion is of measure zero.

11.1.16 Describe completely the class of sets E with the following property: For every $\varepsilon > 0$ there is a *finite* collection of open intervals

$$(a_1, b_1), (a_2, b_2), (a_3, b_3), (a_4, b_4), \dots (a_N, b_N)$$

that covers the set E and so that

$$\sum_{k=1}^{N}(b_k - a_k) < \varepsilon.$$

(These sets are said to have *zero content*.)

11.1.17 Show that a set E has measure zero if and only if there is a sequence of intervals

$$(a_1, b_1), (a_2, b_2), (a_3, b_3), (a_4, b_4), \dots$$

so that every point in E belongs to infinitely many of the intervals and $\sum_{k=1}^{\infty}(b_k - a_k)$ converges.

11.1.18 Suppose that $\{(a_i, b_i)\}$ is a sequence of open intervals for which

$$\sum_{i=1}^{\infty}(b_i - a_i) < \infty.$$

Show that the set

$$E = \bigcap_{n=1}^{\infty}\bigcup_{i=n}^{\infty}(a_i, b_i)$$

has measure zero. What relation does this exercise have with the preceding exercise?

11.1.19 By altering the construction of the Cantor set, construct a nowhere dense closed subset of $[0, 1]$ so that the sum of the lengths of the intervals removed is not equal to 1. Will this set have measure zero?

11.2 Full null sets

Sets of measure zero are defined using open sets that contain them. There is a variant on this using full covers instead. This has the advantage that, since it is defined using full covers, this definition is closely related to the differentiation and integration properties of functions. It has the disadvantage that, unlike the measure zero sets, it is not constructive; full covers themselves are not necessarily constructive. We shall show later that the definitions are equivalent.

Definition 11.11: A set E of real numbers is said to be *full null* if for every $\varepsilon > 0$ there is a full cover β of the set E with the property that

$$\sum_{([u,v],w)\in\pi}(v-u) < \varepsilon \tag{1}$$

for every subpartition π chosen from β.

We will show that the two definitions, full null and measure zero, are equivalent later. For the moment one direction is easy.

Theorem 11.12: *Every set of measure zero is also full null.*

Proof. Assume that a set E measure zero and let $\varepsilon > 0$. Choose an open set G containing E for which $\mathcal{L}(G) < \varepsilon$. Let $\{(a_i, b_i)\}$ be the component intervals of G. Define β to be the collection of all pairs $([u,v], w)$ with the property that $w \in [u,v] \subset G$. It is easy to check that β is a full cover of E.

Consider any subpartition π chosen from β. For each $([u,v],w) \in \pi$, $[u,v]$ is a subinterval of some component (a_i, b_i) of G. Holding i fixed, the sum of the lengths of those intervals $[u,v] \subset (a_i, b_i)$ would certainly be smaller than $(b_i - a_i)$. It follows that

$$\sum_{([u,v],w)\in\pi} (v-u) \leq \sum_{i=1}^{\infty}(b_i - a_i) = \mathcal{L}(G) < \varepsilon.$$

This verifies that E is full null. ■

Exercises

11.2.1 Show that every subset of a full null set is also a full null set.

11.2.2 Show that the union of any two full null sets is also a full null set.

11.2.3 Show that the union of any sequence of full null sets is also a full null set.

11.2.4 Define a set E to be *uniformly full null* if for every $\varepsilon > 0$ there is a uniformly full cover β of the set E with the property that

$$\sum_{([u,v],w)\in\pi} (v-u) < \varepsilon \tag{2}$$

for every subpartition π chosen from β. Show that uniformly full null sets are the same as sets of zero content. (cf. Exercise 11.1.16).

11.3 Fine null sets

Sets of measure zero are defined with attention to the open sets that contain them. Full null sets are defined using full covers. There is a third variant on this using fine covers instead. This offers yet a more delicate way of working with measure zero sets, since fine covers can express very subtle properties of derivatives and integrals. We will show in Section 11.4 that all three notions are equivalent.

Definition 11.13: A set E of real numbers is said to be *fine null* if for every $\varepsilon > 0$ there is a fine cover β of the set E with the property that

$$\sum_{([u,v],w)\in\pi} (v-u) < \varepsilon \tag{3}$$

for every subpartition π chosen from β.

Exercises

11.3.1 Show that every set of measure zero is also fine null.

11.3.2 Show that every full null set is also fine null.

11.3.3 Show that every subset of a fine null set is also a fine null set.

11.3.4 Show that the union of any two fine null sets is also a full null set.

11.3.5 Show that the union of any sequence of fine null sets is also a fine null set.

11.4 The Mini-Vitali Covering Theorem

The original Vitali covering theorem asserts that the Lebesgue measure of an arbitrary set can be determined either by open coverings of E, or by full covers of E, or by fine covers of E. Our goals in this chapter are narrower. We want to establish these same facts, but only for sets of measure zero. Later, in Chapter 17 we will return and complete the Vitali covering theorem.

Theorem 11.14: *For a set $E \subset \mathbb{R}$ the following are equivalent:*

1. *E is a set of measure zero.*
2. *E is a full null set.*
3. *E is a fine null set.*

As a result of this theorem we can now simply call these sets *null sets* and use any of the three characterizations that is convenient. The proof requires some simple geometric arguments and an application of the Heine-Borel theorem; it is give in the sections that now follow.

11.4.1 Covering lemmas for families of compact intervals

We begin with some simple covering lemmas for finite and infinite families of compact intervals.

Lemma 11.15: *Let $\mathcal{C}$ be a finite family of compact intervals. Then there is a pairwise disjoint subcollection $[c_i, d_i]$ $(i = 1, 2, \ldots, m)$ of that family with*[a]

$$\bigcup_{[u,v]\in\mathcal{C}} [u, v] \subset \bigcup_{i=1}^{m} 3 * [c_i, d_i].$$

[a]By $3 * [u, v]$ we mean the interval centered at the same point as $[u, v]$ but with three times the length.

Proof. For $[c_1, d_1]$ simply choose the largest interval. Note that $3 * [c_1, d_1]$ will then include any other interval $[u, v] \in \mathcal{C}$ that intersects $[c_1, d_1]$. See Figure 11.1.

For $[c_2, d_2]$ choose the largest interval from among those that do not intersect $[c_1, d_1]$. Note that together $3 * [c_1, d_1]$ and $3 * [c_2, d_2]$ include any interval of the family that intersects either $[c_1, d_1]$ or $[c_2, d_2]$. Continue inductively, choosing $[c_{k+1}, d_{k+1}]$ as the largest interval in $\mathcal{C}$ that does not intersect one the previously chosen intervals $[c_1, d_1]$, $[c_2, d_2]$, …, $[c_k, d_k]$. Stop when you run out of intervals to select. ■

The next covering lemma addresses arbitrary families of compact intervals.

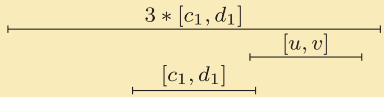

Figure 11.1. Note that $3 * [c_1, d_1]$ will then include any shorter interval $[u, v]$ that intersects $[c_1, d_1]$.

Lemma 11.16: *Let $\mathcal{C}$ be any collection of compact intervals. Then the set*

$$G = \bigcup_{[u,v]\in\mathcal{C}} (u, v)$$

is an open set that contains all but countably many points of the set

$$E = \bigcup_{[u,v]\in\mathcal{C}} [u, v]$$

Proof. Let

$$C = \{x : \ x \notin G \text{ and } x = c \text{ or } x = d \text{ for at least one } [c, d] \in \mathcal{C} \ \}.$$

We observe that G is open, being a union of a family of open intervals. Clearly G contains all of E except for points that are in the set C. To complete the proof of the lemma, we show that C is countable. Write, for $n = 1, 2, 3, \dots,$

$$C_n = \{x : \ x \notin G, \ \ x = c \text{ for at least one } [c, d] \in \mathcal{C} \text{ with } d - c > 1/n\}.$$

$$C'_n = \{x : \ x \notin G, \ \ x = d \text{ for at least one } [c, d] \in \mathcal{C} \text{ with } d - c > 1/n\}.$$

We easily show that each C_n and C'_n is countable. For example if $c \in C_n$ then the interval $(c, c + 1/n)$ can contain no other point of C. This is because there is at least one interval $[c, d]$ from $\mathcal{C}$ with $d - c > 1/n$. Thus $(c, c+1/n) \subset (c, d) \subset G$. Consequently there can be only countably many such points. It follows that the set $C = \bigcup_{n=1}^{\infty}(C_n \cup C'_n)$ is a countable subset of E. ■

11.4.2 Proof of the Mini-Vitali covering theorem

We begin with a simple lemma that is the key to the argument, both for our proof of the mini version as well as the proof of the full Vitali covering theorem.

Lemma 11.17: *Let β be a covering relation and write*

$$G = \bigcup_{([u,v],w)\in\beta} (u,v).$$

Then G is an open set and, if $g = \mathcal{L}(G)$, is finite then there must exist a subpartition $\pi \subset \beta$ for which

$$\sum_{([u,v],w)\in\pi} (v-u) \geq g/6. \tag{4}$$

In particular

$$G' = G \setminus \bigcup_{([u,v],w)\in\pi} [u,v]$$

is an open subset of G and $\mathcal{L}(G') \leq 5g/6$.

Proof. It is clear that the set G of the lemma, expressed as the union of a family of open intervals, must be an open set. Let $\{(a_i, b_i)\}$ be the sequence of component intervals of G. Thus, by definition,

$$g = \mathcal{L}(G) = \sum_{i=1}^{\infty} (b_i - a_i).$$

Choose an integer N large enough that

$$\sum_{i=1}^{N} (b_i - a_i) > 3g/4.$$

Inside each open interval (a_i, b_i), for $i = 1, 2, \ldots, N$, choose a compact interval $[c_i, d_i]$ so that

$$\sum_{i=1}^{N}(d_i - c_i) > g/2.$$

Write

$$K = \bigcup_{i=1}^{N}[c_i, d_i]$$

and note that it is a compact set covered by the family

$$\{(u, v) : ([u, v], w) \in \beta\}.$$

By the Heine-Borel theorem there must be a finite subset

$$([u_1, v_1], w_1),\ ([u_2, v_2], w_2),\ ([u_3, v_3], w_3),\ \ldots, ([u_m, v_m], w_m)$$

from β for which

$$K \subset \bigcup_{i=1}^{m}(u_i, v_i).$$

By Lemma 11.15 we can extract a subpartition π from this list so that

$$K \subset \bigcup_{([u,v],w)\in\pi} 3 * [u, v].$$

and so

$$\sum_{([u,v],w\in\pi} 3(v - u) \geq \sum_{i=1}^{N}(d_i - c_i) > g/2.$$

Statement (4) then follows. [Not that we need it here, but recall that Lemma 11.15 allows us to claim that the intervals in the subpartition π are disjoint, not merely nonoverlapping.]

The final statement of the lemma requires just checking the length of a finite number of the components of G'. We have removed all the intervals $[u,v]$ from G for which $([u,v],w) \in \pi$. Since the total length removed is greater than $g/6$ what remains cannot be larger than $5g/6$. ■

Proof of the Mini-Vitali covering theorem: A set E is of *measure zero* if and only if for every $\varepsilon > 0$ there is an open set G containing E for which $\mathcal{L}(G) < \varepsilon$. We will later on investigate the measure $\mathcal{L}$ in greater detail; here we are interested only in applying it to an understanding of sets of measure zero and the proof then is really just about open sets.

We already know that every set of measure zero is full null, and that every full null set is fine null. To complete the proof we show that every fine null set is a set of measure zero. Let us suppose that E is not a set of measure zero. We show that it is not fine full then. Define

$$\varepsilon_0 = \inf\{\mathcal{L}(G) : G \text{ open and } G \supset E\}.$$

Since E is not measure zero, $\varepsilon_0 > 0$.

Let β be an arbitrary fine cover of E. Define

$$G = \bigcup_{([u,v],w)\in\beta} (u,v).$$

This is an open set and, by Lemma 11.16, G covers all of E except for a countable set. It follows that $\mathcal{L}(G) \geq \varepsilon_0$, since if $\mathcal{L}(G) < \varepsilon_0$ we could add to G a small open set G' that contains the missing countable set of points and for which the combined set $G \cup G'$ is an open set containing E but with measure smaller than ε_0.

By Lemma 11.17 there must exist a subpartition $\pi \subset \beta$ for which

$$\sum_{([u,v],w)\in\pi} (v-u) \geq \varepsilon_0/6.$$

But that means that E is not a fine null set, since this is true for *every* fine cover β.

11.5 Null functions

A function is a null function if it is equal to zero at every point with only a small set of exceptions.

Definition 11.18: A function $f : [a, b] \to \mathbb{R}$ is said to be a null function if the set

$$\{x \in [a, b] : f(x) \neq 0\}$$

is a set of measure zero.

Exercises

11.5.1 Show that the sum of two null functions is again a null function.

11.5.2 Show that the sum of a convergent series of null functions is again a null function.

11.5.3 Show that the absolute value of a null function is again a null function.

11.5.4 Give an example of a null function on the interval $[0, 1]$ that is not constant on any subinterval.

11.5.5 What continuous functions are also null functions?

11.5.6 Can a null function be monotone?

11.6 Integral of null functions

Our main interest in null functions is in the very special role they play for the integration theory. The following three theorems complete that theory.

Theorem 11.19: *Let $f : [a, b] \to \mathbb{R}$ be a null function. Then f is integrable on $[a, b]$ and*

$$\int_a^b f(x)\, dx = 0.$$

Corollary 11.20: *Let f, $g : [a,b] \to \mathbb{R}$ be an arbitrary functions and suppose that $f(x) = g(x)$ for every value of x in $[a,b]$ except possibly in a null set. Then f is integrable on $[a,b]$ if and only if g is integrable on $[a,b]$ and, moreover,*

$$\int_a^b f(x)\,dx = \int_a^b g(x)\,dx.$$

11.7 Functions with a zero integral

We have seen that null functions have a zero integral. We now show that *only* null functions have zero integrals.

Theorem 11.21: *Let $f : [a,b] \to \mathbb{R}$ be an integrable function with the property that*

$$\int_c^d f(x)\,dx = 0 \quad \textit{for all } [c,d] \subset [a,b].$$

Then f is a null function.

Proof. Let $\varepsilon > 0$. The indefinite integral F is zero. Thus, by Theorem 10.9, there exists a full cover β such that

$$\sum_{([u,v],w))\in\pi} |F(v) - F(u) - f(w)(v-u)| = \sum_{([u,v],w))\in\pi} |f(w)|(v-u) < \varepsilon \tag{5}$$

for every subpartition π of $[a,b]$ contained in β. Let $\beta_1 = \beta((a,b))$.

For each integer n let E_n be the set of points x in (a,b) for which $|f(x)| > 1/n$. The pruned covering relation $\beta_2 = \beta_1[E_n]$ is a full cover of E_n. Choose any subpartition π contained in β_2.

Check that

$$\sum_{([u,v],w))\in\pi} (v-u) \le \sum_{([u,v],w))\in\pi} n|f(w)|(v-u)$$

$$\leq n \sum_{([u,v],w))\in\pi} |f(w)|(v-u) < n\varepsilon$$

because of (5). From this it follows that E_n is full null (and hence a set of measure zero). The set of points x in $[a,b]$ at which $f(x) \neq 0$ is the union of the null sets $\{E_n\}$ together, possibly, with the points a and b. Since this is evidently a null set, f must be a null function. ■

Corollary 11.22: *Let $f : [a,b] \to \mathbb{R}$ be a nonnegative integrable function with the property that*

$$\int_a^b f(x)\,dx = 0.$$

Then f is a null function.

Exercises

11.7.1 Let f, $g : [a,b] \to \mathbb{R}$ be an arbitrary functions and suppose that $f(x) = g(x)$ for every value of x in $[a,b]$ except possibly in a null set. Then show that f is absolutely integrable on $[a,b]$ if and only if g is absolutely integrable on $[a,b]$ and that, moreover,

$$\int_a^b f(x)\,dx = \int_a^b g(x)\,dx \quad \text{and} \quad \int_a^b |f(x)|\,dx = \int_a^b |g(x)|\,dx$$

11.7.2 Let f be the characteristic function of the rationals on the interval $[0,1]$. Show directly that $\int_0^1 f(x)\,dx$. Then prove the same thing by choosing an appropriate function $g : [0,1] \to \mathbb{R}$ that is equal to f outside of a null set.

11.7.3 For an absolutely integrable function $f : [a,b] \to \mathbb{R}$ we define the L_1-norm as

$$\|f\|_1 = \int_a^b |f(x)|\,dx.$$

Describe the family of all absolutely integrable functions whose L_1-norm is zero.

11.7.4 For an absolutely integrable function $f : [a, b] \to \mathbb{R}$ and any $1 \leq p < \infty$ we define the L_p-norm as

$$\|f\|_p = \left(\int_a^b |f(x)|\,dx \right)^{1/p}.$$

Describe the family of all absolutely integrable functions whose L_p-norm is zero.

11.7.5 For an absolutely integrable function $f : [a, b] \to \mathbb{R}$ we define the L_∞-norm as

$$\|f\|_\infty = \inf\{t : \text{the set } \{x : |f(x)| > t\} \text{ has measure zero}\}.$$

Describe the family of all absolutely integrable functions whose L_∞-norm is zero.

11.8 Almost everywhere language

Some commonly used language is used in discussions of null sets. Let $P(x)$ be a property that may or not be possessed by a point $x \in \mathbb{R}$. We say that

$P(x)$ is true almost everywhere.

or

$P(x)$ is true for almost every x.

if the set

$$\{x \in \mathbb{R} : P(x) \text{ is not true}\}$$

is a null set.

Almost everywhere is frequently abbreviated "a.e."; thus, for example in the next section we show that bounded a.e. continuous functions are integrable. This describes functions that must be continuous at every point with the possible exception of some null set.

Exercises

11.8.1 What would it mean to say that a function is almost everywhere discontinuous?

11.8.2 What would it mean to say that a function is almost everywhere differentiable? Give an example of function that is almost everywhere differentiable, but not everywhere differentiable.

11.8.3 What would it mean to say that almost every point in $\mathbb{R}$ is irrational? Is this a true statement?

11.8.4 What would it mean to say that almost everywhere point in a set A belongs to a set B? Give an example for which this is true and an example for which this is false.

11.8.5 What would it mean to say that a function is almost everywhere equal to zero?

11.8.6 What would it mean to say that a function is almost everywhere different from zero?

11.8.7 Suppose that the function $f : [a, b] \to \mathbb{R}$ is integrable and is almost everywhere in $[a, b]$ nonnegative. Show that $\int_a^b f(x)\,dx \geq 0$.

11.8.8 Suppose that the functions f, $g : [a, b] \to \mathbb{R}$ are integrable and that $f(x) \leq g(x)$ for almost every x in $[a, b]$. Show that $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$.

11.8.9 Suppose that the functions F, $G : [a, b] \to \mathbb{R}$ are continuous almost everywhere in $[a, b]$. Is the sum function $F(x) + G(x)$ also continuous almost everywhere in $[a, b]$.

11.8.10 Suppose that the functions F, $G : [a, b] \to \mathbb{R}$ are differentiable almost everywhere in $[a, b]$. Is the sum function $F(x) + G(x)$ also differentiable almost everywhere in $[a, b]$.

11.9 Integration conventions on ignoring points

In order for an integral

$$\int_a^b f(x)\,dx$$

to make sense the function f must be defined at least at every point of the interval $[a, b]$. This is because the possible Riemann sums

$$\sum_{([u,v],w))\in\pi} f(w)(v-u)$$

that define the integral require values of $f(w)$ at every point $a \leq w \leq b$.

But we now know that we can freely change f on any null set without in any way affecting the integrability of the function or the values of the integral. Since we can make such changes at will it makes sense to announce that integrable functions need only be defined on the interval $[a, b]$ less some null set.

Thus our convention will be that if the domain of a real-valued function f is $[a, b] \setminus N$ for some null set N then we will interpret

- f is integrable if some (or any) function $g : [a, b] \to \mathbb{R}$ that is equal to f on $[a, b] \setminus N$ is integrable.
- If f is integrable in this sense then
$$\int_a^b f(x)\,dx =: \int_a^b g(x)\,dx$$
for that choice of g.

Thus, according to this convention, in order for us to examine whether a function f has an integral

$$\int_a^b f(x)\,dx$$

we need only be assured that the function is defined almost everywhere in the interval $[a, b]$.

Exercises

11.9.1 Compute

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx.$$

Does the fact that the integrand is undefined at $x = 0$ influence your argument?

11.9.2 Compute

$$\int_{-1}^1 \frac{1}{\sqrt{|x|}}\,dx.$$

Does the fact that the integrand is undefined at $x = 0$ influence your argument?

11.9.3 Suppose that $\sum_{k=1}^{\infty} |a_k| < \infty$ and that $\{r_k\}$ is an enumeration of the rational numbers in $[0, 1]$. It is possible to prove that the function

$$f(x) = \sum_{k=1}^{\infty} \frac{a_k}{\sqrt{|x - r_k|}}$$

is absolutely integrable on $[0, 1]$ and that

$$\int_0^1 f(x)\,dx = \sum_{k=1}^{\infty} \left(\int_0^1 \frac{a_k\,dx}{\sqrt{|x - r_k|}} \right).$$

How does one interpret these statements according to our convention?

11.10 Bounded a.e. continuous functions are absolutely integrable

Lebesgue proved that the class of Riemann integrable functions can be characterized as those that are bounded and almost everywhere continuous. We show directly that such functions are absolutely integrable. The class of absolutely integrable functions is, of course, much larger than this.

Theorem 11.23: *If $f : [a, b] \to \mathbb{R}$ is bounded and almost everywhere continuous then f is absolutely integrable on $[a, b]$.*

Proof. Let M be an upper bound for the values of $|f(x)|$ in the interval. Let N be the null set that allows us to say that f is continuous at every point in $[a, b] \setminus N$. We will assume that f is constant on $(-\infty, a]$ and on $[b, \infty)$. This just allows us to ignore what is happening outside of the interval $[a, b]$.

Let $\varepsilon > 0$ and define

$$\beta_1 = \{([x, y], z) : \omega f([x, y]) < \varepsilon/4(b - a).\}$$

Check, using the continuity of f, that β is a full cover of $\mathbb{R} \setminus N$. Verify that if (I, z) and (I', z') both belong to β then, either I and I' have no points in common or else $|f(z) - f(z')| < \varepsilon/(b - a)$.

Choose an open set G containing N with $\mathcal{L}(G) < \varepsilon/(2M)$. Let β_2 be the collection of all pairs $([u,v],w)$ for which $w \in N$, $u \leq w \leq v$, and $[u,v] \subset G$. It is easy to check that β_2 is a full cover of N. Thus $\beta = \beta_1 \cup \beta_2$ is a full cover of the whole real line.

Complete the proof by checking that

$$\sum_{(I,x)\in\pi} \sum_{(I',x')\in\pi'} \left|f(x) - f(x')\right| \mathcal{L}(I \cap I') < \varepsilon$$

for any pair of partitions π and π' of $[a,b]$. This just requires handling the pairs of items (I,x) and (I',x') differently depending on whether both came originally from β_1 or one of the pair is in β_2. The first case we have already done in the preceding theorem. The second case should present no difficulties if the reader will remember the minor point that

$$\left|f(x) - f(x')\right| \leq 2M$$

in the sum.

Thus f satisfies McShane's criterion on $[a,b]$. It follows that f is absolutely integrable there. ■

Exercises

11.10.1 Give an example of a function with a dense set of discontinuities that is integrable.

11.10.2 Give an example of a function with a dense set of discontinuities that is not integrable.

See Note 242

11.10.3 Show that an unbounded function with merely one point of discontinuity need not be integrable.

11.10.4 A careless student argues: If a bounded function f is almost everywhere continuous that means that there is a continuous function g that is almost everywhere equal to f. Obviously this gives a much easier proof of Theorem 11.23. Your comments?

11.10.5 Prove that if $E \subset [a,b]$ is a closed set then χ_E is integrable on $[a,b]$.

11.10.6 Prove that every bounded measurable function on an interval $[a, b]$ is absolutely integrable, using the following definition[2]:

> $f : [a, b] \to \mathbb{R}$ is *measurable* if, for every $\varepsilon > 0$, there is an open set G and a continuous function $g : [a, b] \to \mathbb{R}$ such that $\mathcal{L}(G) < \varepsilon$ and $f(x) = g(x)$ for every x in $[a, b]$ that is not in G.

11.10.7 Prove that if $E \subset [a, b]$ is a measurable set then χ_E is integrable on $[a, b]$, using the following definition:

> E is *measurable* if, for every $\varepsilon > 0$, there is an open set G such that $\mathcal{L}(G) < \varepsilon$ and such that set $E \setminus G$ is closed.

11.10.8 (Riemann-Lebesgue) Show that a function f on an interval $[a, b]$ that is Riemann integrable [i.e., integrable but using uniformly full covers (cf. Exercise 10.1.6)] *must* be bounded and a.e. continuous. [This is an historically interesting fact, showing exactly the limitations of the Riemann integration theory.]

SEE NOTE 243

Notes

[242]Exercise 11.10.2. To avoid working too hard on this, make your function unbounded. (To construct a bounded, nonintegrable function requires use of a special logical principle.

[243]Exercise 11.10.7. Use the oscillation $\omega_f(x)$ of a function f at a point x as discussed in Section 6.7. (This value is positive if and only if f is discontinuous at x.) Check first that f must be bounded (Exercise 10.1.5). Fix $e > 0$ and consider the set $N(e)$ of points x such that the oscillation of f at x is greater than e; that is, so that

$$\omega_f(x) > e.$$

Any interval (c, d) that contains a point $x \in N(e)$ will certainly have

$$\omega f([c, d]) \geq e.$$

[2]In Chapter 17 we give the more familiar definition of measurable; in advanced courses it is shown that they are equivalent

Let $\varepsilon > 0$ and use Exercise 10.1.6 to find

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

such that

$$\sum_{k=1}^{n} \omega f([x_{k-1}, x_k](x_k - x_{k-1}) < \varepsilon e/2.$$

Select just those intervals that contain a point from $N(e)$ in their interior. The total length of these intervals cannot exceed $(e\varepsilon)/(2e)$ since $\omega f([x_{k-1}, x_k]) \geq e$ for each interval $[x_{k-1}, x_k]$.

This covers the set $N(e)$ by a sequence of intervals $[x_{k-1}, x_k]$ of total length less than $\varepsilon/2$, except that possibly we have missed a point x_i that happens to be in $N(e)$. In any case, argue that $N(e)$ has measure zero. But the set of points of discontinuity of f is the union of the sets $N(1)$, $N(1/2)$, $N(1/4)$, $N(1/8)$, $\ldots$.

Chapter 12

VARIATION OF A FUNCTION

Dripped Chapter[1]

The notion of variation of a function was introduced by Camille Jordan near the end of the 19th century. For a function F defined on a compact interval $[a,b]$ he considered sums of the form

$$\sum_{([u,v],w)\in\pi} |F(v)-F(u)|$$

taken over all possible partitions π of $[a,b]$. The supremum of such sums was called the *total variation* of F. It plays a key role in many parts of analysis, particularly in the study of derivatives and integrals.

We begin by examining functions with small variation on a set. One of the main goals of this chapter is to prove the Lebesgue differentiation theorem asserting that functions that have finite total variation on an interval are differentiable almost everywhere there. In Chapter 18 we will return to this study using the tool known as the Stieltjes integral.

[1]Note to the instructor: For a *modest* course in integration theory you would have skipped the previous ***dripped*** chapter. If not then this chapter is a bit compelling. Having defined full null sets it is only a minor adjustment in the definition to generalize to the concept of zero variation. Should we then go on to the Lebesgue differentiation theorem, whose proof is given here using completely elementary arguments? Absolute continuity comes into the picture naturally. This material, familiar enough at the graduate level, uses only tools that should be accessible to students who have successfully got this far.

12.1 Functions having zero variation

Definition 12.1: Let $F : \mathbb{R} \to \mathbb{R}$ and let E be a set of real numbers. We say that F *has zero variation* on the set E provided that for every $\varepsilon > 0$ there is a full cover β of the set E so that

$$\sum_{([u,v],w)\in\pi} |F(v) - F(u)| < \varepsilon$$

whenever π is a subpartition, $\pi \subset \beta$.

Lemma 12.2: *Let $F : \mathbb{R} \to \mathbb{R}$. Then F has zero variation on an open interval (a, b) if and only if F is constant on (a, b).*

Proof. One direction is obvious; the other direction is an application of the Cousin covering lemma. Suppose that F has zero variation on (a, b). Let $\varepsilon > 0$ and choose a full cover β of the set (a, b) so that

$$\sum_{([u,v],w)\in\pi} |F(v) - F(u)| < \varepsilon$$

whenever π is a subpartition, $\pi \subset \beta$. If $[c, d] \subset (a, b)$ then there is a partition $\pi \subset \beta$ of the whole interval $[c, d]$. Consequently

$$|F(d) - F(c)| \leq \sum_{([u,v],w)\in\pi} |F(v) - F(u)| < \varepsilon.$$

This holds for every such interval $[c, d]$ and every positive ε. It follows that F must be constant on (a, b). ■

Lemma 12.3: *Let $F : \mathbb{R} \to \mathbb{R}$, let E_1, E_2, E_3, ... be a sequence of sets and suppose that F has zero variation on each E_i $(i = 1, 2, 3, \dots)$. Then F has zero variation on any subset of the union $\bigcup_{i=1}^{\infty} E_i$.*

Proof. Let $\varepsilon > 0$ and, for each integer i, choose a full cover β_i of E_i so that

$$\sum_{([u,v],w)\in\pi} |F(v) - F(u)| < 2^{-i}\varepsilon \tag{1}$$

whenever π is a subpartition, $\pi \subset \beta_i$. Construct β as the union of the sequence $\beta_i[E_i]$. This is a full cover of any subset E of the union $\bigcup_{i=1}^{\infty} E_i$. Now simply check that, if π is a subpartition, $\pi \subset \beta$ then

$$\sum_{([u,v],w)\in\pi} |F(v) - F(u)| \leq \sum_{i=1}^{\infty} \sum_{([u,v],w)\in\pi[E_i]} |F(v) - F(u)| < \sum_{i=1}^{\infty} 2^{-i}\varepsilon = \varepsilon. \tag{2}$$

It follows that F has zero variation on E. ■

Exercises

12.1.1 Show that a constant function has zero variation on any set.

12.1.2 Show that if F has zero variation on a set E then it has zero variation on any subset of E.

12.1.3 Let E contain a single point x_0. What does it mean for F to have zero variation on E?

12.1.4 Let E have countably many points. Show that F has zero variation on the set E if and only if F has zero variation on the singleton sets $\{e\}$ for each $e \in E$.

12.1.5 Show that N is a null set if and only if the function $F(x) = x$ has zero variation on N.

12.1.6 Suppose that both the functions F and G have zero variation on a set E. Show that so too does every linear combination $rF + sG$.

12.1.7 Suppose that both the functions F and G have zero variation on a set E. Does it follow that the product FG must have zero variation on E?

12.1.8 Show that a continuous function has variation zero on every countable set.

12.1.9 Show that a function that has variation zero on every countable set must be continuous.

12.2 Zero variation and zero derivatives

There is an intimate connection between the notion of zero variation and the fact of zero derivatives. The following two theorems are central to our theory.

Theorem 12.4: *Let $F : \mathbb{R} \to \mathbb{R}$ and suppose that $F'(x) = 0$ at every point of the set E. Then F has zero variation on E.*

Proof. Fix an integer n and write $E_n = (-n, n) \cap E$. Let $\varepsilon > 0$ and consider the collection

$$\beta = \{([u, v], w) : w \in E,\ w \in [u, v] \subset (-n, n),\ |F(v) - F(u)| < \varepsilon(v - u)\}.$$

By our assumption that $F'(x) = 0$ at every point of E we see easily that β is a full cover of E_n. But if $\pi \subset \beta$ is any subpartition we must have

$$\sum_{([u,v],w)\in\pi} |F(v) - F(u)| < \sum_{([u,v],w)\in\pi} \varepsilon(v - u) < 2\varepsilon n.$$

This proves that F has zero variation on each set E_n. It follows from Lemma 12.3 that F has zero variation on the set E which is, evidently, the union of the sequence of sets $\{E_n\}$. ■

Theorem 12.5: *Let $F : \mathbb{R} \to \mathbb{R}$ and suppose that F has zero variation on a set E. Then $F'(x) = 0$ at almost every point of the set E.*

This theorem is deeper than the preceding and will require, for us, an appeal to our version of the Vitali covering theorem. We present the proof in the next subsection because the techniques there will be used in a number of different computations.

12.2.1 Proof of the zero variation/derivative theorem

Zero variation implies a zero derivative almost everywhere.

Our proof of Theorem 12.5 is given now. We will want, also, to generalize it in the next section. But let us start with the simple version here.

Let N be the set of points x in E at which $F'(x) = 0$ is false. A fine covering argument allows us to analyze this. There must be *some* positive number $\varepsilon(x)$ for each $x \in N$ so that

$$\beta_1 = \{([u,v],w) : w \in E,\ |F(v) - F(u)| \geq \varepsilon(w)(v-u)\} \tag{3}$$

is a fine cover of N. This is how the full/fine arguments work. For, if not, then there would be some point x in E so that, for *every* $\varepsilon > 0$,

$$\beta_2 = \{([u,v],w) : w \in E,\ |F(v) - F(u)| < \varepsilon(v-u)\} \tag{4}$$

would have to be full at x. But that says exactly that $F'(x) = 0$.

Let $\eta > 0$. Since F has zero variation on E we can find a full cover β_2 of N so that there is a full cover β of the set E so that

$$\sum_{([u,v],w)\in\pi} |F(v) - F(u)| < \eta \tag{5}$$

whenever π is a subpartition, $\pi \subset \beta_2$. The intersection $\beta = \beta_1 \cap \beta_2$ is a fine cover of N.

Write $N_i = \{w \in N : \varepsilon(w) > 1/i\}$ for each integer i and note that N is the union of the sequence of sets $\{N_i\}$. For each set N_i and any subpartition $\pi \subset \beta[N_i]$ we compute, with some help from (3) and (5), that

$$\sum_{([u,v],w)\in\pi} (v-u) < \sum_{([u,v],w)\in\pi} \varepsilon(w)|F(v) - F(u)|$$

$$\leq i \sum_{([u,v],w)\in\pi} |F(v) - F(u)| < i\eta.$$

This verifies that each set N_i is a fine null set and so, by the Mini-Vitali covering theorem, also a set of measure zero. Consequently N itself, as the union of a sequence of measure zero sets, is also a set of measure zero. This completes the proof.

12.2.2 Generalization of the zero derivative/variation

We wish to interpret this result in a much more general manner. Let h be any real-valued function that assigns values $h(([u,v],w))$ to pairs $([u,v],w))$. We can define zero variation and zero derivative for h just as easily as we can for a function $F:\mathbb{R}\to\mathbb{R}$.

- h has *zero variation* on a set E if for every $\varepsilon>0$ there is a full cover β of E so that

  $$\sum_{([u,v],w)\in\pi} |h(([u,v],w))| < \eta$$

  whenever π is a subpartition, $\pi\subset\beta$.

- h has a *zero derivative* at a point w if

  $$\lim_{\delta\to0+}\sup\left\{\left|\frac{h(([u,v],w))}{v-u}\right| : u\le w\le v,\ 0<v-u<\delta\right\}=0.$$

A repeat of the proof just given, with minor changes, allows us to claim that

$\star$ *Zero variation for h on a set E implies h has a zero derivative almost everywhere in E.*

We use $\star$ to prove all versions of the fundamental theorem of the calculus in the sequel, both for ordinary integrals and for Stieltjes integrals.

12.3 Functions of bounded variation

Definition 12.6: The *total variation* of a function $F : [a, b] \to \mathbb{R}$ on that interval is the number $V(F, [a, b])$ defined as the supremum of the values

$$\sum_{i=1}^{n} |F(s_i) - F(s_{i-1})|$$

taken over all choices of points

$$a = s_0 < s_1 < \cdots < s_{n-1} < s_n = b.$$

Definition 12.7: A function F defined on a compact interval $[a, b]$ is said to have *bounded variation* on that interval provided that $V(F, [a, b]) < \infty$.

Note that, should F be monotonic on $[a, b]$ then

$$V(F([a, b]) = |F(b) - F(a)|.$$

Thus all monotonic functions have bounded variation.

Exercises

12.3.1 Compute $V(F, [a, b])$ if F is monotonic.

12.3.2 Estimate $V(F_1 + F_2, [a, b])$.

12.3.3 Estimate $V(rF_1 + sF_2, [a, b])$.

12.3.4 Estimate $V(F_1 \cdot F_2, [a, b])$.

12.3.5 Compute $V(F, [0, 1])$ if F is the continuous function given by the formula $F(x) = x \sin(1/x)$.

12.3.6 Show that $V(F, [0, 1]) < \infty$ if F is the continuous function given by the formula $F(x) = x^2 \sin(1/x)$.

12.3.7 Show that every function that has bounded variation on an interval is bounded there.

12.3.8 Let $\{F_k\}$ be a sequence of functions on a compact interval $[a,b]$ such that $\sup_k V(F_k,[a,b]) < \infty$. If $F(x) = \lim_{k\to\infty} F_k(x)$ for all x in $[a,b]$ show that F has bounded variation on $[a,b]$.

12.3.9 Give an example of a sequence of functions $\{F_k\}$ such that $V(F_k,[a,b]) < \infty$ for each k and for which $F(x) = \lim_{k\to\infty} F_k(x)$ exists at every point, but for which F does not have bounded variation on $[a,b]$.

12.4 Lebesgue differentiation theorem

Theorem 12.8: *Let $F : [a,b] \to \mathbb{R}$ be a function of bounded variation. Then F is differentiable at almost every point in (a,b).*

Corollary 12.9: *Let $F : [a,b] \to \mathbb{R}$ be a monotonic function. Then F is differentiable at almost every point in (a,b).*

The proof of the theorem will require an introduction, first, to the upper and lower derivates and then a simple geometric lemma that allows us to use a fine covering argument to show that the set of points where $F'(x)$ does not exist is measure zero.

12.4.1 Upper and lower derivates

The proof uses the *upper and lower derivates.* To analyze how a derivative $F'(x)$ may fail to exist we split that failure into two pieces, an upper and a lower, defined as

$$\overline{D}F(x) = \inf_{\delta>0} \sup \left\{ \frac{F(v)-F(u)}{v-u} : x \in [u,v],\ 0 < v-u < \delta \right\}$$

and

$$\underline{D}F(x) = \sup_{\delta>0} \inf \left\{ \frac{F(v)-F(u)}{v-u} : x \in [u,v],\ 0 < v-u < \delta \right\}$$

We will prove that, for almost every point x in (a,b),

$$\overline{D}F(x) > -\infty, \qquad \underline{D}F(x) < \infty,$$

and
$$\overline{D}F(x) = \underline{D}F(x).$$
From these three assertions it follows that F has a finite derivative $F'(x)$ at almost every point x in (a,b).

The proof will depend on a fine covering argument. For that we need to recognize the following connection between derivates and covers. The proof is trivial; it is only a matter of interpreting the statements.

Lemma 12.10: *Let $F : [a,b] \to \mathbb{R}$, $\alpha \in \mathbb{R}$, and let*
$$\beta = \left\{([u,v],w) : \frac{F(v)-F(u)}{v-u} > \alpha,\ w \in [u,v] \subset [a,b]\right\}.$$
Then, β is a full cover of the set
$$E_1 = \{x \in (a,b) : \underline{D}F(x) > \alpha\}$$
and a fine cover of the larger set
$$E_2 = \{x \in (a,b) : \overline{D}F(x) > \alpha\}.$$

12.4.2 Geometrical lemmas

The proof employs an elementary geometric lemma that Donald Austin[2] used in 1965 to give a simple proof of this theorem. Our proof of the differentiation theorem is essentially his, but written in different language. See also the version of Michael Botsko[3].

[2] D. Austin, *A geometric proof of the Lebesgue differentiation theorem.* Proc. Amer. Math. Soc. 16 (1965) 220–221.

[3] M. W. Botsko, *An elementary proof of Lebesgue's differentiation theorem.* Amer. Math. Monthly 110 (2003), no. 9, 834–838.

Lemma 12.11 (Austin's lemma) *Let* $G : [a, b] \to \mathbb{R}$, $\alpha > 0$ *and suppose that* $G(a) \le G(b)$. *Let*

$$\beta = \left\{ ([u, v], w) : \frac{G(v) - G(u)}{v - u} < -\alpha,\ w \in [u, v] \subset [a, b] \right\}.$$

Then, for any nonempty subpartition $\pi \subset \beta$,

$$\alpha \left(\sum_{([u,v],w)\in\pi} (v - u) \right) < V(G, [a, b]) - |G(b) - G(a)|.$$

Proof. To prove the lemma, let π_1 be a partition of $[a, b]$ that contains the subpartition π. Just write

$$|G(b) - G(a)| = G(b) - G(a) = \sum_{([u,v],w)\in\pi_1} [G(v) - G(u)]$$

$$= \sum_{([u,v],w)\in\pi} [G(v) - G(u)] + \sum_{([u,v],w)\in\pi_1\setminus\pi} [G(v) - G(u)]$$

$$< -\alpha \left(\sum_{([u,v],w)\in\pi} [v - u] \right) + V(G, [a, b]).$$

The statement of the lemma follows. ■

As a corollary we can replace F with $-F$ to obtain a similar statement.

Corollary 12.12: *Let $G : [a,b] \to \mathbb{R}$, $\alpha > 0$ and suppose that $G(b) \le G(a)$. Let*

$$\beta = \left\{ ([u,v],w) : \frac{G(v)-G(u)}{v-u} > \alpha,\ \ w \in [u,v] \subset [a,b] \right\}.$$

Then, for any nonempty subpartition $\pi \subset \beta$,

$$\alpha \left(\sum_{([u,v],w)\in\pi} (v-u) \right) < V(G,[a,b]) - |G(b)-G(a)|.$$

12.4.3 Proof of the Lebesgue differentiation theorem

We now prove the theorem. The first step in the proof is to show that at almost every point t in (a,b),

$$\underline{D}F(t) = \overline{D}F(t).$$

If this is not true then there must exist a pair of rational numbers r and s for which the set

$$E_{rs} = \{t \in (a,b) : \underline{D}F(t) < r < s < \overline{D}F(t)\}$$

is not a set of measure zero. This is because the union of the countable collection of sets E_{rs} contains all points t for which $\underline{D}F(t) \ne \overline{D}F(t)$.

Let us show that each such set E_{rs} is fine null. By the Mini-Vitali theorem we then know that E_{rs} is a set of measure zero. Write $\alpha = (s-r)/2$, $B = (r+s)/2$, $G(t) = F(t) - Bt$. Note that

$$E_{rs} = \{t \in (a,b) : \underline{D}G(t) < -\alpha < 0 < \alpha < \overline{D}G(t)\}.$$

Since F has bounded variation on $[a,b]$, so too does the function G. In fact

$$V(G,[a,b]) \le V(F[a,b]) + B(b-a).$$

Let $\varepsilon > 0$ and select points

$$a = s_0 < s_1 < \cdots < s_{n-1} < s_n = b$$

so that

$$\sum_{i=1}^{n} |G(s_i) - G(s_{i-1})| > V(G, [a, b]) - \alpha\varepsilon.$$

Let $E'_{rs} = E_{rs} \setminus \{s_1, s_2, \dots, s_{n-1}\}$. Let us call an interval $[s_{i-1}, s_i]$ *black* if $G(s_i) - G(s_{i-1}) \geq 0$ and call it *red* if $G(s_i) - G(s_{i-1}) < 0$.

For each $i = 1, 2, 3, \dots, n$ we define a covering relation β_i as follows. If $[s_{i-1}, s_i]$ is a black interval then

$$\beta_i = \left\{([u, v], w) : \frac{G(v) - G(u)}{v - u} < -\alpha,\ w \in [u, v] \subset [s_{i-1}, s_i]\right\}.$$

If, instead, $[s_{i-1}, s_i]$ is a red interval then

$$\beta_i = \left\{([u, v], w) : \frac{G(v) - G(u)}{v - u} > \alpha,\ w \in [u, v] \subset [s_{i-1}, s_i]\right\}.$$

Let $\beta = \bigcup_{i=1}^{n} \beta_i$. Because of Lemma 12.10 we see that this collection β is a fine cover of E'_{rs}.

Let π be any nonempty subpartition contained in β. Write $\pi_i = \pi \cap \beta_i$. By Lemma 12.11 applied to the black intervals and Corollary 12.11 applied to the red intervals we obtain that

$$\alpha \left(\sum_{([u,v],w)\in\pi_i} (v - u) \right) < V(G, [s_{i-1}, s_i]) - |G(s_i) - G(s_{i-1})|.$$

Consequently

$$\alpha \left(\sum_{([u,v],w)\in\pi} (v - u) \right) = \alpha \left(\sum_{i=1}^{n} \sum_{([u,v],w)\in\pi_i} (v - u) \right)$$

$$\leq \sum_{i=1}^{n} V(G, [s_{i-1}, s_i]) - \sum_{i=1}^{n} |G(s_i - G(s_{i-1})|$$

$$\leq V(G,[a,b]) - [V(G,[a,b]) - \alpha\varepsilon] = \alpha\varepsilon.$$

We have proved that β is a fine cover of E'_{rs} with the property that

$$\sum_{([u,v],w)\in\pi} (v-u) < \varepsilon$$

for every subpartition $\pi \subset \beta$. It follows that E'_{rs} is fine null, and hence a set of measure zero. So too then is E_{rs} since the two sets differ by only a finite number of points.

We know now that the function F has a derivative, finite or infinite, almost everywhere in (a,b). We wish to exclude the possibility of the infinite derivative, except on a set of measure zero.

Let

$$E_\infty = \{t \in (a,b) : \underline{D}F(t) = \infty\}.$$

Choose any B so that $F(b) - F(a) \leq B(b-a)$ and set $G(t) = F(t) - Bt$. Note that $G(b) \leq G(a)$ which will allow us to apply Corollary 12.12.

Let $\varepsilon > 0$ and choose a positive number α large enough so that

$$V(G,[a,b]) - |G(b) - G(a)| < \alpha\varepsilon.$$

Define

$$\beta = \left\{([u,v],w) : \frac{G(v)-G(u)}{v-u} > \alpha, [u,v] \subset [a,b]\right\}.$$

This is a fine cover of E_∞. Let π be any subpartition $\pi \subset \beta$. By our corollary then

$$\alpha \sum_{([u,v],w)\in\pi} (v-u) < V(G,[a,b]) - |G(b) - G(a)| < \alpha\varepsilon.$$

We have proved that β is a fine cover of E_∞ with the property that

$$\sum_{([u,v],w)\in\pi_i} (v-u) < \varepsilon$$

for every subpartition $\pi \subset \beta$. It follows that E_∞ is fine null, and hence a set of measure zero. The same arguments will handle the set

$$E_{-\infty} = \{t \in (a,b) : \overline{D}F(t) = -\infty\}.$$

12.5 Continuity and absolute continuity

The notions of continuity and absolute continuity[4] arise from the focus on functions that have zero variation on small sets. (As part of this definition we include the singular and saltus definitions, although we will not need them for some time.)

Definition 12.13: Let $F : \mathbb{R} \to \mathbb{R}$.

1. We say that F is *continuous at a point* x_0 provided that F has zero variation on the singleton set $E = \{x_0\}$.

2. We say that F is *continuous* at points of a set E provided that F has zero variation on any countable subset of E.

3. We say that F is *absolutely continuous* in E provided that F has zero variation on any null subset of E.

4. We say that F is *singular* in E if there is a null subset N of E so that F has zero variation on $E \setminus N$.

5. We say that a monotonic function $F : [a,b] \to \mathbb{R}$ is *a saltus function* if there is a nonempty finite or countable set $N \subset (a,b)$ so that F has zero variation on $(a,b) \setminus N$ and F does not have zero variation in any subset of N.

[4] Note to instructors: This version of absolute continuity is the measure version (zero variation on measure zero sets). The less general, but more familiar, ε-δ–version of most real analysis course is discussed in Chapter 18 along with functions of bounded variation.

The terms continuous, absolutely continuous, and singular without reference to any set E would mean that the function is continuous, absolutely continuous, or singular in all of $\mathbb{R}$. The term saltus[5] comes from the Latin for "jump." We are interested only in monotone saltus functions, which you may prefer to call, as some do, *jump functions*.

Theorem 12.14: *Let $F : \mathbb{R} \to \mathbb{R}$ and let E be a set of real numbers. Then F is singular in E if and only if $F'(x) = 0$ at almost every point of E.*

Proof. In one direction this follows directly from the Theorem 12.5. If F is singular in E then there is a null set $N \subset E$ and F has zero variation on $E \setminus N$. Thus $F'(x) = 0$ at almost every point of the set $E \setminus N$, hence $F'(x) = 0$ at almost every point of E. Conversely if $F'(x) = 0$ for all $x \in E' \subset E$ and $E \setminus E'$ is a null set, then F has zero variation on E'. ■

Corollary 12.15: *Let $F : [a, b] \to \mathbb{R}$ be a saltus function. Then $F'(x) = 0$ at almost every point in (a, b).*

Proof. A saltus function is singular in (a, b). ■

Exercises

12.5.1 Show that the definition of continuity here is identical with the definition of continuity elsewhere in the text:

> Let $F : \mathbb{R} \to \mathbb{R}$ and let x_0 be a real number. Then F is continuous at x_0 if and only if for every $\varepsilon > 0$ there exists a $\delta > 0$ so that
> $$|F(x) - F(x_0)| < \varepsilon$$
> for all $|x - x_0| < \delta$.

12.5.2 Show that every absolutely continuous function is continuous.

[5] Easily remembered, for English speakers, as part of the word "somersault" from Middle French *sombresaut* leap, ultimately from Latin *super* over + *saltus* leap, from *salire* to jump.

12.5.3 Show that if F is an absolutely continuous function in a set E then F is absolutely continuous in every subset of E.

12.5.4 Let $F : \mathbb{R} \to \mathbb{R}$. Show that, according to the definition here, F is continuous if and only if F is continuous at every point in $\mathbb{R}$.

12.5.5 If $F : \mathbb{R} \to \mathbb{R}$ is both absolutely continuous and singular then show that F is constant.

12.5.6 Show that if F is absolutely continuous in each of the sets E_1 and the set E_2 then F is absolutely continuous in every subset of $E_1 \cup E_2$.

12.5.7 Show that if F is absolutely continuous in each of the sets E_1, E_2, E_3, ... then F is absolutely continuous in every subset of the union $E = \bigcup_{n=1}^{\infty} E_n$.

12.5.8 Show that if F has a *bounded* derivative at every point of a set E then F is absolutely continuous in E.

12.5.9 Show that if F has a derivative at every point of a set E then F is absolutely continuous in E.

SEE NOTE 244

12.5.10 Using the definition in this section, suppose that each of F, F_1 $F_2 : [a, b] \to \mathbb{R}$ is continuous at a point x_0 and show that $|F|$ and any linear combination $rF_1 + rF_2$ are also continuous at x_0.

12.5.11 Using the definition in this section, suppose that each of F, F_1 $F_2 : [a, b] \to \mathbb{R}$ is absolutely continuous and show that $|F|$ and any linear combination $rF_1 + rF_2$ are also absolutely continuous .

12.5.12 Show that a saltus function is singular, but not necessarily conversely.

12.5.13 Show that a saltus function is definitely not continuous, but must have "many" points of continuity.

12.5.14 A careless student claims that he can improve on Corollary 12.15. If $F : [a, b] \to \mathbb{R}$ is a saltus function and C is the countable set of the "jumps" then, clearly, near any x not in C the function must be constant and so has a zero derivative. Thus, not merely is $F'(x) = 0$ outside of a null set, it is zero outside of a countable set. Your comments?

12.5.15 ✂ Let $F : [a, b] \to \mathbb{R}$ be a continuous, nondecreasing function. Show that F is absolutely continuous on (a, b) if and only if F has zero variation on the set of points where $F'(x) = \infty$.

12.5.1 Decompositions of monotone functions

Theorem 12.16: *Let $F : [a, b] \to \mathbb{R}$ be a nondecreasing function. Then there is a nondecreasing saltus function S and a continuous, nondecreasing function G so that*

$$F(x) = G(x) + S(x)$$

for all $a \le x \le b$ and

$$F'(x) = G'(x) + S'(x) = G'(x)$$

for almost all $a < x < b$.

Proof. The identity $F'(x) = G'(x) + S'(x) = G'(x)$ for almost all $a < x < b$ follows easily once we have established that $F(x) = G(x) + S(x)$. Almost everywhere G has a derivative and almost everywhere the saltus function (being singular) has a zero derivative.

We now show how to construct S. Let $\{x_n\}$ be the sequence of discontinuity points of F in (a, b). (We will suppose that F is continuous at a and b, but a similar argument would handle these points too.) The reader will recall that monotone functions have only jump discontinuities, and only countably many of them (see Theorems 5.61 and 5.64).

Define $c_n = F(x_n) - F(x_n-)$ and $d_n = F(x_n+) - F(x_n)$. These are nonnegative numbers, at least one of which is positive, and

$$\sum_{n=1}^{\infty} (c_n + d_n) < \infty.$$

Define the saltus function

$$S(x) = \sum_{a<x_n \le x} c_n + \sum_{a<x_n<x} d_n.$$

Write $G(x) = F(x) - S(x)$. Check that G is continuous everywhere (easy). Thus the proof is completed by showing that G is nondecreasing, which will take one more step.

Take any points $a \leq p < q \leq b$ at which F is continuous. Compute

$$S(q) - S(p) = \sum_{p<x_n<q} S(x_n+) - S(x_n-)$$

$$= \sum_{p<x_n<q} F(x_n+) - F(x_n-) \leq F(q) - F(p).$$

This gives us

$$F(p) - S(p) \leq F(q) - S(q).$$

In particular we now know that $G(p) \leq G(q)$ if $a \leq p < q \leq b$ and x and y are points of continuity of F. But such points are dense in $[a, b]$ and hence $G(x) \leq G(y)$ for *all* $a \leq x < y \leq b$. ■

12.6 Absolute continuity of the indefinite integral

In order for a function $F : [a, b] \to \mathbb{R}$ to be the indefinite integral of an integrable function it is necessary[6] that F be absolutely continuous.[7]

Theorem 12.17: *A necessary condition for a function $F : [a, b] \to \mathbb{R}$ to be the indefinite integral of an integrable function is that F is absolutely continuous in (a, b) and continuous on $[a, b]$.*

Proof. For convenience we will agree that $f(x) = 0$ for $x \notin [a, b]$ and that F is constant on $(-\infty, a]$ and $[b, \infty)$. Then, using the Henstock's zero variation criterion (see Section 10.2.6), we know that the function

$$h(([u, v], w)) = F(v) - F(u) - f(w)(v - u)$$

[6] At this level, we can prove only the necessity part. In fact *every* function that is absolutely continuous is the integral of its derivative. That can only be proved with much deeper tools.

[7] A necessary condition and sufficient condition for a function $F : [a, b] \to \mathbb{R}$ to be the indefinite integral of an *absolutely* integrable function is that F is absolutely continuous in the narrower sense of Vitali on $[a, b]$. This must wait for Chapter 18.

has zero variation. This is the key to the proof. From this the continuity of F follows (it has zero variation on all countable sets) and the absolute continuity of F follows (it has zero variation on all null sets).

Here are the details of the latter. Let $N \subset (a,b)$ be an arbitrary null set and write for $n = 1, 2, 3, \ldots$

$$N_n = \{x \in N : |f(x)| < n\}.$$

We wish to show that F has zero variation on N. It is enough to show that F has zero variation on each set N_n since it then follows that F has zero variation on the set N which is the union of the sequence $\{N_n\}$.

Fix an integer n and let $\varepsilon > 0$. There exists by Henstock's zero variation criterion a full cover β_1 such that

$$\sum_{(([u,v],w)\in\pi} |F(v) - F(u) - f(w)(v-u)| < \varepsilon/2, \tag{6}$$

for every partition π of $[a,b]$ contained in β_1. In note that this inequality holds for *any* subpartition π from $\beta_1(a,b)$.

Since N_n is a null set there is a full cover β_2 of N_n so that

$$\sum_{(([u,v],w)\in\pi} (v-u) < \varepsilon/(2n), \tag{7}$$

for every subpartition π contained in β_2. Let $\beta = \beta_1((a,b)) \cap \beta_2$. This too is a full cover of N_n .

Suppose now that π is a subpartition contained in β. Then we deduce that

$$\sum_{(([u,v],w)\in\pi} |F(v) - F(u)| \le \sum_{(([u,v],w)\in\pi} |f(w)(v-u)| + \varepsilon/2$$

$$\le \sum_{(([u,v],w)\in\pi} n(v-u) + \varepsilon/2 < \varepsilon.$$

By definition this shows that F has zero variation on the set N_n. ■

12.7 Lipschitz functions

Definition 12.18: A function $F : [a, b] \to \mathbb{R}$ is said to be *Lipschitz* on $[a, b]$ if there is a nonnegative number M so that

$$|F(y) - F(x)| \ \leq M|y - x|$$

for all real x and y.

The smallest possible number M for a Lipschitz function is called its *Lipschitz constant*.

Exercises

12.7.1 Show that every Lipschitz function $F : [a, b] \to \mathbb{R}$ is continuous.

12.7.2 Show that every Lipschitz function $F : [a, b] \to \mathbb{R}$ has bounded variation on $[a, b]$.

12.7.3 Show that every Lipschitz function $F : [a, b] \to \mathbb{R}$ is absolutely continuous on (a, b).

12.7.4 Show that a continuously differentiable function $F : [a, b] \to \mathbb{R}$ is Lipschitz.

12.7.5 Show that every differentiable function $F : [a, b] \to \mathbb{R}$ is Lipschitz provided the derivative $F'(x)$ is bounded.

12.7.6 Suppose that each of F, F_1 $F_2 : [a, b] \to \mathbb{R}$ is Lipschitz and show that $|F|$ and any linear combination $rF_1 + rF_2$ are also Lipschitz.

12.7.7 Suppose that both of F_1 $F_2 : [a, b] \to \mathbb{R}$ are Lipschitz. Is the product F_1F_2 also Lipschitz?

12.8 Monotonicity theorems

A basic principle in calculus regarding the monotonicity of functions is this: a set of points E where the derivative is unknown can be introduced into a monotonicity theorem provided that the function has zero variation on E.

Compare the following two theorems. These can be made the model for many others of this type. The proof of the first theorem is best obtained by using the mean-value theorem, but that proof does not generalize if an exceptional set is allowed. The proof of the second theorem is a straightforward application of our covering arguments.

Theorem 12.19: *Let $F : \mathbb{R} \to \mathbb{R}$ and suppose that $F'(x) > m$ for each x in a compact interval $[c,d]$. Then*

$$F(d) - F(c) > m(d-c).$$

Theorem 12.20: *Let $F : \mathbb{R} \to \mathbb{R}$ and suppose that $F'(x) > m$ for each x in a compact interval $[c,d]$ with the exception of points x in a null set E. Assume that F has zero variation on E. Then*

$$F(d) - F(c) \geq m(d-c).$$

Proof. Let $\varepsilon > 0$. Define β_1 to be the collection of all pairs $([x,y],z)$ subject only to the conditions that if $a \leq z \leq b$, $x \notin E$, and $[x,y] \subset [c,d]$ then $F(y) - F(x) > m(y-x)$. Because our assumption that $f'(x) > m$ for each x not in E but in the compact interval $[c,d]$ this must be a full cover. Using the fact that E is a null set choose a full cover β_2 of the set E so that

$$\sum_{([u,v],w)\in\pi} (v-u) < \varepsilon$$

for every subpartition π of β_2.

Using the fact that F has zero variation in E choose a full cover β_3 of the set E so that

$$\sum_{([u,v],w)\in\pi} |F(v) - F(u)| < \varepsilon$$

for every subpartition π of β_2.

Define $\beta = \beta_1[\mathbb{R}\setminus E] \cup (\beta_2[E] \cap \beta_3[E])$. This is a full cover. Take a partition π of the interval $[c,d]$ contained in β. Write $\pi_1 = \pi[\mathbb{R}\setminus E]$ and $\pi_2 = \pi[E]$. Note that

$$F(d) - F(c) = \sum_{([u,v],w)\in\pi} [F(v) - F(u)] =$$

$$\sum_{([u,v],w)\in\pi_1} [F(v) - F(u)] + \sum_{([u,v],w)\in\pi_2} [F(v) - F(u)]$$

$$> \sum_{([u,v],w)\in\pi} m(v-u) - \sum_{([u,v],w)\in\pi_2} m(v-u) - \sum_{([u,v],w)\in\pi_2} |F(v)-F(u)|$$

$$> m(d-c) - m\varepsilon - \varepsilon.$$

Since ε is arbitrary the inequality of the theorem follows. ■

Exercises

12.8.1 Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and suppose that $f'(x) \geq m$ for each x in a compact interval $[c, d]$ with countably many exceptions. Then $f(d) - f(c) \geq m(d - c)$.

12.8.2 Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and suppose that $f'(x) = 0$ for each x in a compact interval $[c, d]$ with countably many exceptions. Show that f is constant. [Prove this using the preceding exercise (Exercise 12.8.1) and also construct a covering argument to prove it.]

12.8.3 Let $f : \mathbb{R} \to \mathbb{R}$ and suppose that $f'(x) \geq m$ for every x in $[a, b]$ excepting possibly a null set, but that f is absolutely continuous.

Show that
$$f(b) - f(a) \geq m(b - a).$$

12.8.4 Let $f : \mathbb{R} \to \mathbb{R}$ be an absolutely continuous function and suppose that $f'(x) = 0$ for almost every x in a compact interval $[c, d]$. Show that f is constant. [Prove this using the preceding exercise and also construct a covering argument to prove it.]

12.8.5 Let $F : \mathbb{R} \to \mathbb{R}$ be continuous and suppose that F has zero variation on a set E. Show that the image set
$$F(E) = \{y : F(x) = y \text{ for some } x \in E\}$$
has measure zero. Is the converse true?

12.8.6 Let the function $F : \mathbb{R} \to \mathbb{R}$ be continuous and monotone nondecreasing. Show that that F has zero variation on a set E if and only if the image set
$$F(E) = \{y : F(x) = y \text{ for some } x \in E\}$$
has measure zero.

12.8.7 A function $F : \mathbb{R} \to \mathbb{R}$ is said to satisfy *Lusin's condition* N if it maps zero measure sets to zero measure sets. Show that every differentiable function satisfies Lusin's condition N.

12.8.8 A function $F : \mathbb{R} \to \mathbb{R}$ is said to satisfy *Lusin's condition* N if it maps zero measure sets to zero measure sets. Show that every absolutely continuous function satisfies Lusin's condition N.

12.8.9 Let the function $F : \mathbb{R} \to \mathbb{R}$ be continuous and monotone nondecreasing. Show that that F is absolutely continuous if and only it satisfies Lusin's condition N.

Notes

[244]Exercise 12.5.9. If F has a derivative at every point of a set E then E can be split into a sequence of subsets E_n so that F has a bounded derivative in each E_n.

Chapter 13

FUNDAMENTAL THEOREM OF THE CALCULUS

Dripped Chapter

The fundamental theorem of the calculus asserts that integration and differentiation are inverse operations. Expressed in the language of a calculus course, and expressed at the level of the calculus integral, the student should recognize this theorem in the two following slogans.

Derivative of the integral: $\frac{d}{dt}\int_a^t f(x)\,dx = f(t).$

Integral of the derivative: $\int_a^b \left[\frac{d}{dx}F(x)\right]\,dx = F(b) - F(a).$

We have already established, in Chapter 10, that the first statement holds at all points of continuity of the integrand f. We know, too, that the second statement holds if F is continuous and differentiable at every point inside (a,b). In this way we have shown that the calculus integral and the Newton integral are included in our integration theory.

Our goal in this chapter is to give a *complete* account of the fundamental theorem of the calculus. We will drop continuity in the first part and show that the derivative of the integral is indeed the integrand at almost every point. In the second part we introduce exceptional sets where the derivative $F'(x)$ need not exist; we give exactly those conditions which are necessary and sufficient to be able to write

$$\int_a^b \frac{d}{dx}F(x)\,dx = F(b) - F(a)$$

13.1 Derivative of the integral

The correct version of the derivative theorem shows that, even if the function being integrated is discontinuous everywhere (which may happen), the derivative is equal to the integrand *almost everywhere*.

Theorem 13.1: *Let $f : [a, b] \to \mathbb{R}$ be an integrable function on the interval $[a, b]$. Let*

$$F(t) = \int_a^t f(x)\,dx \qquad (a \le t \le b).$$

Then $F'(t) = f(t)$ for almost every point t in (a, b).

Proof. Using the Henstock's zero variation criterion (see Section 10.2.6), we can check that the function

$$h(([u, v], w)) = F(v) - F(u) - f(w)(v - u)$$

has zero variation on (a, b). It follows from the statement $\star$ in Section 12.2.1 that h has a zero derivative almost everywhere in (a, b). In particular

$$\lim_{y\to x+} \frac{F(y) - F(x) - f(x)(y - x))}{y - x} = 0$$

and

$$= \lim_{y\to x-} \frac{F(x) - F(y) - f(x)(x - y))}{y - x} = 0$$

for almost every $a < x < b$. Interpreting these we have that $F'(x) = f(x)$ almost everywhere in the interval (a, b). ■

13.2 Integral of the derivative

We now go in the opposite direction. Can we integrate derivatives? We already know that the integral includes the simple Newton integral, but what if we allow an exceptional set? There are two controls needed on the exceptional set: it must be a null set and the variation of the function that we are proposing as an indefinite integral should be zero on that set.

Theorem 13.2: *Let F, $f : [a, b] \to \mathbb{R}$ be functions defined on the interval $[a, b]$. Suppose that there is a set $N \subset [a, b]$ with the following properties:*

1. *$F'(x) = f(x)$ for all x in $[a, b]$ excepting possibly the set N.*
2. *N is a null set.*
3. *F has zero variation on the set N.*

Then f is integrable on $[a, b]$ and

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Proof. We use the first item to define (as in the proof of Theorem 10.17) β_1 to be the collection of all pairs $([u, v], w)$ subject only to the conditions that if $a \le w \le b$, $w \notin N$ and $[u, v] \subset [a, b]$ then

$$|F(v) - F(u) - f(w)(v - u)| \le \varepsilon(v - u).$$

This must be a full cover.

Let, for each integer $n = 1, 2, 3, \ldots,$

$$N_n = \{x \in N : n - 1 \le |f(t)| < n\}.$$

We use the second item here to choose a full cover α_n of the measure zero set N_n so that

$$\sum_{([u,v],w)\in\pi} (v-u) < n^{-1}2^{-n}\varepsilon \tag{1}$$

whenever π is a subpartition, $\pi \subset \alpha_n$. Let

$$\beta_2 = \bigcup_{n=1}^{\infty} \alpha_n[N_n]$$

and observe that β_2 is a full cover of N for which

$$\sum_{([u,v],w)\in\pi} |f(w)|(v-u) \leq \sum_{n=1}^{\infty} \sum_{([u,v],w)\in\pi[N_n]} n(v-u) < \varepsilon \tag{2}$$

whenever π is a subpartition, $\pi \subset \beta_2[N]$.

We use the third item here to choose a full cover β_3 of the set N so that

$$\sum_{([u,v],w)\in\pi} |F(v)-F(u)| < \varepsilon \tag{3}$$

whenever π is a subpartition, $\pi \subset \beta_3$.

Now we tailor a full cover β from these three covering relations. Set

$$\beta = \beta_1[\mathbb{R}\setminus N] \cup (\beta_2 \cap \beta_3)\,.$$

Now we verify the Henstock criterion for this full cover β. Let $\pi \subset \beta$ be a partition of the interval $[a,b]$. Write $\pi_1 = \pi[\mathbb{R}\setminus N]$ and $\pi_2 = \pi[N]$. Then

$$\sum_{([u,v],w)\in\pi} |F(v)-F(u)-f(w)(v-u)| =$$

$$\sum_{([u,v],w)\in\pi_1} |F(v) - F(u) - f(w)(v-u)|$$
$$+ \sum_{([u,v],w)\in\pi_2} |F(v) - F(u)| + \sum_{([u,v],w)\in\pi_2} |f(w)|(v-u) < 3\varepsilon.$$

Since this inequality holds for every partition π of $[a,b]$ contained in β the criterion in (Theorem 10.9) is satisfied and the proof is complete. ■

13.2.1 Relation to the Newton integral

The integral as defined here includes all variants of the Newton integral. A precise statement of this is given in the following theorem:

Theorem 13.3: *Let F be a continuous function on an interval $[a,b]$ and let f be defined on that interval. Then f is integrable on $[a,b]$ and*

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

under any of the following conditions:

1. *$F'(x) = f(x)$ at every point of (a,b).*
2. *$F'(x) = f(x)$ at all but countably many points of (a,b).*
3. *F is Lipschitz and $F'(x) = f(x)$ at almost every point of (a,b).*
4. *F is absolutely continuous on (a,b) and $F'(x) = f(x)$ at almost every point of (a,b).*

Proof. Theorem 13.2 supplies each of these statements. ■

Exercises

13.2.1 State a variant of the Newton integral that the theorem includes and construct a proof without explicitly using Theorem 13.2.

13.2.2 Show that under the third condition of Theorem 13.3, the function f must be even *absolutely* integrable.

13.2.3 Show that under conditions one, two, or four of Theorem 13.3 the function may be possibly nonabsolutely integrable.

Notes

[244]Exercise 12.5.9. If F has a derivative at every point of a set E then E can be split into a sequence of subsets E_n so that F has a bounded derivative in each E_n.

Chapter 14

SEQUENCES AND SERIES OF FUNCTIONS

✂ If the material on series in Chapter 3 was omitted in a first reading, then Sections 3.4, 3.5, and parts of 3.6 should be studied before attempting this chapter.

14.1 Introduction

We have seen that a function f that is the sum of two or more functions will share certain desirable properties with those functions. For example, our study of continuity, differentiation, and integration allows us to state that if

$$f = f_1 + f_2 + \cdots + f_n$$

on an interval $I = [a, b]$, then

(1) If $f_1, f_2, \ldots, f_n$ are continuous on I, so is f.

(2) If $f_1, f_2, \ldots, f_n$ are differentiable on I, so is f, and

$$f' = f_1' + f_2' + \cdots + f_n'.$$

(3) If $f_1, f_2, \dots, f_n$ are integrable on I, so is f, and

$$\int_a^b f(x)\,dx = \int_a^b f_1(x)\,dx + \int_a^b f_2(x)\,dx + \cdots + \int_a^b f_n(x)\,dx.$$

It is natural to ask whether the corresponding results hold when f is the sum of an *infinite* series of functions,

$$f = \sum_{k=0}^{\infty} f_k.$$

If each term of the series is continuous, is the sum function also continuous? Can the derivative be obtained by summing the derivatives? Can the integral be obtained by summing the integrals? We study such questions in this chapter.

These problems are of considerable practical importance. For example, if we are allowed to take limits, integrate, and differentiate freely, then the computations in the following example would all be valid.

Example 14.1: From the formula for the sum of a geometric series we know that

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \dots \tag{1}$$

on the interval $(-1, 1)$. Differentiation of both sides of (1) leads immediately to

$$\frac{-1}{(1+x)^2} = -1 + 2x - 3x^2 + 4x^3 - 5x^4 + \dots.$$

Repeated differentiation would give formulas for $(1+x)^{-n}$ for all positive integers n.

On the other hand, integration of both sides of (1) from 0 to t leads immediately to

$$\ln(1+t) = t - \frac{1}{2}t^2 + \frac{1}{3}t^3 - \frac{1}{4}t^4 + \frac{1}{5}t^5 - \dots.$$

Taking limits as $t \to 1$ in the latter yields the intriguing formula for the sum of the alternating harmonic series:

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$$

◀

The conclusions in the example are all true and useful. But have we used illegitimate means to find them? If we use such methods freely might we find situations where our conclusions are wrong?

We first formulate our questions in the language of sequences of functions (rather than series). We do this in Section 14.2, where we see that the answer to our questions is "not necessarily." Then in Sections 14.3–14.6 we see that if we require a bit more of convergence, the answer to each of our questions is "yes."

14.2 Pointwise Limits

Suppose $f_1, f_2, f_3, \ldots$ is a sequence of functions, each of which is defined on a common domain D. What should we mean by the sum

$$f = \sum_{k=0}^{\infty} f_k$$

or by the limit

$$f = \lim_{n\to\infty} f_n?$$

For sequence limits it would seem natural to require that the values of the functions $f_n(x)$ should converge to the values of the function $f(x)$.

Definition 14.2: Let $\{f_n\}$ be a sequence of functions defined on a common domain D. If $\lim_{n\to\infty} f_n(x)$ exists (as a real number) for all $x \in D$, we say that the sequence $\{f_n\}$ *converges pointwise* on D. This limit defines a function f on D by the equation

$$f(x) = \lim_{n} f_n(x).$$

We write $\lim_n f_n = f$ or $f_n \to f$.

For the infinite sum, the simplest idea is to extend the definition of finite sum using our familiar interpretation of convergence of an infinite series of numbers as a limit of the sequence of partial sums.

Definition 14.3: For each x in D and $n \in \mathbb{N}$ let

$$S_n(x) = f_1(x) + \cdots + f_n(x).$$

If $\lim_{n\to\infty} S_n(x)$ exists (as a real number), we say the series $\sum_{k=1}^{\infty} f_k$ *converges at* x and we write

$$\sum_{k=1}^{\infty} f_k(x)$$

for $\lim_{n\to\infty} S_n(x)$. If the series converges for all $x \in D$, we say the series *converges pointwise* on D to the function f defined by

$$f(x) = \sum_{k=1}^{\infty} f_k(x) \quad (= \lim_{n\to\infty} \sum_{k=1}^{n} f_k(x)).$$

We would like such infinite sums of functions to behave like finite sums of functions (as our three questions in Section 14.1 suggest): If $f = \sum_1^{\infty} f_k$ on an interval $I = [a,b]$, is it true that

(1) If f_k is continuous on I for all $k \in \mathbb{N}$, then so is f?

(2) If f_k is differentiable on I for all $k \in \mathbb{N}$, then so is f and

$$f'(x) = \sum_{k=1}^{\infty} f_k'(x)?$$

(3) If f_k is integrable on I for all $k \in \mathbb{N}$, then so is f, and

$$\int_a^b f(x)\,dx = \sum_{k=1}^{\infty} \int_a^b f_k(x)\,dx?$$

In the special case that D is an interval $I = [a,b]$ our questions then become the following: Is it true that

1. If f_n is continuous on I for all n, then is f continuous on I?

2. If f_n is differentiable on I for all n, then is f differentiable on I and, if so, does $f' = \lim_n f_n'$?

3. If f_n is integrable on I for all n, then is f integrable on I and, if so, does $\int_a^b f(x)\,dx = \lim_n \int_a^b f_n(x)\,dx$?

These questions have *negative* answers in general, as the three examples that follow show.

Example 14.4: (A discontinuous limit of continuous functions) For each $n \in \mathbb{N}$ and $x \in [0,1]$, let $f_n(x) = x^n$. Each of the functions is continuous on $[0,1]$. Notice, however, that for each $x \in (0,1)$, $\lim_n f_n(x) = 0$ and yet $\lim_n f_n(1) = 1$. This is easy to see, but it is instructive to check the details since we can use them later to see what is going wrong in this example. At the right-hand endpoint it is clear that, for $x = 1$, $\lim_n f_n(x) = 1$. For $0 < x_0 < 1$ and $\varepsilon > 0$, let $N \geq \ln\varepsilon / \ln x_0$. Then $(x_0)^N \leq \varepsilon$, so for $n \geq N$

$$|f_n(x_0) - 0| = (x_0)^n < (x_0)^N \leq \varepsilon.$$

Thus

$$f(x) = \lim_n f_n(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1, \end{cases}$$

so the pointwise limit f of the sequence of continuous functions $\{f_n\}$ is discontinuous at $x = 1$. (Figure 14.1 shows the graphs of several of the functions in the sequence.) ◀

Example 14.5: (The derivative of the limit is not the limit of the derivative.) Let $f_n(x) = x^n/n$. Then $f_n \to 0$ on $[0,1]$. Now $f_n'(x) = x^{n-1}$, so by the previous example, Example 14.4,

$$\lim_n f_n'(x) = \lim_n x^{n-1} = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1, \end{cases}$$

while the derivative of the limit function, $f \equiv 0$, equals zero on $[0,1]$. Thus

$$\lim_{n\to\infty} \frac{d}{dx}(f_n(x)) \neq \frac{d}{dx}\left(\lim_{n\to\infty} f_n(x)\right)$$

at $x = 1$. ◀

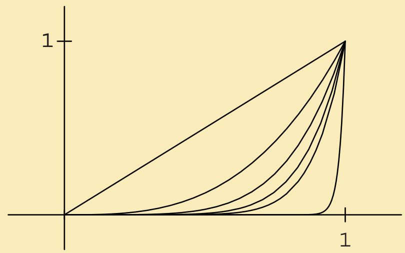

Figure 14.1. Graphs of x^n on $[0, 1]$ for $n = 1$, 3, 5, 7, 9, and 50.

Example 14.6: (The integral of the limit is not the limit of the integrals.) In this example we consider a sequence of continuous functions, each of which has the same integral over the domain. For each $n \in \mathbb{N}$ let f_n be defined on $[0, 1]$ as follows: $f_n(0) = 0$, $f_n(1/(2n)) = 2n$, $f_n(1/n) = 0$, f_n is linear on $[0, 1/(2n)]$ and on $[1/(2n), 1/n]$, and $f_n = 0$ on $[1/n, 1]$. (See Figure 14.2.)

It is easy to verify that $f_n \to 0$ on $[0, 1]$. Now, for each $n \in \mathbb{N}$,

$$\int_0^1 f_n(x)\ dx = 1.$$

But

$$\int_0^1 (\lim_n f_n(x))\, dx = \int_0^1 0\, dx = 0.$$

Thus

$$\lim_n \int_0^1 f_n(x)\ dx \neq \int_0^1 \lim_n f_n(x)\, dx$$

so that the limit of the integrals is not the integral of the limit. ◀

These examples show that the answer to each of our three questions is negative, in general. We present some additional examples that illustrate similar phenomena in the exercises.

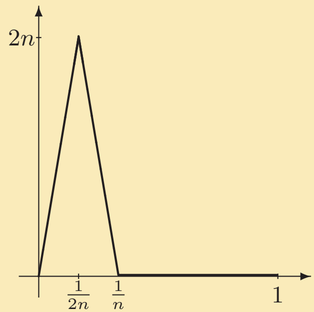

Figure 14.2. Graph of $f_n(x)$ on $[0,1]$ in Example 14.6.

We shall see in the next few sections that by replacing pointwise convergence in appropriate places with a stronger form of convergence, the answers to our questions become affirmative. The form of convergence in question is called *uniform convergence*.

Interchange of Limit Operations Before turning to uniform convergence, let us first try to get an insight into a difficulty we must overcome if we wish affirmative answers to our questions. If f_n is a sequence of continuous functions converging to a function f, must f be continuous? Continuity of f at a point x_0 would mean that

$$\lim_{x\to x_0} f(x) = f(x_0)$$

and this would require that

$$\lim_{x\to x_0}\big(\lim_{n\to\infty} f_n(x)\big) = \lim_{n\to\infty} f_n(x_0) = \lim_{n\to\infty}\big(\lim_{x\to x_0} f_n(x)\big).$$

Apparently, to verify the continuity of f at x_0 we need to use two limit operations and be assured that the order of passing to the limits is immaterial.

You will remember situations in which two limit operations are involved and the order of taking the limit does not affect the result. For example, in elementary calculus one finds conditions under which the value of a double integral can be obtained by iterating "single integrals" in either order. By way of contrast, we present an example in the setting of double sequences in which the order of taking limits *is* important.

Example 14.7: In this example we illustrate that an interchange of limit operations may not give a correct result. Let

$$S_{mn} = \begin{cases} 0, & \text{if } m \leq n \\ 1, & \text{if } m > n. \end{cases}$$

Viewed as a matrix,

$$[S_{mn}] = \begin{bmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 1 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

where we are placing the entry S_{mn} in the mth row and nth column. For each row m, we have $\lim_{n\to\infty} S_{mn} = 0$, so

$$\lim_{m\to\infty} \left(\lim_{n\to\infty} S_{mn} \right) = 0.$$

On the other hand, for each column n, $\lim_{m\to\infty} S_{mn} = 1$, so

$$\lim_{n\to\infty} \left(\lim_{m\to\infty} S_{mn} \right) = 1.$$

◀

Exercises

14.2.1 Examine the pointwise limiting behavior of the sequence of functions

$$f_n(x) = \frac{x^n}{1+x^n}.$$

14.2.2 Show that the logarithm function can be expressed as the pointwise limit of a sequence of "simpler" functions,

$$\ln x = \lim_{n\to\infty} n \left(\sqrt[n]{x} - 1 \right)$$

for every point in its domain. If the answer to our three questions for this particular limit is affirmative, what can you say about the continuity of the logarithm function? What would be its derivative? What would be $\int_1^2 \ln x\,dx$?

14.2.3 Let $x_1, x_2, \dots$ be an enumeration of $\mathbb{Q}$, let

$$f_n(x) = \begin{cases} 1, & \text{if } x \in \{x_1, \dots, x_n\} \\ 0, & \text{otherwise,} \end{cases}$$

and let

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{otherwise.} \end{cases}$$

Show that $f_n \to f$ pointwise on $[0,1]$, but $\int_0^1 f_n(x)\,dx = 0$ for all $n \in \mathbb{N}$, while f is not integrable on $[0,1]$.

14.2.4 Let $f_n(x) = \sin nx/\sqrt{n}$. Show that $\lim_n f_n = 0$ but $\lim_n f_n'(0) = \infty$.

14.2.5 Each of Examples 14.4, 14.5 and 14.6 can be interpreted as a statement that the order of taking the limit operation does matter. Verify this.

14.2.6 Refer to Example 14.7. What should we mean by the statement that a "double sequence" $\{t_{mn}\}$ converges; that is, that

$$\lim_{m\to\infty, n\to\infty} t_{mn}$$

exists? Does the double sequence $\{S_{mn}\}$ of Example 14.7 converge?

14.2.7 Let $f_n \to f$ pointwise at every point in the interval $[a,b]$. We have seen that even if each f_n is continuous it does not follow that f is continuous. Which of the following statements are true?

(a) If each f_n is increasing on $[a,b]$, then so is f.

(b) If each f_n is nondecreasing on $[a,b]$, then so is f.

(c) If each f_n is bounded on $[a,b]$, then so is f.

(d) If each f_n is everywhere discontinuous on $[a,b]$, then so is f.

(e) If each f_n is constant on $[a,b]$, then so is f.

(f) If each f_n is positive on $[a,b]$, then so is f.

(g) If each f_n is linear on $[a,b]$, then so is f.

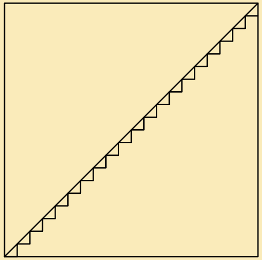

Figure 14.3. Construction in Exercise 14.2.8.

(h) If each f_n is convex on $[a, b]$, then so is f.

SEE NOTE 245

14.2.8 A careless student[1] once argued as follows: "It seems to me that one can construct a curve without a tangent in a very elementary way. We divide the diagonal of a square into n equal parts and construct on each subdivision as base a right isosceles triangle. In this way we get a kind of delicate little saw. Now I put $n = \infty$. The saw becomes a continuous curve that is infinitesimally different from the diagonal. But it is perfectly clear that its tangent is alternately parallel now to the x-axis, now to the y-axis." What is the error? (Figure 14.3 illustrates the construction.)

SEE NOTE 246

14.2.9 As yet another illustration that some properties are not preserved in the limit, compute the length of the curves in Exercise 14.2.8 (Fig. 14.3) and compare with the length of the limiting curve.

[1]In this case the "careless student" was the great Russian analyst N. N. Luzin (1883–1950), who recounted in a letter [reproduced in *Amer. Math. Monthly*, 107, (2000), pp. 64–82] how he offered this argument to his professor after a lecture on the Weierstrass continuous nowhere differentiable function.

14.2.10 ✂ If $f_n \to f$ pointwise at every real number, then prove that

$$\{x : f(x) > \alpha\} = \bigcup_{m=1}^{\infty} \bigcup_{r=1}^{\infty} \bigcap_{n=r}^{\infty} \{x : f_n(x) \geq \alpha + 1/m\}.$$

14.2.11 Let $\{f_n\}$ be a sequence of real functions. Show that the set E of points of convergence of the sequence can be written in the form

$$E = \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \bigcap_{m=N}^{\infty} \left\{x : |f_n(x) - f_m(x)| \leq \tfrac{1}{k}\right\}.$$

14.3 Uniform Limits

Pointwise limits do not allow the interchange of limit operations. In many situations, uniform limits will. To see how the definition of a uniform limit needs to be formulated, let us return to the sequence of Example 14.4. That sequence illustrated the fact that a pointwise limit of continuous functions need not be continuous. The difficulty there was that

$$\lim_{x\to 1-} \left(\lim_{n\to\infty} f_n(x)\right) \neq \lim_{n\to\infty} \left(\lim_{x\to 1-} f_n(x)\right).$$

A closer look at the limits involved here shows what went wrong and suggests what we need to look for in order to allow an interchange of limits.

Example 14.8: Consider again the sequence $\{f_n\}$ of functions $f_n(x) = x^n$. We saw that $f_n \to 0$ pointwise on $[0,1)$, and that for every fixed $x_0 \in (0,1)$ and $\varepsilon > 0$,

$$|x_0|^n < \varepsilon \text{ if and only if } n > \ln\varepsilon/\ln x_0.$$

Now fix ε but let the point x_0 vary. Observe that, when x_0 is relatively small in comparison with ε, the number $\ln x_0$ is large in absolute value compared with $\ln\varepsilon$, so relatively small values of n suffice for the inequality $|x_0|^n < \varepsilon$. On the other hand, when x_0 is near 1, $\ln x_0$ is small in absolute value, so $\ln\varepsilon/\ln x_0$ will be large. In fact,

$$\lim_{x_0\to 1-} \frac{\ln\varepsilon}{\ln x_0} = \infty. \tag{2}$$

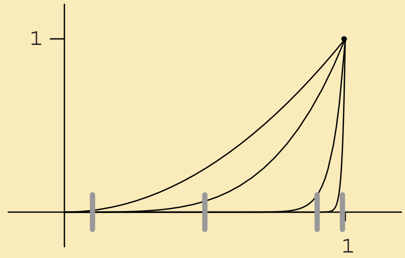

Figure 14.4. The sequence $\{x^n\}$ converges infinitely slowly on $[0,1]$. The functions $y = x^n$ are shown with $n = 2$, 4, 22, and 100, with $x_0 = .1$, .5, .9, and .99, and with $\varepsilon = .1$.

The following table illustrates how large n must be before $|x_0^n| < \varepsilon$ for $\varepsilon = .1$.

x_0	n
.1	2
.5	4
.9	22
.99	230
.999	2,302
.9999	23,025

Note that for $\varepsilon = .1$, there is no single value of N such that $|x_0|^n < \varepsilon$ for every value of $x_0 \in (0,1)$ and $n > N$. (Figure 14.4 illustrates this.) ◀

Some nineteenth-century mathematicians would have described the varying rates of convergence in the example by saying that "the sequence $\{x^n\}$ converges *infinitely slowly* on $(0,1)$." Today we would say that this sequence, which does converge pointwise, does *not* converge uniformly. Our definition is formulated precisely to avoid this possibility of infinitely slow convergence.

Definition 14.9: Let $\{f_n\}$ be a sequence of functions defined on a common domain D. We say that $\{f_n\}$ *converges uniformly* to a function f on D if, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \varepsilon \text{ for all } n \geq N \text{ and } x \in D.$$

We write

$$f_n \to f \text{ [unif] on } D \text{ or } \lim_n f_n = f \text{ [unif] on } D$$

to indicate that the sequence $\{f_n\}$ converges uniformly to f on D. If the domain D is understood from the context, we may delete explicit reference to D and write

$$f_n \to f \text{ [unif] or } \lim_n f_n = f \text{ [unif]}.$$

Uniform convergence plays an important role in many parts of analysis. In particular, it figures in questions involving the interchanging of limit processes such as those we discussed in Section 14.2. This was not apparent to mathematicians in the early part of the nineteenth century. As late as 1823, Cauchy believed a convergent series of continuous functions could be integrated term by term. Similarly, Cauchy believed that a convergent series of continuous functions has a continuous sum. Abel provided a counterexample in 1826. It may have been Weierstrass who first recognized the importance of uniform convergence in the middle of the nineteenth century.[2]

Example 14.10: Let $f_n(x) = x^n$, $D = [0, \eta]$, $0 < \eta < 1$. We observed that the sequence $\{f_n\}$ converges pointwise, but not uniformly, on $(0, 1)$ (or on $[0, 1]$). We realized that the difficulty arises from the fact that the convergence near 1 is very "slow." But for any fixed η with $0 < \eta < 1$, the convergence *is* uniform on $[0, \eta]$.

To see this, observe that for $0 \leq x_0 < \eta$, $0 \leq (x_0)^n < \eta^n$. Let $\varepsilon > 0$. Since $\lim_n \eta^n = 0$, there exists N such that if $n \geq N$, then $0 < \eta^n < \varepsilon$. Thus, if $n \geq N$, we have

$$0 \leq x_0^n < \eta^n < \varepsilon,$$

so the same N that works for $x = \eta$, also works for all $x \in [0, \eta)$. ◀

[2] More on the history of uniform convergence can be found in Thomas Hawkins' interesting historical book *Lebesgue's Theory of Integration: Its Origins and Development*, Univ. of Wisconsin Press (Madison, 1970)

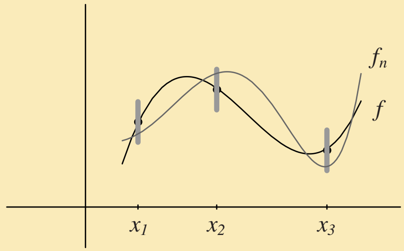

Figure 14.5. Uniform convergence on the finite set $\{x_1, x_2, x_3\}$.

Suppose that $f_n \to f$ on $[0,1]$. It follows easily from the definition that the convergence is uniform on any *finite* subset D of $[0,1]$ (Exercise 14.3.3). Thus given any $\varepsilon > 0$ and any finite set $x_1, x_2, \dots, x_m$ in [0,1], we can find $n \in \mathbb{N}$ such that

$$|f_n(x_i) - f(x_i)| < \varepsilon$$

for all $n \geq N$ and all $i = 1, 2, \dots, m$. (Figure 14.5 illustrates this.)

The vertical line segments over the points $x_1, \dots, x_m$ are centered on the graph of f and are of length 2ε. In simple geometric language, we can go sufficiently far out in the sequence to guarantee that the graphs of all the functions f_n intersect all of these finitely many vertical segments.

In contrast, uniform convergence on $[0,1]$ requires that we can go sufficiently far out in the sequence to guarantee that the graphs of the functions go through such vertical segments at *all* points of $[0,1]$; that is, that the graph of f_n for n sufficiently large lies in the "ε-band" centered on the graph of f. (See Figure 14.6.)

14.3.1 The Cauchy Criterion

Suppose now that we are given a sequence of functions $\{f_n\}$ on an interval I, and we wish to know whether it converges uniformly to some function on I. We are not told what that limit function might be. The

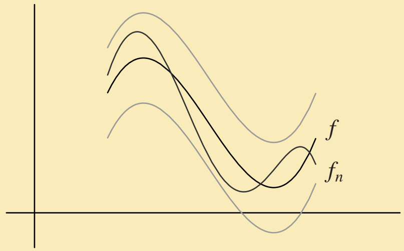

Figure 14.6. Uniform convergence on the whole interval.

problem is similar to one we faced for a sequence of *numbers* $\{a_n\}$ in our study of sequences. There we saw that $\{a_n\}$ converges if and only if it is a Cauchy sequence. We can formulate a similar criterion for uniform convergence of a sequence of functions.

Definition 14.11: Let $\{f_n\}$ be a sequence of functions defined on a set D. The sequence is said to be *uniformly Cauchy* on D if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if $n \geq N$ and $m \geq N$, then $|f_m(x) - f_n(x)| < \varepsilon$ for all $x \in D$.

Theorem 14.12 (Cauchy Criterion) *Let $\{f_n\}$ be a sequence of functions defined on a set D. Then there exists a function f defined on D such that $f_n \to f$ uniformly on D if and only if $\{f_n\}$ is uniformly Cauchy.*

Proof. We leave the proof of Theorem 14.12 as Exercise 14.3.15. ■

Example 14.13: In Example 14.10 we showed that the sequence $f_n(x) = x^n$ converges uniformly on any interval $[0, \eta]$, for $0 < \eta < 1$. Let us prove this again, but using the Cauchy criterion.

Fix $n \geq m$ and compute

$$\sup_{x \in [0,\eta]} |x^n - x^m| \leq \eta^m. \tag{3}$$

Let $\varepsilon > 0$ and choose an integer N so that $\eta^N < \varepsilon$. Equivalently we require that $N > \ln\varepsilon/\ln\eta$. Then it follows from (3) for all $n \geq m \geq N$ and all $x \in [0, \eta]$ that

$$|x^n - x^m| \leq \eta^m < \varepsilon.$$

We conclude, by the Cauchy criterion, that the sequence $f_n(x) = x^n$ converges uniformly on any interval $[0, \eta]$, for $0 < \eta < 1$. Here there was no computational advantage over the argument in Example 14.10. Frequently, though, we do not know the limit function and *must* use the Cauchy criterion rather than the definition. ◀

Cauchy Criterion for Series The Cauchy criterion can be expressed for uniformly convergent series too. We say that a series $\sum_1^\infty f_k$ converges *uniformly* to the function f on D if the sequence $\{S_n\} = \{\sum_{k=1}^n f_k\}$ of partial sums converges uniformly to f on D.

Theorem 14.14 (Cauchy Criterion) *Let $\{f_k\}$ be a sequence of functions defined on a set D. Then the series $\sum_1^\infty f_k$ converges uniformly to some function f on D if and only if for every $\varepsilon > 0$ there is an integer N so that*

$$\left|\sum_{j=m}^{n} f_j(x)\right| < \varepsilon$$

for all $n \geq m \geq N$ and all $x \in D$.

Proof. This follows immediately from Theorem 14.12. ■

Example 14.15: Let us show that the series

$$1 + x + x^2 + x^3 + x^4 + \ldots$$

converges uniformly on any interval $[0, \eta]$, for $0 < \eta < 1$. Our computations could be based on the fact that the sum of this series is known to us; it is $(1 - x)^{-1}$. We could prove the uniform convergence directly from the definition. Instead let us use the Cauchy criterion.

Fix $n \geq m$ and compute

$$\sup_{x\in[0,\eta]} \left| \sum_{j=m}^{n} x^j \right| \leq \sup_{x\in[0,\eta]} \left| \frac{x^m}{1-x} \right| \leq \frac{\eta^m}{1-\eta}. \tag{4}$$

Let $\varepsilon > 0$. Since

$$\eta^m (1-\eta)^{-1} \to 0$$

as $m \to \infty$ we may choose an integer N so that

$$\eta^N (1-\eta)^{-1} < \varepsilon.$$

Then it follows from (4) for all $n \geq m \geq N$ and all $x \in [0, \eta]$ that

$$\left| x^m + x^{m+1} + \cdots + x^n \right| \leq \frac{\eta^m}{1-\eta} < \varepsilon.$$

It follows now, by the Cauchy criterion, that the series converges uniformly on any interval $[0, \eta]$, for $0 < \eta < 1$. Observe, however, that the series does not converge uniformly on $(-1, 1)$, though it does converge pointwise there. (See Exercise 14.3.16.) ◀

14.3.2 Weierstrass M-Test

It is not always easy to determine whether a sequence of functions is uniformly convergent. In the settings of *series* of functions, a certain simple test is often useful. This will certainly become one of the most frequently used tools in your study of uniform convergence.

Theorem 14.16 (M-Test) *Let $\{f_k\}$ be a sequence of functions defined on a set D and let $\{M_k\}$ be a sequence of positive constants. If*

$$\sum_0^\infty M_k < \infty$$

and if

$$|f_k(x)| \leq M_k$$

for each $x \in D$ and $k = 0, 1, 2, \ldots,$ then the series $\sum_0^\infty f_k$ converges uniformly on D.

Proof. Let $S_n(x) = \sum_{k=0}^n f_k(x)$. We show that $\{S_n\}$ is uniformly Cauchy on D. Let $\varepsilon > 0$. For $m < n$ we have

$$S_n(x) - S_m(x) = f_{m+1}(x) + \cdots + f_n(x),$$

so

$$|S_n(x) - S_m(x)| \leq M_{m+1} + \cdots + M_n.$$

Since the series of constants $\sum_{k=0}^\infty M_k$ converges by hypothesis, there exists an integer N such that if $n > m \geq N$,

$$M_{m+1} + \cdots + M_n < \varepsilon.$$

This implies that for $n > m \geq N$,

$$|S_n(x) - S_m(x)| < \varepsilon$$

for all $x \in D$. Thus the sequence $\{S_n\}$ is uniformly convergent on D; that is, the series $\sum_0^\infty f_k$ is uniformly convergent on D. ■

Example 14.17: Consider again the geometric series $1 + x + x^2 + \ldots$ on the interval $[-a, a]$, for any $0 < a < 1$ (as we did in Example 14.15). Then $|x^k| \leq a^k$ for every $k = 0, 1, 2 \ldots$ and $x \in [-a, a]$. Since $\sum_{k=0}^\infty a^k$ converges, by the M-test the series $\sum_{k=0}^\infty x^k$ converges uniformly on $[-a, a]$. ◀

Example 14.18: Let us investigate the uniform convergence of the series

$$\sum_{k=1}^{\infty} \frac{\sin k\theta}{k^p}$$

for values of $p > 0$. The crudest estimate on the size of the terms in this series is obtained just by using the fact that the sine function never exceeds 1 in absolute value. Thus

$$\left|\frac{\sin k\theta}{k^p}\right| \le \frac{1}{k^p} \quad \text{for all } \theta \in \mathbb{R}.$$

Since the series $\sum_{k=1}^{\infty} 1/k^p$ converges for $p > 1$, we obtain immediately by the M-test that our series converges uniformly (and absolutely) for all real θ provided $p > 1$. In particular, as we shall see in subsequent sections, this series represents a continuous function, one that could be integrated term by term in any bounded interval.

We seem to have been particularly successful here, but a closer look also reveals a limitation in the method. The series is also pointwise convergent for $0 < p \le 1$ (use the Dirichlet test) for all values of θ, but it converges nonabsolutely. The M-test cannot be of any help in this situation since it can address only absolutely convergent series. ◀

Because of the remark at the end of this example, it is perhaps best to conclude, when using the M-test, that the series tested "converges absolutely and uniformly" on the set given. This serves, too, to remind us to use a different method for checking uniform convergence of nonabsolutely convergent series (see the next section).

14.3.3 Abel's Test for Uniform Convergence

✂ *Adv.*

The M-test is a highly useful tool for checking the uniform convergence of a series. By its nature, though, it clearly applies only to absolutely convergent series. For a more delicate test that will apply to some nonabsolutely convergent series we should search through our methods in Chapter 3 for tests that handled nonabsolute convergence. Two of these, the Dirichlet test and Abel's test, can be modified so as to give uniform convergence.

A number of nineteenth century authors (including Abel, Dirichlet, Dedekind, and du Bois-Reymond) arrived at similar tests for uniform convergence. We recall that Abel's test for convergence of a series $\sum_{k=1}^{\infty} a_k b_k$ required the sequence $\{b_k\}$ to be convergent and monotone and for the series $\sum_{k=1}^{\infty} a_k$ to converge. Dirichlet's variant weakened the latter requirement so that $\sum_{k=1}^{\infty} a_k$ had bounded partial sums but required of the sequence $\{b_k\}$ that it converge monotonically to zero. Here we seek similar conditions on a series

$$\sum_{k=1}^{\infty} a_k(x) b_k(x)$$

of functions in order to obtain uniform convergence. The next theorem is one variant; others can be found in the Exercises.

Theorem 14.19 (Abel) *Let $\{a_k\}$ and $\{b_k\}$ be sequences of functions on a set $E \subset \mathbb{R}$. Suppose that there is a number M so that*

$$-M \leq s_N(x) = \sum_{k=1}^{N} a_k(x) \leq M$$

for all $x \in E$ and every $N \in \mathbb{N}$. Suppose that the sequence of functions $\{b_k\} \to 0$ converges monotonically to zero at each point and that this convergence is uniform on E. Then the series

$$\sum_{k=1}^{\infty} a_k b_k$$

converges uniformly on E.

Proof. We will use the Cauchy criterion applied to the series to obtain uniform convergence. We may assume that the $b_k(x)$ are nonnegative and decrease to zero. Let $\varepsilon > 0$. We need to estimate the sum

$$\left| \sum_{k=m}^{n} a_k(x) b_k(x) \right| \tag{5}$$

for large n and m and all $x \in E$. Since the sequence of functions $\{b_k\}$ converges uniformly to zero on E, we can find an integer N so that for all $k \geq N$ and all $x \in E$

$$0 \leq b_k(x) \leq \frac{\varepsilon}{2M}.$$

The key to estimating the sum (5), now, is the summation by parts formula that we have used earlier (see Section 3.2). This is just the elementary identity

$$\sum_{k=m}^{n} a_k b_k = \sum_{k=m}^{n} (s_k - s_{k-1}) b_k$$

$$= s_m(b_m - b_{m+1}) + s_{m+1}(b_{m+1} - b_{m+2}) \cdots + s_{n-1}(b_{n-1} - b_n) + s_n b_n.$$

This provides us with

$$\left| \sum_{k=m}^{n} a_k(x) b_k(x) \right| \leq 2M \left(\sup_{x \in E} |b_m(x)| \right) < \varepsilon$$

for all $n \geq m \geq N$ and all $x \in E$ which is exactly the Cauchy criterion for the series and proves the theorem. ■

It is worth pointing out that in many applications of this theorem the sequence $\{b_k\}$ can be taken as a sequence of numbers, in which case the statement and the conditions that need to be checked are simpler.

Corollary 14.20: *Let $\{a_k\}$ be a sequence of functions on a set $E \subset \mathbb{R}$. Suppose that there is a number M so that*

$$\left|\sum_{k=1}^{N} a_k(x)\right| \leq M$$

for all $x \in E$ and every integer N. Suppose that the sequence of real numbers $\{b_k\}$ converges monotonically to zero. Then the series

$$\sum_{k=1}^{\infty} b_k a_k$$

converges uniformly on E.

Proof. Consider that $\{b_k\}$ is a sequence of constant functions on E and then apply the theorem. ■

In the exercises there are several other variants of Theorem 14.19, all with similar proofs and all of which have similar applications.

Example 14.21: As an interesting application of Theorem 14.19, consider a series that arises in Fourier analysis:

$$\sum_{k=1}^{\infty} \frac{\sin k\theta}{k}.$$

It is possible by using Dirichlet's test (see Section 3.6.13) to prove that this series converges for all θ.

Questions about the uniform convergence of this series are intriguing. In Figure 14.7 we have given a graph of some of the partial sums of the series.

The behavior near $\theta = 0$ is most curious. Apparently, if we can avoid that point (more precisely if we can stay a small distance away from that point) we should be able to obtain uniform convergence. Theorem 14.19 will provide a proof. We apply that theorem with $b_k(\theta) = 1/k$ and $a_k(\theta) = \sin k\theta$. All that is required is to obtain an estimate for the sums

$$\left|\sum_{k=1}^{n} \sin k\theta\right|$$

Figure 14.7. Graph of $\sum_{k=1}^{n} (\sin k\theta)/k$ on $[0, 2\pi]$ for, clockwise from upper left, $n = 1$, 4, 7, and 10.

for all n and all θ in an appropriate set. Let $0 < \eta < \pi/2$ and consider making this estimate on the interval $[\eta, 2\pi - \eta]$. From Exercise 3.2.11 we obtain the formula

$$\sin\theta + \sin 2\theta + \sin 3\theta + \sin 4\theta + \cdots + \sin n\theta = \frac{\cos\theta/2 - \cos(2n+1)\theta/2}{2\sin\theta/2}$$

and using this we can see that

$$\left|\sum_{k=1}^{n} \sin k\theta\right| \leq \frac{1}{\sin(\eta/2)}.$$

Now Theorem 14.19 immediately shows that

$$\sum_{k=1}^{\infty} \frac{\sin k\theta}{k}$$

converges uniformly on $[\eta, 2\pi - \eta]$.

Figure 14.7 illustrates graphically why the convergence cannot be expected to be uniform near to 0. A computation here is instructive. To check the Cauchy criterion on $[0,\pi]$ we need to show that the sums

$$\sup_{\theta\in[0,\pi]}\left|\sum_{k=m}^{n}\frac{\sin k\theta}{k}\right|$$

are small for large m, n. But in fact

$$\sup_{\theta\in[0,\pi]}\left|\sum_{k=m}^{2m}\frac{\sin k\theta}{k}\right| \geq \sum_{k=m}^{2m}\frac{\sin(k/2m)}{k} \geq \sum_{k=m}^{2m}\frac{\sin 1/2}{2m} > \frac{\sin 1/2}{2},$$

obtained by checking the value at points $\theta = 1/2m$. Since this is not arbitrarily small, the series cannot converge uniformly on $[0,\pi]$. ◀

Exercises

14.3.1 Examine the uniform limiting behavior of the sequence of functions

$$f_n(x) = \frac{x^n}{1+x^n}.$$

On what sets can you determine uniform convergence?

14.3.2 Examine the uniform limiting behavior of the sequence of functions

$$f_n(x) = x^2 e^{-nx}.$$

On what sets can you determine uniform convergence? On what sets can you determine uniform convergence for the sequence of functions $n^2 f_n(x)$?

14.3.3 Prove that if $f_n \to f$ pointwise on a finite set D, then the convergence is uniform.

14.3.4 Prove that if $f_n \to f$ uniformly on a set E_1 and also on a set E_2, then $f_n \to f$ uniformly on $E_1 \cup E_2$.

14.3.5 Prove or disprove that if $f_n \to f$ uniformly on each set E_1, E_2, E_3, $\ldots$, then $f_n \to f$ uniformly on the union of all these sets $\bigcup_{k=1}^{\infty} E_k$.

14.3.6 Prove that if $f_n \to f$ uniformly on a set E, then $f_n \to f$ uniformly on every subset of E.

14.3.7 Prove or disprove that if $f_n \to f$ uniformly on each set $E \cap [a,b]$ for every interval $[a,b]$, then $f_n \to f$ uniformly on E.

14.3.8 Prove or disprove that if $f_n \to f$ uniformly on each closed interval $[a,b]$ contained in an open interval (c,d), then $f_n \to f$ uniformly on (c,d).

14.3.9 Prove that if $\{f_n\}$ and $\{g_n\}$ both converge uniformly on a set D, then so too does the sequence $\{f_n + g_n\}$.

14.3.10 Prove or disprove that if $\{f_n\}$ and $\{g_n\}$ both converge uniformly on a set D, then so too does the sequence $\{f_n g_n\}$.

14.3.11 Prove or disprove that if f is a continuous function on $(-\infty,\infty)$, then

$$f(x+1/n) \to f(x)$$

uniformly on $(-\infty,\infty)$. (What extra condition, stronger than continuity, would work if not?)

14.3.12 Prove that $f_n \to f$ converges uniformly on D if and only if

$$\lim_n \sup_{x\in D} |f_n(x) - f(x)| = 0.$$

14.3.13 Show that a sequence of functions $\{f_n\}$ fails to converge to a function f uniformly on a set E if and only if there is some positive ε_0 so that a sequence $\{x_k\}$ of points in E and a subsequence $\{f_{n_k}\}$ can be found such that

$$|f_{n_k}(x_k) - f(x_k)| \geq \varepsilon_0.$$

14.3.14 Apply the criterion in the preceding exercise to show that the sequence $f_n(x) = x^n$ does not converge uniformly to zero on $(0,1)$.

14.3.15 Prove Theorem 14.12.

See Note 247

14.3.16 Verify that the geometric series $\sum_{k=0}^{\infty} x^k$, which converges pointwise on $(-1,1)$, does not converge uniformly there.

See Note 248

14.3.17 Do the same for the series obtained by differentiating the series in Exercise 14.3.16; that is, show that $\sum_{k=1}^{\infty} kx^{k-1}$ converges pointwise but not uniformly on $(-1,1)$. Show that this series does converge uniformly on every closed interval $[a,b]$ contained in $(-1,1)$.

14.3.18 Verify that the series

$$\sum_{k=1}^{\infty} \frac{\cos kx}{k^2}$$

converges uniformly on all of $\mathbb{R}$.

14.3.19 If $\{f_n\}$ is a sequence of functions converging uniformly on a set E to a function f, what conditions on the function g would allow you to conclude that $g \circ f_n$ converges uniformly on E to $g \circ f$?

14.3.20 Prove that the series $\displaystyle\sum_{k=0}^{\infty} \frac{x^k}{k}$ converges uniformly on $[0, b]$ for every $b \in [0, 1)$ but does not converge uniformly on $[0, 1)$.

14.3.21 Prove that if $\sum_0^\infty f_k$ converges uniformly on a set D, then the sequence of terms $\{f_k\}$ converges uniformly on D.

14.3.22 A sequence of functions $\{f_n\}$ is said to be *uniformly bounded* on an interval $[a, b]$ if there is a number M so that

$$|f_n(x)| \leq M$$

for every n and also for every $x \in [a, b]$. Show that a uniformly convergent sequence $\{f_n\}$ of continuous functions on $[a, b]$ must be uniformly bounded. Show that the same statement would not be true for pointwise convergence.

14.3.23 Suppose that $f_n \to f$ on $(-\infty, +\infty)$. What conditions would allow you to compute that

$$\lim_{n\to\infty} f_n(x + 1/n) = f(x)?$$

14.3.24 Suppose that $\{f_n\}$ is a sequence of continuous functions on the interval $[0, 1]$ and that you know that $\{f_n\}$ converges uniformly on the set of rational numbers inside $[0, 1]$. Can you conclude that $\{f_n\}$ uniformly on $[0, 1]$? (Would this be true without the continuity assertion?)

14.3.25 Prove the following variant of the Weierstrass M-test: Let $\{f_k\}$ and $\{g_k\}$ be sequences of functions on a set $E \subset \mathbb{R}$. Suppose that $|f_k(x)| \leq g_k(x)$ for all k and $x \in E$ and that $\sum_{k=1}^\infty g_k$ converges uniformly on E. Then the series $\sum_{k=1}^\infty f_k$ converges uniformly on E.

14.3.26 Prove the following variant on Theorem 14.19: Let $\{a_k\}$ and $\{b_k\}$ be sequences of functions on a set $E \subset \mathbb{R}$. Suppose that $\sum_{k=1}^{\infty} a_k(x)$ converges uniformly on E. Suppose that $\{b_k\}$ is monotone for each $x \in E$ and uniformly bounded on E. Then the series $\sum_{k=1}^{\infty} a_k b_k$ converges uniformly on E.

14.3.27 Prove the following variant on Theorem 14.19: Let $\{a_k\}$ and $\{b_k\}$ be sequences of functions on a set $E \subset \mathbb{R}$. Suppose that there is a number M so that

$$\left|\sum_{k=1}^{N} a_k(x)\right| \leq M$$

for all $x \in E$ and every integer N. Suppose that

$$\sum_{k=1}^{\infty} |b_k - b_{k+1}|$$

converges uniformly on E and that $b_k \to 0$ uniformly on E. Then the series $\sum_{k=1}^{\infty} a_k b_k$ converges uniformly on E.

14.3.28 Prove the following variant on Theorem 14.19: Let $\{a_k\}$ and $\{b_k\}$ be sequences of functions on a set $E \subset \mathbb{R}$. Suppose that $\sum_{k=1}^{\infty} a_k$ converges uniformly on E. Suppose that the series

$$\sum_{k=1}^{\infty} |b_k - b_{k+1}|$$

has uniformly bounded partial sums on E. Suppose that the sequence of functions $\{b_k\}$ is uniformly bounded on E. Then the series $\sum_{k=1}^{\infty} a_k b_k$ converges uniformly on E.

14.3.29 Suppose that $\{f_n\}$ is a sequence of continuous functions on an interval $[a, b]$ converging uniformly to a function f on the open interval (a, b). If f is also continuous on $[a, b]$, show that the convergence is uniform on $[a, b]$.

14.3.30 Suppose that $\{f_n\}$ is a sequence of functions converging uniformly to zero on an interval $[a, b]$. Show that $\lim_{n\to\infty} f_n(x_n) = 0$ for every convergent sequence $\{x_n\}$ of points in $[a, b]$. Give an example to show that this statement may be false if $f_n \to 0$ merely pointwise.

14.3.31 Suppose that $\{f_n\}$ is a sequence of functions on an interval $[a, b]$ with the property that $\lim_{n\to\infty} f_n(x_n) = 0$ for every convergent sequence $\{x_n\}$ of points in $[a, b]$. Show that $\{f_n\}$ converges uniformly to zero on $[a, b]$.

14.4 Uniform Convergence and Continuity

We can now address the questions we asked at the beginning of this chapter. We begin with continuity. We know that the pointwise limit of a sequence of continuous functions need not be continuous. We now show that the *uniform* limit of a sequence of continuous functions must be continuous.

Theorem 14.22: *Let $\{f_n\}$ be a sequence of functions defined on an interval I, and let $x_0 \in I$. If the sequence $\{f_n\}$ converges uniformly to some function f on I and if each of the functions f_n is continuous at x_0, then the function f is also continuous at x_0. In particular, if each of the functions f_n is continuous on I, then so too is f.*

Proof. Let $\varepsilon > 0$. We must show there exists $\delta > 0$ such that

$$|f(x) - f(x_0)| < \varepsilon$$

if $|x - x_0| < \delta$, $x \in I$. For each $x \in I$ we have

$$f(x) - f(x_0) = (f(x) - f_n(x)) + (f_n(x) - f_n(x_0)) + (f_n(x_0) - f(x_0)),$$

so

$$|f(x) - f(x_0)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)|. \tag{6}$$

Since $f_n \to f$ *uniformly*, there exists $N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \frac{\varepsilon}{3} \tag{7}$$

for all $x \in I$ and all $n \geq N$. We infer from inequalities (6) and (7) that

$$|f(x) - f(x_0)| < |f_N(x) - f_N(x_0)| + \frac{2}{3}\varepsilon. \tag{8}$$

We now use the continuity of the function f_N. We choose $\delta > 0$ such that if $x \in I$ and $|x - x_0| < \delta$, then

$$|f_N(x) - f_N(x_0)| < \frac{\varepsilon}{3}. \tag{9}$$

Combining (8) and (9), we have

$$|f(x) - f(x_0)| < \frac{\varepsilon}{3} + \frac{2}{3}\varepsilon = \varepsilon,$$

for each $x \in I$ for which $|x - x_0| < \delta$, as was to be shown. ■

Note. Let us look a bit more closely at the proof of Theorem 14.22. We first obtained $N \in \mathbb{N}$ such that the function f_N approximated f closely (within $\varepsilon/3$) on all of I. This function f_N served as a "stepping stone" toward verifying the continuity of f at x_0. There are three small "steps" involved:

1. $|f_N(x) - f(x)|$ is small (*for all* $x \in I$) because of uniform convergence.
2. $|f_N(x) - f_N(x_0)|$ is small (for all x near x_0) because of the continuity of f_N.
3. $|f_N(x_0) - f(x_0)|$ is small because $\{f_n(x_0)\} \to f(x_0)$.

If we tried to imitate the proof under the assumption of pointwise convergence, the first of these steps would fail. You may wish to observe the failure by working Exercise 14.4.2.

Theorem 14.22 can be stated in terms of series. Recall that a series $\sum_1^\infty f_k$ converges uniformly to the function f on D if the sequence

$$\{S_n\} = \{\sum_{k=1}^{n} f_k\}$$

of partial sums converges uniformly to f on D.

Corollary 14.23: *If $\sum_1^\infty f_k$ converges uniformly to f on an interval I and if each of the functions f_k is continuous on I, then f is continuous on I.*

Proof. This follows immediately from Theorem 14.22. ■

14.4.1 Dini's Theorem

Adv.

Observe that Theorem 14.22 provides a *sufficient* condition for continuity of the limit function f. The condition is not necessary. (The sequence in Example 14.6 converges to the zero function, which is continuous, even though the convergence is not uniform.)

Under certain circumstances, however, uniform convergence *is* necessary, as Theorem 14.24 shows. (See also Exercise 14.4.6.) This theorem is due to Ulisse Dini (1845–1918) and gives a condition under which pointwise convergence of a sequence of continuous functions to a continuous function must be uniform.

Theorem 14.24 (Dini) *Let $\{f_n\}$ be a sequence of continuous functions on an interval $[a,b]$. Suppose for each $x \in [a,b]$ and for all $n \in \mathbb{N}$,*

$$f_n(x) \geq f_{n+1}(x).$$

Suppose in addition that for all $x \in [a,b]$

$$f(x) = \lim_n f_n(x).$$

If f is continuous, then the convergence is uniform.

Proof. Suppose the convergence were *not* uniform. Then

$$\max_{x\in[a,b]} (f_n(x) - f(x))$$

does not approach zero as $n \to \infty$ (see Exercise 14.3.12). Hence there exists $c > 0$ such that for infinitely many $n \in \mathbb{N}$,

$$\max_{x\in[a,b]} (f_n(x) - f(x)) > c > 0.$$

Now, for each $n \in \mathbb{N}$, $f_n - f$ is continuous, so it achieves a maximum value at a point $x_n \in [a,b]$. By the Bolzano-Weierstrass theorem we can thus choose a subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$ such that $\{x_{n_k}\}$ converges to a point $x_0 \in [a,b]$. Note that we must have

$$f_{n_k}(x_{n_k}) - f(x_{n_k}) > c$$

for all $k \in \mathbb{N}$.

Because of our assumption that $f_n(x) \geq f_{n+1}(x)$ for all $n \in \mathbb{N}$ and $x \in [a,b]$, we infer

$$f_n(x_{n_k}) - f(x_{n_k}) > c \text{ for each } n \leq n_k.$$

Now fix n and let $k \to \infty$. Using the continuity of the functions $f_n - f$ we obtain $f_n(x_0) - f(x_0) \geq c$

for all $n \in \mathbb{N}$. But this is impossible since $f_n(x_0) \to f(x_0)$ by hypothesis. Thus our assumption that the convergence is not uniform has led to a contradiction. ■

Example 14.25: The sequence of continuous functions $f_n(x) = x^n$ is converging monotonically to a function f on the interval $[0, 1]$. But that function f is (as we have seen before) discontinuous at $x = 1$, so immediately we know that the convergence cannot be uniform. Dini's theorem implies that the convergence is uniform on $[0, b]$ for any $0 < b < 1$ since the function f is continuous there. ◀

Exercises

14.4.1 Can a sequence of discontinuous functions converge uniformly on an interval to a continuous function?

14.4.2 Let $f_n(x) = x^n$, $0 \le x \le 1$. Try to imitate the proof of Theorem 14.22 for $x_0 = 1$ and observe where the proof breaks down.

14.4.3 Let $\{f_n\}$ be a sequence of functions each of which is uniformly continuous on an open interval (a, b). If $f_n \to f$ uniformly on (a, b) can you conclude that f is also uniformly continuous on (a, b)?

14.4.4 Give an example of a sequence of continuous functions $\{f_n\}$ on the interval $(0, 1)$ that is monotonic decreasing and converges pointwise to a continuous function f on $(0, 1)$ but for which the convergence is not uniform. Why does this not contradict Theorem 14.24?

14.4.5 Give an example of a sequence of continuous functions $\{f_n\}$ on the interval $[0, \infty)$ that is monotonic decreasing and converges pointwise to a continuous function f on $[0, \infty)$ but for which the convergence is not uniform. Why does this not contradict Theorem 14.24?

14.4.6 Let $\{f_n\}$ be a sequence of continuous nondecreasing functions defined on an interval $[a, b]$. Suppose $f_n \to f$ pointwise on $[a, b]$. Prove that if f is continuous on $[a, b]$, then the convergence is uniform. Observe that, in this exercise, the *functions* are assumed monotonic, whereas in Theorem 14.24 it is the *sequence* that we assume monotonic.

14.4.7 The proof of Theorem 14.24 depends on the compactness of the interval $[a, b]$. The compactness argument used here relied on the Bolzano-Weierstrass theorem. Attempt another proof using one of our other strategies from Section 4.5.

14.4.8 Prove this variant on Dini's theorem (Theorem 14.24). Let $\{f_n\}$ be a sequence of continuous functions on an interval $[a,b]$. Suppose for each $x \in [a,b]$ and for all $n \in \mathbb{N}$, $f_n(x) \leq f_{n+1}(x)$. Suppose in addition that for all $x \in [a,b]$ $\lim_n f_n(x) = \infty$. Show that for all $M > 0$ there is an integer N so that $f_n(x) > M$ for all $x \in [a,b]$ and all $n \geq N$. Show that this conclusion would not be valid without the monotonicity assumption.

14.4.9 Show that if, in Exercise 14.4.8, the interval $[a,b]$ is replaced by the unbounded interval $[0,\infty)$ or the nonclosed interval $(0,1)$, then the conclusion need not be valid.

14.4.10 Let $\{f_n\}$ be a sequence of Lipschitz functions on $[a,b]$ with common Lipschitz constant M. (This means that $|f_n(x) - f_n(y)| \leq M|x-y|$ for all $n \in \mathbb{N}$, $x, y \in [a,b]$.)

(a) If $f = \lim_n f_n$ pointwise, then f is continuous and, in fact, satisfies a Lipschitz condition with constant M.

(b) If $f = \lim_n f_n$ pointwise the convergence is uniform.

(c) Show by example that the results in (a) and (b) fail if we weaken our hypotheses by requiring only that each function is a Lipschitz function. (Here, the constant M may depend on n.)

SEE NOTE 249

14.4.11 (Continuous convergence and uniform convergence) A sequence of functions $\{f_n\}$ defined on an interval I is said to *converge continuously* to the function f if $f_n(x_n) \to f(x_0)$ whenever $\{x_n\}$ is a sequence of points in the interval I that converges to a point x_0 in I. Prove the following theorem:

Let $\{f_n\}$ be a sequence of continuous functions on an interval $[a,b]$. Then $\{f_n\}$ converges continuously on $[a,b]$ if and only if $\{f_n\}$ converges to f uniformly on $[a,b]$.

Does the theorem remain true if the interval $[a,b]$ is replaced with (a,b) or $[a,\infty)$?

14.4.12 Show that the sequence $f_n(x) = x^n/n$ converges uniformly on [0,1]:

(a) By direct computation using the definition of uniform convergence

(b) By using Theorem 14.24

(c) By using Exercise 14.4.6

(d) By using Exercise 14.4.11

14.5 Uniform Convergence and the Integral

Calculus students often learn the following simple computation. The geometric series

$$\frac{1}{1-t} = \sum_{k=0}^{\infty} t^k \tag{10}$$

is valid on the interval $(-1,1)$. An integration of both sides for t in the interval $[0,x]$, and any choice of $x<1$ will yield

$$-\log(1-x) = \int_0^x \frac{1}{1-t}\,dt = \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1}.$$

Indeed this identity is valid and provides a series expansion for the logarithm function. But can this really be justified?

In general, do we know that if $f(x) = \sum_0^\infty f_n(x)$ on an interval $[a,b]$, then

$$\int_a^b f(x)\,dx = \sum_0^\infty \int_a^b f_n(x)\,dx?$$

In fact, we already observed in Section 14.3 that during the early part of the nineteenth century, some prominent mathematicians took for granted the permissibility of term by term integration of convergent infinite series of functions. This was true of Fourier, Cauchy, and Gauss. Example 14.6, cast in the setting of sequences of integrable functions, shows that these mathematicians were mistaken.

14.5.1 Sequences of Continuous Functions

Around the middle of the nineteenth century, Weierstrass showed that term by term integration is permissible when the series of integrable functions converges *uniformly*. Let us first verify this result for sequences of continuous functions.

Theorem 14.26: *Suppose that $f(x) = \lim_{n\to\infty} f_n(x)$ for all $x \in [a,b]$, that each function f_n is continuous on $[a,b]$, and that the convergence is uniform. Then*

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\int_a^b f_n(x)\,dx.$$

Proof. By Theorem 14.22, f is continuous, so $\int_a^b f(x)\,dx$ exists. We must show that $\int_a^b f_n(x)\,dx \to \int_a^b f(x)\,dx$.

Let $\varepsilon > 0$. We wish to obtain $N \in \mathbb{N}$ such that

$$\left|\int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx\right| < \varepsilon \text{ for all } n \geq N.$$

We calculate that for any $n \in \mathbb{N}$

$$\left|\int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx\right| = \left|\int_a^b [f_n(x) - f(x)]\,dx\right|$$

$$\leq \int_a^b |f_n(x) - f(x)|\,dx \leq \int_a^b \max_{x\in[a,b]} |f_n(x) - f(x)|\,dx$$

$$\leq (b-a)\left(\max_{x\in[a,b]} |f_n(x) - f(x)|\right).$$

Since $f_n \to f$ uniformly on $[a,b]$, there exists $N \in \mathbb{N}$ such that

$$\max_{x\in[a,b]} |f_n(x) - f(x)| < \frac{\varepsilon}{b-a} \text{ for all } n \geq N.$$

Thus, for $n \geq N$, we have

$$\left|\int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx\right| \leq (b-a)\frac{\varepsilon}{b-a} = \varepsilon$$

as was to be shown. ∎

Applying the theorem to the partial sums S_n of a series allows us to express this result for series.

Corollary 14.27: *If an infinite series of continuous functions $\sum_0^\infty f_k$ converges uniformly to a function f on an interval $[a,b]$, then f is also continuous and*

$$\int_a^b f(x)\,dx = \sum_0^\infty \int_a^b f_k(x)\,dx.$$

Example 14.28: Let us justify the computations that we made in our introduction to this topic. The geometric series

$$\frac{1}{1-t} = \sum_{k=0}^\infty t^k \tag{11}$$

converges pointwise on the interval $(-1,1)$. Let $0 < x < 1$. By the M-test we see that this series converges uniformly on $[0,x]$. Each of the terms in the sum is continuous. As a result we may apply our theorem to integrate term by term just as we might have seen in a calculus course. Thus

$$\int_0^x \frac{1}{1-t}\,dt = \sum_{k=0}^\infty \frac{x^{k+1}}{k+1}.$$

◀

14.5.2 Sequences of Integrable Functions

✂ *Enrich.*

In Theorem 14.26 we required that the functions f_n be continuous. Suppose we now weaken our hypotheses for these functions by requiring only that they be integrable, but still requiring the sequence $\{f_n\}$ to converge uniformly to f. This latter hypothesis is very strong and suffices for a simple convergence theorem. Later we will prove much stronger results under very weak assumptions.

Our theorem shows that a uniform limit of integrable functions must be integrable and provides the following extension of Theorem 14.26.

Theorem 14.29: *Let $\{f_n\}$ be a sequence of functions integrable on an interval $[a,b]$. If $f_n \to f$ uniformly on $[a,b]$, then f is integrable on $[a,b]$ and*

$$\int_a^b f(x)\,dx = \lim_n \int_a^b f_n(x)\,dx.$$

Proof. The proof of Theorem 14.26 can be repeated if we happen to know that f is integrable. Thus we need only show that the limit function f is integrable on $[a,b]$.

Let $\varepsilon > 0$, choose a positive number η so that $\eta + 2\eta(b-a) < \varepsilon$, and choose an integer N so that $|f_n(x) - f(x)| < \eta$ for all $n \geq N$ and for all x in the interval $[a,b]$. The integrability criterion of Theorem 10.4 assures us that there must exist a full cover β so that

$$\left| \sum_{(I,z)\in\pi} \sum_{(I',z')\in\pi'} [f_N(z) - f_N(z')]\mathcal{L}(I\cap I') \right| < \eta \tag{12}$$

for all partitions π, π' of $[a,b]$ contained in β. It is easy to take the inequality (12) along with the inequalities $|f_N(z) - f(z)| < \eta$ and $|f_N(z') - f(z')| < \eta$ to deduce that

$$\left| \sum_{(I,z)\in\pi} \sum_{(I',z')\in\pi'} [f(z) - f(z')]\mathcal{L}(I\cap I') \right| < \eta + 2\eta(b-a) < \varepsilon \tag{13}$$

for all partitions π, π' of $[a,b]$ contained in β. But the condition (13) is a sufficient condition for the integrability of f by another application of Theorem 10.4. ■

Corollary 14.30: *Let $\{f_n\}$ be a sequence of functions absolutely integrable on an interval $[a,b]$. If $f_n \to f$ uniformly on $[a,b]$, then f is absolutely integrable on $[a,b]$,*

$$\int_a^b f(x)\,dx = \lim_n \int_a^b f_n(x)\,dx \ \ \textit{and} \ \ \int_a^b |f(x)|\,dx = \lim_n \int_a^b |f_n(x)|\,dx$$

Corollary 14.31: *If an infinite series of integrable functions $\sum_0^\infty f_k$ converges uniformly to a function f on an interval $[a,b]$, then f is also integrable and*

$$\int_a^b f(x)\,dx \int_a^b f\,(x)\ dx = \sum_0^\infty \int_a^b f_k\,(x)\ dx.$$

Exercises

14.5.1 Let $\{f_n\}$ be a sequence of functions on an interval $[a,b]$ and suppose that $f_n \to f$ uniformly on $[a,b]$. Show that

$$\overline{\int_a^b} f_n(x)\,dx \to \overline{\int_a^b} f(x)\,dx \quad \text{and} \quad \underline{\int_a^b} f_n(x)\,dx \to \underline{\int_a^b} f(x)\,dx.$$

14.5.2 Use the preceding exercise to construct another proof for Theorem 14.29.

14.5.3 Prove Corollary 14.30 by checking that the uniform convergence of the sequence $\{f_n\}$ would imply the uniform convergence of the sequence $\{|f_n|\}$.

14.5.3 Sequences of Improper Integrals

Thus far we have studied limits of ordinary integrals of functions on a finite interval $[a,b]$. What if the integrals are to be taken on an infinite interval?

More narrowly, let us just ask for the validity of the formulas:

$$\lim_{n\to\infty} \int_a^\infty f_n(t)\,dt = \int_a^\infty f(t)\,dt$$

in case $f_n \to f$ or

$$\sum_{k=1}^\infty \int_a^\infty g_k(t)\,dt = \int_a^\infty f(t)\,dt$$

in case $f = \sum_{k=1}^\infty g_k$. A fast and glib answer would be that we hardly expect these to be true for pointwise convergence but certainly uniform convergence will suffice.

But these integrals involve an extra limit operation and we therefore need extra caution. Indeed, the following example shows that uniform convergence is far from enough. It is not just the "smoothness" of the convergence that is an issue here.

Example 14.32: Let $f_n(x)$ be defined as $1/n$ for all values of $x \in [0, n]$ but as zero for $x > n$. Then the sequence $\{f_n\}$ converges to zero uniformly on the interval $[0, \infty)$. But the integrals do not converge to zero (as we would have hoped) since

$$\int_0^\infty f_n(t)\, dt = 1$$

for all n. ◀

What further condition can we impose so that, together with uniform convergence, we will be able to take the limit operation inside the integral

$$\lim_{n\to\infty} \int_0^\infty f_n(t)\, dt?$$

The condition we impose in the theorem just requires that all the functions are controlled or dominated by some function that is itself integrable. In Example 14.32 note that there is no possibility of an integrable function g on $[0, \infty)$ such that $f_n(x) \le g(x)$ for all n and x. Theorems of this kind are called *dominated convergence theorems.*

Theorem 14.33: *Let $\{f_n\}$ be a sequence of continuous functions on the interval $[a, \infty)$ such that $f_n \to f$ uniformly on any interval $[a, b]$. Suppose that there is a continuous function g on $[a, \infty)$ such that*

$$|f_n(x)| \le g(x)$$

for all $a \le x$ and all n. Suppose that the integral $\int_a^\infty g(x)\, dx$ exists. Then

$$\lim_{n\to\infty} \int_a^\infty f_n(t)\, dt = \int_a^\infty f(t)\, dt.$$

Proof. As a first step let us show that f is integrable on $[a, \infty)$. Certainly f is continuous since it is a uniform limit of a sequence of continuous functions. Since each $|f_n(x)| \le g(x)$ it follows that $|f(x)| \le g(x)$.

We check then

$$\left|\int_c^d f(t)\,dt\right| \le \int_c^d |f(t)|\,dt \le \int_c^d g(t)\,dt.$$

Since g is integrable, it follows by the Cauchy criterion for improper integrals (see Exercise 8.2.10) that the integral $\int_c^d g(t)\,dt$ can be made arbitrarily small for large c and d. But then so also is the integral $\int_c^d f(t)\,dt$, and a further application of the Cauchy criterion for improper integrals shows that f is integrable. (Indeed this argument shows that f is absolutely integrable in fact.)

Now let $\varepsilon > 0$. Choose L_0 so large that

$$\int_{L_0}^{\infty} g(t)\,dt < \varepsilon/4.$$

Choose N so large that

$$|f_n(t) - f(t)| < \frac{\varepsilon}{2(L_0 - a)}$$

if $n \ge N$ and $t \in [a, L_0]$. This is possible because $f_n \to f$ uniformly on $[a, L_0]$. Then we have

$$\left|\int_a^{\infty} f_n(t)\,dt - \int_a^{\infty} f(t)\,dt\right| \le \int_a^{L_0} |f_n(t) - f(t)|\,dt + \int_{L_0}^{\infty} 2g(t)\,dt$$

$$\le \frac{\varepsilon}{2(L_0 - a)}(L_0 - a) + \frac{2\varepsilon}{4} = \varepsilon$$

for all $n \ge N$. This proves the assertion of the theorem. ■

Exercises

14.5.1 Prove that

$$\lim_{n\to\infty} \int_{\frac{\pi}{2}}^{\pi} \frac{\sin nx}{nx}\,dx = 0.$$

14.5.2 Prove that

$$\int_0^{\pi} \sum_{n=1}^{\infty} \frac{\sin nx}{n^2}\,dx = \sum_{n=1}^{\infty} \frac{2}{(2n-1)^3}.$$

14.5.3 Show that if $f_n \to f$ uniformly on $[a, b]$ and each f_n is continuous then the sequence of functions

$$F_n(x) = \int_a^x f_n(t)\, dt$$

also converges uniformly on $[a, b]$.

14.5.4 Show that if $f_n \to f$ uniformly on $[a, b]$ and each f_n is continuous then

$$\lim_{n\to\infty} \int_a^b \left(\int_a^x f_n(t)\, dt \right) dx = \int_a^b \left(\int_a^x f(t)\, dt \right) dx.$$

14.5.5 Show that the series

$$\sum_{k=0}^{\infty} \frac{x^k}{k!}$$

converges uniformly on $[-a, a]$ for every $a \in \mathbb{R}$ but does not converge uniformly on all of the real line. (Does it converge pointwise on the real line?) Obtain a series representation for

$$\int_{-a}^{a} \sum_{k=0}^{\infty} \frac{x^k}{k!}\, dx.$$

14.5.6 Let $\{f_n\}$ be a sequence of continuous functions on an interval $[a, b]$ that converges uniformly to a function f. What conditions on g would allow you to conclude that

$$\lim_{n\to\infty} \int_a^b f_n(t)g(t)\, dt = \int_a^b f(t)g(t)\, dt?$$

14.5.7 Let $p > -1$. Show that

$$\lim_{n\to\infty} \int_1^n \left(1 - \frac{t}{n}\right)^n t^p\, dt = \int_1^\infty e^{-t} t^p\, dt.$$

14.5.8 Compute the limit

$$\lim_{n\to\infty} \int_0^1 \frac{e^{-nt}}{\sqrt{t}}\, dt.$$

14.6 Uniform Convergence and Derivatives

We saw in Section 14.5 that a uniformly convergent sequence (or series) of continuous functions can be integrated term by term. This allows an easy proof of a theorem on term by term differentiation.

Theorem 14.34: *Let $\{f_n\}$ be a sequence of functions each with a continuous derivative on an interval $[a,b]$. If the sequence $\{f'_n\}$ of derivatives converges uniformly to a function on $[a,b]$ and the sequence $\{f_n\}$ converges pointwise to a function f, then f is differentiable on $[a,b]$ and*

$$f'(x) = \lim_n f'_n(x) \quad \text{for all } x \in [a,b].$$

Proof. Let $g = \lim_n f'_n$. Since each of the functions f'_n is assumed continuous and the convergence is uniform, the function g is also continuous (Theorem 14.22). From Theorem 14.29 we infer

$$\int_a^x g(t)\,dt = \lim_n \int_a^x f'_n(t)\,dt \text{ for all } x \in [a,b]. \tag{14}$$

Applying the fundamental theorem of calculus we see that

$$\int_a^x f'_n(t)\,dt = f_n(x) - f_n(a) \text{ for all } x \in [a,b] \tag{15}$$

for all $n \in \mathbb{N}$.

But $f_n(x) \to f(x)$ for all $x \in [a,b]$ by hypothesis, so letting $n \to \infty$ in equation (15) and noting (14), we obtain

$$\int_a^x g(t)\,dt = f(x) - f(a)$$

or

$$f(x) = \int_a^x g(t)\,dt + f(a).$$

It follows from the continuity of g and the fundamental theorem of calculus that f is differentiable and that

$$f'(x) = g(x)$$

for all $x \in [a,b]$. ■

For series, the theorem takes the following form:

Corollary 14.35: *Let $\{f_k\}$ be a sequence of functions each with a continuous derivative on $[a,b]$ and suppose $f = \sum_0^\infty f_k$ on $[a,b]$. If the series $\sum_{k=0}^\infty f_k'$ converges uniformly on $[a,b]$, then $f' = \sum_{k=0}^\infty f_k'$ on $[a,b]$.*

Example 14.36: Starting with the geometric series

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \quad \text{on } (-1,1), \tag{16}$$

we obtain from Corollary 14.35 that

$$\frac{1}{(1-x)^2} = \sum_{k=1}^{\infty} kx^{k-1} \quad \text{on } (-1,1) \tag{17}$$

To justify (17) we observe first that the series (16) converges pointwise on $(-1,1)$. Next we note (Exercise 14.3.17) that the series (17) converges pointwise on $(-1,1)$ and uniformly on any closed interval $[a,b] \subset (-1,1)$. Thus, if $x \in (-1,1)$ and $-1 < a < x < b < 1$, then (17) converges uniformly on $[a,b]$, so (17) holds at x. ◀

14.6.1 Limits of Discontinuous Derivatives

The hypotheses of Theorem 14.34 are somewhat more restrictive than necessary for the conclusion to hold. We need not assume that $\{f_n\}$ converges on all of $[a,b]$; convergence at a single point suffices. Nor need we assume that each of the derivatives f_n' is continuous. (We cannot, however, replace uniform convergence of the sequence $\{f'_n\}$ with pointwise convergence, as Example 14.5 shows.) The theorem that follows applies in a number of cases in which Theorem 14.34 does not apply.

Theorem 14.37: *Let $\{f_n\}$ be a sequence of continuous functions defined on an interval $[a,b]$. Suppose that $f_n'(x)$ exists for each n and each $x \in [a,b]$. Suppose that the sequence $\{f'{}_n\}$ of derivatives converges uniformly on $[a,b]$ and that there exists a point $x_0 \in [a,b]$ such that the sequence of numbers $\{f_n(x_0)\}$ converges. Then the sequence $\{f_n\}$ converges uniformly to a function f on the interval $[a,b]$, f is differentiable, and*

$$f'(x) = \lim_{n\to\infty} f_n'(x)$$

at each point $x \in [a,b]$.

Proof. Let $\varepsilon > 0$. Since the sequence of derivatives converges uniformly on $[a,b]$, there is an integer N_1 so that

$$|f_n'(x) - f_m'(x)| < \varepsilon$$

for all n, $m \geq N_1$ and all $x \in [a,b]$. Also, since the sequence of numbers $\{f_n(x_0)\}$ converges, there is an integer $N > N_1$ so that

$$|f_n(x_0) - f_m(x_0)| < \varepsilon$$

for all n, $m \geq N$. Let us, for any $x \in [a,b]$, $x \neq x_0$, apply the mean value theorem to the function $f_n - f_m$ on the interval $[x_0, x]$ (or on the interval $[x, x_0]$ if $x < x_0$). This gives us the existence of some point ξ between x and x_0 so that

$$f_n(x) - f_m(x) - [f_n(x_0) - f_m(x_0)] = (x - x_0)[f_n'(\xi) - f_m'(\xi)]. \tag{18}$$

From this we deduce that

$$\begin{aligned} |f_n(x) - f_m(x)| &\leq |f_n(x_0) - f_m(x_0)| + |(x - x_0)(f_n'(\xi) - f_m'(\xi)| \\ &< \varepsilon(1 + (b - a)) \end{aligned}$$

for any n, $m \geq N$. Since this N depends only on ε this assertion is true for all $x \in [a,b]$ and we have verified that the sequence of continuous functions $\{f_n\}$ is uniformly Cauchy on $[a,b]$ and hence converges uniformly to a continuous function f on $[a,b]$.

Let us now show that $f'(x_0)$ is the limit of the derivatives $f_n'(x_0)$. Again, for any $\varepsilon > 0$, equation (18) implies that

$$|f_n(x) - f_m(x) - [f_n(x_0) - f_m(x_0)]| \leq |x - x_0|\varepsilon \tag{19}$$

for all $n, m \geq N$ and any $x \neq x_0$ in the interval $[a, b]$. In this inequality let $m \to \infty$ and, remembering that $f_m(x) \to f(x)$ and $f_m(x_0) \to f(x_0)$, we obtain

$$|f_n(x) - f_n(x_0) - [f(x) - f(x_0)]| \leq |x - x_0|\varepsilon \tag{20}$$

if $n \geq N$. Let C be the limit of the sequence of numbers $\{f_n'(x_0)\}$. Thus there exists $M > N$ such that

$$|f_M'(x_0) - C| < \varepsilon. \tag{21}$$

Since the function f_M is differentiable at x_0, there exists $\delta > 0$ such that if $0 < |x - x_0| < \delta$, then

$$\left| \frac{f_M(x) - f_M(x_0)}{x - x_0} - f_M'(x_0) \right| < \varepsilon. \tag{22}$$

From Equation (20) and the fact that $M > N$, we have

$$\left| \frac{f_M(x) - f_M(x_0)}{x - x_0} - \frac{f(x) - f(x_0)}{x - x_0} \right| < \varepsilon.$$

This, together with the inequalities (21) and (22), shows that

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - C \right| < 3\varepsilon$$

for $0 < |x - x_0| < \delta$. This proves that $f'(x_0)$ exists and is the number C, which we recall is $\lim_{n\to\infty} f_n'(x_0)$.

In this argument x_0 may be taken as any point inside the interval $[a, b]$ and so the theorem is proved. ■

For infinite series Theorem 14.37 takes the following form:

Corollary 14.38: *Let $\{f_k\}$ be a sequence of differentiable functions on an interval $[a,b]$. Suppose that the series $\sum_{k=0}^{\infty} f_k'$ converges uniformly on $[a,b]$. Suppose also that there exists $x_0 \in [a,b]$ such that the series $\sum_{k=0}^{\infty} f_k(x_0)$ converges. Then the series $\sum_{k=0}^{\infty} f_k(x)$ converges uniformly on $[a,b]$ to a function F, F is differentiable, and*

$$F'(x) = \sum_{k=0}^{\infty} f_k'(x)$$

for all $a \leq x \leq b$.

Note. In the statement of Theorem 14.37 we hypothesized the existence of a single point x_0 at which the sequence $\{f_n(x_0)\}$ converges. It then followed that the sequence $\{f_n\}$ converges on all of the interval I. If we drop that requirement but retain the requirement that the sequence $\{f_n'\}$ converges uniformly to a function g on I, we cannot conclude that $\{f_n\}$ converges on I [e.g. let $f_n(x) \equiv n$], but we can still conclude that there exists f such that $f' = g = \lim_n f_n'$ on I. (To see this, fix $x_0 \in I$, let $F_n = f_n - f_n(x_0)$ and apply Theorem 14.37 to the sequence $\{F_n\}$.) Thus, the uniform limit of a sequence of derivatives $\{f_n'\}$ is a derivative even if the sequence of primitives $\{f_n\}$ does not converge.

Exercises

14.6.1 Can the sequence of functions $f_n(x) = \dfrac{\sin nx}{n^3}$ be differentiated term by term? How about the series $\displaystyle\sum_{k=1}^{\infty} \frac{\sin kx}{k^3}$?

14.6.2 Verify that the function

$$y(x) = 1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \dots$$

is a solution of the differential equation $y' = 2xy$ on $(-\infty, \infty)$ without first finding an explicit formula for $y(x)$.

14.7 Fubini differentiation theorem

Enrich.

Added Dripped Section

If we have the Lebesgue differentiation theorem (Theorem 12.8) available to us we can get a nice series differentiation result.

We would like to have the formula

$$\frac{d}{dx}\sum_{n=1}^{\infty} F_n(x) = \sum_{n=1}^{\infty} \frac{d}{dx} F_n(x)$$

but we know that there are serious limitations to this. Uniform convergence assumed for the latter series and convergence assumed for the former will work, but that is asking rather a lot. Fubini's theorem says that with some assumptions on the nature of the functions F_n we can have this differentiation formula, not everywhere, but almost everywhere.

Theorem 14.39 (Fubini) *Let $\{F_n\}$ be a sequence of monotonic, nondecreasing functions on the interval $[a, b]$ and suppose that*

$$F(x) = \sum_{n=1}^{\infty} F_n(x)$$

is absolutely convergent for all $a \le x \le b$. Then

$$F'(x) = \sum_{n=1}^{\infty} F_n'(x)$$

for almost every x in (a, b).

Proof. Our main tool, apart from ordinary computations, is the fact that monotonic functions are differentiable almost everywhere. This is proved in Theorem 12.8.

Let us simplify the proof by deciding that $F_n(a) = 0$ for all n, so that F and all functions F_n are nonnegative. We know from the Lebesgue differentiation theorem applied to all of these monotonic functions that, except for x in a set of measure zero, all of the derivatives, $F'(x)$ and $F_n'(x)$ exist. Thus it is only the identity for these values of x that we need to establish.

Note that

$$F(x) \geq \sum_{n=1}^{m} F_n(x)$$

for every integer m so that for almost every x,

$$F'(x) \geq \sum_{n=1}^{m} F_n'(x)$$

and, consequently,

$$F'(x) \geq \sum_{n=1}^{\infty} F_n'(x). \tag{23}$$

To simplify we can assume that

$$F(b) - \sum_{n=1}^{m} F_n(b) \leq 2^{-m}.$$

If this were not the case then we could put parentheses in the series, group terms together, and relabel so that this would be the case. Consider the series

$$G(x) = \sum_{n=1}^{\infty} \left(F(x) - \sum_{k=1}^{n} F_k(x) \right).$$

Note that

$$0 \leq G(x) - \sum_{n=1}^{m} \left(F(x) - \sum_{k=1}^{n} F_k(x) \right) \leq \sum_{n=m+1}^{\infty} 2^{-n} = 2^{-m}.$$

Thus we see that G is also the sum of a series of functions.

A repeat of the argument we just gave to establish (23) will provide the analogous statement for this series:

$$0 \leq \sum_{n=1}^{\infty} \left(F'(x) - \sum_{k=1}^{n} F_k'(x) \right) \leq G'(x) \tag{24}$$

The function G has a finite derivative at almost every point. So in order for the inequality in (24) to hold for this series at a particular value of x the terms must tend to zero. Writing that out we now know that, for almost every x,

$$\lim_{n\to\infty}\left(F'(x)-\sum_{k=1}^{n}F_k'(x)\right)=0.$$

This is exactly the conclusion of the theorem.

■

14.8 Pompeiu's Function

By the end of the nineteenth century analysts had developed enough tools to begin constructing examples of functions that challenged the then prevailing views. One famous mathematician, Henri Poincaré, complained that

> Before when one would invent a new function it was to some practical end; today they are invented to demonstrate the errors in the reasoning of our fathers ....

Many mathematicians were both shocked and appalled that functions could be constructed which possessed, to them, bizarre and unnatural properties. The beautiful and elegant theories of the nineteenth century were being torn to pieces by pathological examples.

Perhaps the earliest shock was the construction by Weierstrass and others of continuous functions that had derivatives at no points. This did indeed demonstrate some earlier errors because not a few mathematicians thought they had succeeded in proving that continuous functions could not be like this. Another famous example is due to Vito Volterra (1860–1940), who produced a differentiable function F with a bounded derivative F' that was not Riemann integrable. Of course, F' is absolutely integrable [i.e., Lebesgue integrable]. This example alone is sufficient to put the Riemann integral in a pretty bad light. Lebesgue claimed that it was this example was one of his motivations for extending the integral to include all bounded derivatives.

In this section we present an example due to D. Pompeiu in 1906. This function h is differentiable and has the remarkable property that h' is discontinuous on a dense set and h' is zero on another dense set. We shall see that this implies that h is a differentiable function that, like Volterra's example, has a derivative that is not Riemann integrable. In fact, it is Riemann integrable on *no* interval while Volterra's example is Riemann integrable on many subintervals.

The example makes use of many theorems that we have established to this point and so offers an excellent review of our techniques. We present the example in a series of steps, each of which is left as a relatively easy exercise. (Exercise 14.8.4 is plausible but messy to verify, and you may prefer not to check the details.)

To begin the example we observe that the function

$$f(x) = \sqrt[3]{x-a}$$

has an infinite derivative at $x = a$ and a finite derivative elsewhere. Let $q_1, q_2, q_3, \ldots$ be an enumeration of $\mathbb{Q} \cap [0,1]$. Let

$$f(x) = \sum_{k=1}^{\infty} \frac{\sqrt[3]{x-q_k}}{10^k}.$$

The Pompeiu function is the inverse of this function, $h = f^{-1}$.

The details appear in the exercises. Note especially that our main goal is to prove that h is differentiable, h' is bounded, $h' = 0$ on a dense set and h' is positive and discontinuous on another dense set, and h' is not Riemann integrable.

For comparisons let us recall that in Exercise 7.4.2, we provided an example of a differentiable function g with g' bounded but discontinuous on a nowhere-dense perfect set P. It is a feature (or bug) in the Riemann integration theory that, if P does not have measure zero, g' will not be Riemann integrable. Thus we cannot write

$$g(x) - g(a) = \int_a^x g'(t)\, dt,$$

that is, the fundamental theorem of calculus does not hold for the function g and its derivative g' if we are restricted to using the Riemann integral. This is essentially how Volterra constructed his example, by ensuring that the set P does not have measure zero.

We also mentioned in Section 7.4 that it is possible for a differentiable function f to have f' discontinuous on a dense set, and so Pompeiu's function justifies this comment.

Exercises

14.8.1 Show that the function $f(x) = (x-a)^{\frac{1}{3}}$ has an infinite derivative at $x = a$ and a finite derivative elsewhere.

14.8.2 Let $q_1, q_2, q_3, \ldots$ be an enumeration of $\mathbb{Q} \cap [0,1]$. For each $k \in \mathbb{N}$ let

$$f_k(x) = (x - q_k)^{\frac{1}{3}} \quad \text{and} \quad f(x) = \sum_{k=1}^{\infty} \frac{f_k(x)}{10^k}.$$

Show that the series defining f converges uniformly.

14.8.3 Show that f is continuous on $[0,1]$.

14.8.4 Check that, for all $x \in \mathbb{R}$,

$$f'(x) = \sum_{k=1}^{\infty} \frac{f_k'(x)}{10^k} = \sum_{k=1}^{\infty} \frac{(x-q_k)^{-\frac{2}{3}}}{3 \times 10^k}.$$

(This part is messy to prove. Indicate why it is that we cannot simply apply Corollary 14.38 and differentiate term by term.)

14.8.5 Show that $f'(x) = \infty$ for all $x \in \mathbb{Q} \cap [0,1]$. (There are also other points at which f' is infinite; see Exercise 14.8.17.)

14.8.6 Show that $f([0,1])$ is an interval. Call it $[a,b]$.

14.8.7 Let $S = f(\mathbb{Q} \cap [0,1])$. Show that S is dense in $[a,b]$.

14.8.8 Show that f has an inverse.

14.8.9 Let $h = f^{-1}$. Show that h is continuous and strictly increasing on $[a,b]$.

14.8.10 Show that $h' = 0$ on the dense set S.

14.8.11 Show that there exists $\gamma > 0$ such that $f'(x) \geq \gamma$ for all $x \in [0,1]$.

14.8.12 Show that h is differentiable and that h' is bounded.

14.8.13 Show that $h' > 0$ on a dense subset of $[a,b]$.

14.8.14 Show that h' is discontinuous on a dense subset of $[a,b]$.

14.8.15 Thus far we know that h is differentiable, has a bounded derivative, $h' = 0$ on a dense set and h' is positive and discontinuous on another dense set. Show that h' is not Riemann integrable.

See Note 250

14.8.16 Show that $\{x : h'(x) \neq 0\}$ does not have measure zero.

14.8.17 Show that there exists $x \notin S$ such that $h'(x) = 0$ and that there exists $t \notin \mathbb{Q}$ such that $f'(t) = \infty$.

14.8.18 Show that the function h is not convex or concave in any interval. Which of the definitions of inflection point given as Exercise 7.10.14 apply to the points x at which $h'(x) = 0$? Do you think that such a point should be called an inflection point?

14.9 Continuity and Pointwise Limits

Adv.

Much of this chapter focused on the concept of uniform convergence because of its role in providing affirmative answers to the questions we raised in Section 14.1. In particular, we saw in Section 14.2 that a pointwise limit of a sequence of continuous functions need not be continuous. On the other hand, these problems will not occur if the convergence is uniform.

There are, however, many situations in which pointwise convergence arises naturally, but uniform convergence doesn't. Consider, for example, a function F that is differentiable on $\mathbb{R}$. Then for $x \in \mathbb{R}$,

$$F'(x) = \lim_{n\to\infty} \frac{F(x+\frac{1}{n}) - F(x)}{\frac{1}{n}}.$$

If we define functions f_n by

$$f_n(x) = \frac{F(x+\frac{1}{n}) - F(x)}{\frac{1}{n}},$$

then each of the functions f_n is continuous on $\mathbb{R}$ and $f_n \to F'$ pointwise.

Now, we have seen examples of derivatives that are discontinuous at many points. For example, the function h' in Section 14.8 is discontinuous on a set that is dense in $[0,1]$ and does not have measure zero. Similarly, Exercise 7.4.2 provides an example of a differentiable function g whose derivative g' is discontinuous at every point of a Cantor set that does not have measure zero. We might ask the question, "Can the derivative of a differentiable function be discontinuous everywhere?" We shall see that the answer is "no." In fact, the set of points of continuity must be large in the sense of category—this set must be dense and of type $\mathcal{G}_\delta$, therefore residual (Theorem 6.17).

We actually prove a more general theorem.

Theorem 14.40: *Let $\{g_n\}$ be a sequence of continuous functions defined on an interval I and converging pointwise to a function g on I. Then the set of points of continuity of g forms a dense set of type $\mathcal{G}_\delta$ in I.*

Proof. Let us first outline the idea of the proof, leaving the formal proof for a moment. In Section 6.7 we defined the oscillation $\omega_f(x_0)$ of a function f at a point x_0 and showed (Theorem 6.25) that f is continuous at x_0 if and only if $\omega_f(x_0) = 0$. We now show that under the hypotheses of Theorem 14.40, $\omega_g(x)$ will be zero on a dense set. That will imply that g is continuous on a dense set. This set must be of type $\mathcal{G}_\delta$ (by Theorem 6.28).

We will argue by contradiction. We suppose that g is discontinuous at every point of some subinterval J. We will then use the Baire category theorem (Theorem 6.11) to show that there exists $n \in \mathbb{N}$ and an interval $H \subset J$ such that $\omega_g(x) \geq 1/n$ at every point of H. (This argument is valid for any function discontinuous at every point of an interval J.) We then use our hypotheses on g to show this is impossible. We do this by applying the Baire category theorem once again to obtain a subinterval K of H that g maps onto a set of diameter less than $1/n$. This implies that $\omega_f(x) < 1/n$ for every $x \in K$, a contradiction.

Now we can begin a formal proof of Theorem 14.40.

In order to obtain a contradiction, we suppose that g is discontinuous everywhere on some interval $J \subset I$. For each $n \in \mathbb{N}$, let

$$E_n = \{x \in J : \omega_g(x) \geq 1/n\}.$$

Each of the sets E_n is closed (by Theorem 6.27)and $J = \bigcup_{n=1}^{\infty} E_n$.

By the Baire category theorem there exists $n \in \mathbb{N}$ and an interval $H \subset J$ such that E_n is dense in H. The interval H has the property that g maps every subinterval of H onto a set of diameter at least $1/n$. We now show this not possible for g, a pointwise limit of continuous functions.

Let $\{I_k = (a_k, b_k)\}$ be a sequence of intervals, each of length less than $1/n$, such that

$$g(H) \subset \bigcup_{k=1}^{\infty} I_k.$$

For each k, let $H_k = g^{-1}(I_k) \cap H$. Then $H = \bigcup_{k=1}^{\infty} H_k$, but none of the sets H_k can contain an interval [since each H_k has length less than $1/n$, but $\omega_g(x) \geq 1/n$ for all $x \in H$].

Now

$$H_k = \{x : g(x) < b_k\} \cap \{x : g(x) > a_k\}.$$

By Exercise 14.9.4, each of these sets is of type $\mathcal{F}_\sigma$, thus $H_k = \bigcup_{j=1}^{\infty} H_{kj}$, with each of the sets H_{kj} closed. It follows that

$$H = \bigcup_{k=1}^{\infty} H_k = \bigcup_{k=1}^{\infty} \bigcup_{j=1}^{\infty} H_{kj}.$$

The interval H is expressed as a countable union of closed sets. It follows from the Baire category theorem that at least one of these sets, say H_{ij}, is dense in some interval $K \subset H$. Since H_{ij} is closed, $H_{ij} \supset K$. But this implies that $H_i \supset K$, which we have seen is not possible (since each of the sets H_k contains no intervals). This contradiction completes the proof. ■

Corollary 14.41: *Let f be differentiable on an interval (a, b). Then f' is continuous on a residual subset of (a, b). Thus the set of points of continuity of f' must be dense in (a, b).*

Note. Theorem 14.40 and Exercise 14.9.4 describe two important properties of functions that are pointwise limits of sequences of continuous functions. Each such function f is continuous on a residual set, and every set of the form $\{x : f(x) > a\}$ or $\{x : f(x) < a\}$ is of type $\mathcal{F}_\sigma$.

Theorem 14.40 can be generalized. If P is a nonempty closed subset of the domain of f, then the function $f|P$ is continuous on a dense $\mathcal{G}_\delta$ subset of P.

The converses are also true[3]: A function f is a pointwise limit of a sequence of continuous functions on an interval I if and only if for every closed set $P \subset I$, f considered as a function defined on the set P is continuous on a dense $\mathcal{G}_\delta$ in P, and this happens if and only if every set of the form $\{x : f(x) > a\}$ or $\{x : f(x) < a\}$ is of type $\mathcal{F}_\sigma$.

These theorems have many applications. Functions that are pointwise limits of sequences of continuous functions are called *Baire* 1 functions. We have seen that this class of functions contains the class of derivatives. It also contains all monotonic functions and many other important classes of functions that arise in analysis.

The following exercises may be instructive. You may need to use one of the unproved statements in this section to work some of these exercises.

Exercises

14.9.1 Give an example of a function F that is differentiable on $\mathbb{R}$ such that the sequence

$$f_n(x) = n(F(x + 1/n) - F(x)),$$

converges pointwise but not uniformly to F'.

See Note 251

14.9.2 Give an example of a function f that is Baire 1 and a real number a so that the sets $\{x : f(x) > a\}$ and $\{x : f(x) < a\}$ are not open. Show that, for your example, these sets are of type $\mathcal{F}_\sigma$.

14.9.3 Give an example of a function f that is Baire 1 and a real number a so that the sets $\{x : f(x) \geq a\}$ and $\{x : f(x) \leq a\}$ are not closed. Show that, for your example, these sets are of type $\mathcal{G}_\delta$.

14.9.4 Show that for any f that is Baire 1 and any real number a the sets

$$\{x : f(x) > a\} \quad \text{and} \quad \{x : f(x) < a\}$$

[3] Proofs of these statements can be found in I. P. Natanson, *Theory of Functions of a Real Variable*, Vol. II, Chapter XV, Fredrick Ungar Pub. Co., New York (1955) [English translation].

are of type $\mathcal{F}_\sigma$.

14.9.5 If f has only countably many discontinuities on an interval I, then f is a Baire 1 function. In particular, this is true for every monotonic function.

14.9.6 Let K be the Cantor set in $[0,1]$. Define

$$f(x) = \begin{cases} 1, & \text{if } x \in K \\ 0, & \text{elsewhere;} \end{cases}$$

and

$$g(x) = \begin{cases} 1, & \text{if } x \text{ is a one-sided limit point of } K \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Show that f and g have exactly the same set of continuity points.

(b) Show that f is a Baire 1 function but g is not.

14.9.7 Let f be the characteristic function of the rationals. Show that f is not a Baire 1 function. Show that f is a pointwise limit of a sequence of Baire 1 functions. (Such functions are called functions of Baire class 2.)

14.10 Challenging Problems for Chapter 14

14.10.1 Let f_n be a sequence of functions converging pointwise to a function f on the interval $[0,1]$. Suppose that each function f_n is convex on $[0,1]$. Show that the convergence is uniform on any interval $[a,b] \subset (0,1)$. Need it be uniform on $[0,1]$?

14.10.2 Let $f_n : [0,1] \to \mathbb{R}$ be a sequence of continuous functions converging pointwise to a function f. If the convergence is uniform, prove that there is a finite number M so that $|f_n(x)| < M$ for all n and all $x \in [0,1]$. Even if the convergence is not uniform, show that there must be a subinterval $[a,b] \subset [0,1]$ and a finite number M so that $|f_n(x)| < M$ for all n and all $x \in [a,b]$.

See Note 252

14.10.3 Let E be a set of real numbers, fixed throughout this exercise. For any function f defined on E write

$$\|f\|_\infty = \sup_{x \in E} |f(x)|.$$

Show that

(a) $\|f\|_\infty = 0$ if and only if f is identically zero on E.
(b) $\|cf\|_\infty = |c|\|f\|_\infty$ for any real number c.
(c) $\|f+g\|_\infty \le \|f\|_\infty + \|g\|_\infty$ for any functions f and g.
(d) $f_n \to f$ uniformly on E if and only if $\|f - f_n\|_\infty \to 0$ as $n \to \infty$.
(e) f_n converges uniformly on E if and only if $\|f_m - f_n\|_\infty \to 0$ as $n, m \to \infty$.
(f) Using $E = (0,1)$ and $f_n(x) = x^n$, compute $\|f_n\|_\infty$ and, hence, show that $\{f_n\}$ is not converging uniformly to zero on $(0,1)$.

Notes

[245]Exercise 14.2.7. The statements that are defined by inequalities (e.g., bounded, convex) or by equalities (e.g., constant, linear) will not lead to an interchange of two limit operations, and you should expect that they are likely true.

[246]Exercise 14.2.8. As the footnote to the exercise explains, this was Luzin's unfortunate attempt as a young student to understand limits. The professor began by saying "What you say is nonsense." He gave him the example of the double sequence $m/(m+n)$ where the limits as $m \to \infty$ and $n \to \infty$ cannot be interchanged and continued by insisting that "permuting two passages to the limit *must not be done*." He concluded with "Give it some thought; you won't get it immediately."

[247]Exercise 14.3.15. Use the Cauchy criterion for convergence of sequences of real numbers to obtain a candidate for the limit function f. Note that if $\{f_n\}$ is uniformly Cauchy on a set D, then for each $x \in D$, the sequence of real numbers $\{f_n(x)\}$ is a Cauchy sequence and hence convergent.

[248]Exercise 14.3.16.

$$S_n(x) = \sum_{k=0}^{n} x^k = \frac{1-x^n}{1-x}.$$

[249]Exercise 14.4.10. For part (b) consider

$$F_n(x) = f_n(x) + Mx$$

and apply Exercise 14.4.6.

[250]Exercise 14.8.15. Suppose h' were Riemann integrable. Explain why

$$h(x) - h(a) = \int_a^x h'(t)\, dt$$

for all $x \in [a, b]$. Now by considering an appropriate Riemann sum, since $h' = 0$ on a dense set, we would have

$$h(x) - h(a) = 0$$

for all $x \in [a, b]$. That should be a contradiction.

[251]Exercise 14.9.1. What properties would F' have to have if the convergence were uniform?

[252]Exercise 14.10.2. You will need to use the Baire category theorem for the second part of this.

Chapter 15

MONOTONE CONVERGENCE THEOREM

Dripped Chapter[1]

For a great many calculus applications we would need to be able to use the formula

$$\int_a^b \left(\sum_{n=1}^{\infty} f_n(x) \right) dx = \sum_{n=1}^{\infty} \left(\int_a^b f_n(x)\, dx \right).$$

Our previous chapter suggests that we should check for uniform convergence. But the integral allows exactly such a computation under many simpler hypotheses. This chapter focusses on the special case of monotone limits, and (for series) sums of nonnegative functions.

[1]Note to the instructor: For a truly elementary course, perhaps uniform convergence is the main and only tool that should be communicated to the student about the interchange of limit operations for integrals. This chapter carries this much further and shows that pointwise a.e. convergence is enough with monotonicity (or sums of nonnegative functions). The usual textbooks proofs would require measure theory and would be presented as part of the development of the Lebesgue integral. The advantage in the ***dripped*** version is that our standard covering arguments are all that are needed (albeit requiring a rather careful argument).

15.1 Summing inside the integral

We establish that the summation

$$\int_a^b \left(\sum_{n=1}^{\infty} f_n(x)\right)\, dx = \sum_{n=1}^{\infty}\left(\int_a^b f_n(x)\,dx\right)$$

is possible for nonnegative functions. Our analysis is made easy by handling the upper and lower integrals separately. In this way we do not have to worry about the integral or integrability assumptions in advance. It is often the case that if we can formulate a theorem with minimal hypotheses, the proof is simplified.

15.1.1 Two lemmas

Lemma 15.1: *Suppose that $f,\ f_1,\ f_2,\ \ldots$ is a sequence of nonnegative functions defined on a compact interval $[a,b]$. If, for almost every x*

$$f(x) \geq \sum_{n=1}^{\infty} f_n(x),$$

then

$$\underline{\int_a^b} f(x)\,dx \geq \sum_{n=1}^{\infty}\left(\underline{\int_a^b} f_n(x)\,dx\right). \tag{1}$$

Proof. We can assume that the inequality assumed is valid for *every* x; simply redefine $f_n(x) = 0$ for those points in the null set where the inequality doesn't work. The resulting functions will have the same lower integrals as f_n.

Let $\varepsilon > 0$. Take any integer N and choose full covers β_n $(n = 1, 2, \ldots, N)$ so that all the Riemann sums[2]

$$\sum_{\pi} f_n(w)(v-u) \geq \underline{\int_a^b} f_n(x)\,dx - \varepsilon 2^{-n}$$

whenever $\pi \subset \beta_n$ is a partition of $[a, b]$. (If the integrals here are not finite then there is nothing to prove, since both sides of the inequality (1) will be infinite.)

Let

$$\beta = \bigcap_{n=1}^{N} \beta_n.$$

This too is a full cover, one that is contained in all of the others.

Take any partition of $[a, b]$ with $\pi \subset \beta$, and compute

$$\sum_{\pi} f(w)(v-u) \geq \sum_{\pi}\left(\sum_{n=1}^{N} f_n(w)(v-u)\right) = \sum_{n=1}^{N}\left(\sum_{\pi} f_n(w)(v-u)\right) \geq$$

$$\sum_{n=1}^{N}\left(\underline{\int_a^b} f_n(x)\,dx - \varepsilon 2^{-n}\right).$$

This gives a lower bound for all Cauchy sums and hence, since ε is arbitrary, shows that

$$\underline{\int_a^b} f(x)\,dx \geq \sum_{n=1}^{N}\left(\underline{\int_a^b} f_n(x)\,dx\right).$$

As this is true for all N the inequality (1) must follow. ■

[2]We simplify our notation for Riemann sums a bit by replacing

$$\sum_{([u,v],w)\in\pi} f(w)(v-u) \quad \text{by} \quad \sum_{\pi} f(w)(v-u).$$

Lemma 15.2: *Suppose that f, f_1, f_2, ... is a sequence of nonnegative functions defined on a compact interval $[a,b]$. If, for almost every x*

$$f(x) \le \sum_{n=1}^{\infty} f_n(x),$$

then

$$\overline{\int_a^b} f(x)\,dx \le \sum_{n=1}^{\infty} \left(\overline{\int_a^b} f_n(x)\,dx \right). \tag{2}$$

Proof. As before, we can assume that the inequality assumed is valid for *every* x; simply redefine $f(x) = 0$ for those points in the null set where the inequality doesn't work. The resulting function will have the same integral and same upper integral as f.

This lemma is similar to the preceding one, but requires a bit of bookkeeping and a new technique with the covers. Let $t < 1$ and choose for each $x \in [a,b]$ the first integer $N(x)$ so that

$$tf(x) \le \sum_{n=1}^{N(x)} f_n(x).$$

Choose, again and using the same ideas as before, full covers β_n $(n = 1, 2, \dots)$ so that $\beta_1 \supset \beta_2 \supset \beta_3 \dots$ and all Riemann sums[3]

$$\sum_{\pi} f_n(w)(v-u) \le \overline{\int_a^b} f_n(x)\,dx + \varepsilon 2^{-n}$$

whenever $\pi \subset \beta_n$ is a partition of $[a,b]$. (Again, if the integrals here are not finite then there is nothing to prove, since the larger side of the inequality (2) will be infinite.)

[3] As before, we simplify our notation for Riemann sums by replacing

$$\sum_{([u,v],w)\in\pi} f(w)(v-u) \quad \text{by} \quad \sum_{\pi} f(w)(v-u).$$

Let

$$E_n = \{x \in [a,b] : N(x) = n\}.$$

We use these sets to carve up the covering relations. Write

$$\beta_n[E_n] = \{([u,v],w) \in \beta_n : w \in E_n\}.$$

There must be a full cover β so that

$$\beta[E_n] \subset \beta_n[E_n]$$

for all $n = 1, 2, 3, \dots$.

Take any partition of $[a,b]$ with $\pi \subset \beta$. Let N be the largest value of $N(x)$ for the finite collection of pairs $(I,x) \in \pi$. We need to carve the partition π into a finite number of disjoint subsets by writing, for $j = 1, 2, 3, \dots, N$,

$$\pi_j = \{([u,v],w) \in \pi : w \in E_j\}$$

and

$$\sigma_j = \pi_j \cup \pi_{j+1} \cup \dots \cup \pi_N.$$

for integers $j = 1, 2, 3, \dots, N$. Note that

$$\sigma_j \subset \beta_j$$

and that

$$\pi = \pi_1 \cup \pi_2 \cup \dots \cup \pi_N.$$

Check the following computations, making sure to use the fact that for $x \in E_i$,

$$tf(x) \le f_1(x) + f_2(x) + \dots + f_i(x).$$

$$\sum_{\pi} tf(w)(v-u) = \sum_{i=1}^{N} \sum_{\pi_i} tf(w)(v-u)$$

$$\le \sum_{i=1}^{N} \sum_{\pi_i} \left(f_1(w) + f_2(w) + \dots + f_i(w)\right)(v-u)$$

$$= \sum_{j=1}^{N} \left(\sum_{\sigma_j} f_j(w)(v-u) \right) \leq$$

$$\sum_{j=1}^{N} \left(\overline{\int_a^b} f_j(x)\,dx + \varepsilon 2^{-j} \right) \leq \sum_{j=1}^{\infty} \left(\overline{\int_a^b} f_j(x)\,dx \right) + \varepsilon.$$

This gives an upper bound for all Cauchy sums and hence, since ε is arbitrary, shows that

$$\overline{\int_a^b} t f(x)\,dx \leq \sum_{n=1}^{\infty} \left(\overline{\int_a^b} f_n(x)\,dx \right).$$

As this is true for all $t < 1$ the inequality (2) must follow too. ■

15.1.2 Integration of series

Theorem 15.3: *Let $f_n : [a,b] \to \mathbb{R}$ $(n = 1, 2, 3, \dots)$ be a sequence of nonnegative integrable functions and suppose that*

$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$

for almost every x. Then

$$\int_a^b f(x)\,dx = \sum_{n=1}^{\infty} \left(\int_a^b f_n(x)\,dx \right).$$

Proof. This follows from Lemmas 15.1 and 15.2. ■

15.1.3 Monotone convergence theorem

Theorem 15.4 (Monotone convergence theorem) *Let $f_n : [a,b] \to \mathbb{R}$ $(n = 1, 2, 3, \dots)$ be a nondecreasing sequence of integrable functions and suppose that*

$$f(x) = \lim_{n\to\infty} f_n(x)$$

for almost every x in $[a,b]$. Then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \int_a^b f_n(x)\,dx.$$

Proof. This follows directly from Theorem 15.3 and the identity

$$f(x) = f_1(x) + \sum_{n=1}^{\infty} \left(f_n(x) - f_{n-1}(x)\right).$$

■

Exercises

15.1.1 Give an example to show that it is possible that $\int_a^b f(x)\,dx = \infty$ in Theorem 15.3.

15.1.2 Give an example to show that it is possible for the Theorem 15.3 to fail if we drop the assumption that the functions are nonnegative in the theorem.

15.1.3 Let $f_n : [a,b] \to \mathbb{R}$ $(n = 1, 2, 3, \dots)$ be a sequence of absolutely integrable functions and suppose that

$$\sum_{n=1}^{\infty} |f_n(x)| < \infty$$

for almost every x and that

$$\sum_{n=1}^{\infty} \left(\int_a^b |f_n(x)|\,dx \right) < \infty.$$

Then show that

$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$

is finite for almost every x in $[a, b]$, is absolutely integrable, and that

$$\int_a^b f(x)\,dx = \sum_{n=1}^{\infty} \left(\int_a^b f_n(x)\,dx \right).$$

Notes

[245]Exercise 14.2.7. The statements that are defined by inequalities (e.g., bounded, convex) or by equalities (e.g., constant, linear) will not lead to an interchange of two limit operations, and you should expect that they are likely true.

[246]Exercise 14.2.8. As the footnote to the exercise explains, this was Luzin's unfortunate attempt as a young student to understand limits. The professor began by saying "What you say is nonsense." He gave him the example of the double sequence $m/(m+n)$ where the limits as $m \to \infty$ and $n \to \infty$ cannot be interchanged and continued by insisting that "permuting two passages to the limit *must not be done*." He concluded with "Give it some thought; you won't get it immediately."

[247]Exercise 14.3.15. Use the Cauchy criterion for convergence of sequences of real numbers to obtain a candidate for the limit function f. Note that if $\{f_n\}$ is uniformly Cauchy on a set D, then for each $x \in D$, the sequence of real numbers $\{f_n(x)\}$ is a Cauchy sequence and hence convergent.

[248]Exercise 14.3.16.

$$S_n(x) = \sum_{k=0}^{n} x^k = \frac{1 - x^n}{1 - x}.$$

[249]Exercise 14.4.10. For part (b) consider

$$F_n(x) = f_n(x) + Mx$$

and apply Exercise 14.4.6.

[250]Exercise 14.8.15. Suppose h' were Riemann integrable. Explain why

$$h(x) - h(a) = \int_a^x h'(t)\, dt$$

for all $x \in [a,b]$. Now by considering an appropriate Riemann sum, since $h' = 0$ on a dense set, we would have

$$h(x) - h(a) = 0$$

for all $x \in [a,b]$. That should be a contradiction.

[251]Exercise 14.9.1. What properties would F' have to have if the convergence were uniform?

[252]Exercise 14.10.2. You will need to use the Baire category theorem for the second part of this.

Chapter 16

POWER SERIES

✂ If the material on lim sups and lim infs in Section 2.13 of Chapter 2 was omitted, that should be studied before attempting this chapter. The notion of a radius of convergence depends naturally on these concepts.

16.1 Introduction

One of the simplest and, arguably, the most important type of series of functions is the power series. This is a series of the form

$$\sum_{0}^{\infty} a_k x^k$$

or, more generally,

$$\sum_{0}^{\infty} a_k (x-c)^k.$$

It represents the notion of an "infinitely long" polynomial

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k + \ldots.$$

The material we developed in Chapter 14 will allow us to show in this chapter that power series can be treated very much as if they were indeed polynomials in the sense that they can be integrated and differentiated term by term.

The main reason for developing this theory is that it allows a representation for functions as series. This enlarges considerably the class of functions that we can work with. Not all functions that arise in applications can be expressed as finite combinations of the elementary functions (i.e., as combinations of e^x, x^p, $\sin x$, $\cos x$, etc.). Thus, if we remain at the level of an elementary calculus class, we would be unable to solve many problems since we cannot express the functions needed for the solution in any way. For a large and important class of problems, the functions that can be represented as power series (the so-called analytic functions) are precisely the functions needed.

16.2 Power Series: Convergence

We begin with the formal definition of power series.

Definition 16.1: Let $\{a_k\}$ be a sequence of real numbers and let $c \in \mathbb{R}$. A series of the form

$$\sum_0^\infty a_k(x-c)^k = a_0 + a_1(x-c) + a_2(x-c)^2 + \dots$$

is called a *power series* centered at c. The numbers a_k are called the *coefficients* of the power series.

What can we say about the set of points on which the power series

$$\sum_0^\infty a_k(x-c)^k$$

converges? It is immediately clear that the series converges at its center c. What possibilities are there? A collection of examples illustrates the methods and also essentially all of the possibilities.

Example 16.2: The simple example

$$\sum_{1}^{\infty} k^k x^k = x + 4x^2 + 27x^3 + \dots$$

shows that a power series can diverge at every point other than its center. Observe that in this example $k^k x^k = (kx)^k$ does not approach 0 unless $x = 0$, so the series diverges for every $x \neq 0$ by the trivial test. Thus the set of convergence of this series is the set $\{0\}$. ◀

Example 16.3: The familiar geometric series

$$\sum_{k=0}^{\infty} x^k$$

should be considered the most elementary of all power series. We know that this series converges precisely on the interval $(-1, 1)$ and diverges everywhere else. ◀

Example 16.4: The series

$$\sum_{k=1}^{\infty} \frac{x^k}{k}$$

has as coefficients $a_k = 1/k$ and the root test[1] supplies

$$s = \limsup_{k \to \infty} \sqrt[k]{|x|^k / k} = |x|.$$

(Verify this!) Thus the series converges on $(-1, 1)$ and diverges for $|x| > 1$. At the two endpoints of the interval $(-1, 1)$ a different test is required. We see that for $x = 1$ this is the familiar harmonic series and so diverges, while for $x = -1$ this is the familiar alternating harmonic series and so converges nonabsolutely. The interval of convergence is $[-1, 1)$. Observe that the series converges at only one of the two endpoints of the interval. ◀

[1] The form of the root test needed to discuss power series uses the limit superior. For that the study of Section 2.13 may be required.

Example 16.5: The series

$$\sum_{k=1}^{\infty} \frac{x^k}{k^2}$$

converges on $[-1,1]$ and diverges otherwise. Again the root test (or the ratio test) is helpful here. Simpler, though, is to notice that

$$\left|\frac{x^k}{k^2}\right| \leq \frac{1}{k^2}$$

for all $|x| \leq 1$ and so obtain convergence on $[-1,1]$ by a comparison test with the convergent series $\sum_{k=0}^{\infty} 1/k^2$. If $|x|>1$ the terms $|x^k/k^2| \to \infty$ and so, trivially, the series diverges. Note here that the series converges on the interval $[-1,1]$ and is absolutely convergent there. ◀

Example 16.6: The root test applied to the series

$$\sum_{k=1}^{\infty} \frac{x^k}{k^k}$$

gives

$$\lim_{k\to\infty} \sqrt[k]{\frac{|x|^k}{k^k}} = |x| \lim_{k\to\infty} \frac{1}{k} = 0.$$

(The ratio test can also be used here.) It follows that the series converges for all $x \in \mathbb{R}$. Perhaps an easier method in this particular example is to use the comparison test and the fact that

$$\left|\frac{x}{k}\right|^k < \frac{1}{2^k} \text{ when } k \geq 2|x|.$$

Thus the series converges at any x by comparison with a geometric series. Thus the set of convergence of this series is $(-\infty,\infty)$, again as in the previous examples an interval. ◀

In general, as these examples seem to suggest, the set of points of convergence of a power series forms an interval and an application of the root test is an essential tool in determining that interval. Let us apply this test to the series

$$\sum_{0}^{\infty} a_k(x-c)^k.$$

Let

$$s = \limsup_{k\to\infty} \sqrt[k]{|a_k|}.$$

Then

$$\limsup_{k\to\infty} \sqrt[k]{|a_k||x-c|^k} = \limsup_{k\to\infty} \sqrt[k]{|a_k|}|x-c| = s|x-c|.$$

By the root test the series converges absolutely if $s|x-c| < 1$ and diverges if $s|x-c| > 1$.

If $0 < s < \infty$, then the series converges on the interval

$$(c-1/s, c+1/s)$$

and diverges for x outside the interval

$$[c-1/s, c+1/s].$$

The root test is inconclusive about convergence at the endpoints

$$x = c \pm 1/s$$

of these intervals. The interval of convergence is thus one of the four possibilities

$$(c-1/s, c+1/s) \text{ or } [c-1/s, c+1/s) \text{ or}$$

$$(c-1/s, c+1/s] \text{ or } [c-1/s, c+1/s].$$

If $s = 0$, then the series converges for all values of x. We could say that the interval of convergence is $(-\infty, \infty)$ in this case. If $s = \infty$, then the series converges for no values of x other than the trivial value $x = c$. We could say that the interval of convergence is the degenerate "interval" $\{c\}$.

Thus the set of convergence is an interval centered at c. This interval might be degenerate (consisting of only the center), might be all of the real line, and might contain none, one, or both of its endpoints.

Our next theorem summarizes the discussion of convergence to this point. We first give a formal definition.

Definition 16.7: Let $\sum_0^\infty a_k(x-c)^k$ be a power series. Then the number

$$R = \frac{1}{\limsup_{k\to\infty} \sqrt[k]{|a_k|}}$$

is called the *radius of convergence* of the series. Here we interpret $R = \infty$ if

$$\limsup_{k\to\infty} \sqrt[k]{|a_k|} = 0$$

and $R = 0$ if

$$\limsup_{k\to\infty} \sqrt[k]{|a_k|} = \infty.$$

Note. This book deals with *real* analysis, but a full theory of power series fits more naturally into the setting of *complex* analysis. In that setting, a power series converges in a "circle of convergence" centered at a complex number c in the complex plane and with radius

$$R = \frac{1}{\limsup_k \sqrt[k]{|a_k|}}.$$

This explains the origin of the term "radius of convergence."

Theorem 16.8: *Let $\sum_0^\infty a_k(x-c)^k$ be a power series with radius of convergence R.*

1. *If $R = 0$, then the series converges only at $x = c$.*
2. *If $R = \infty$, then the series converges absolutely for all x.*
3. *If $0 < R < \infty$, then the series converges absolutely for all x satisfying $|x-c| < R$ and diverges for all x satisfying $|x-c| > R$.*

Proof. We first consider the case $R = 0$. Here $\limsup_k \sqrt[k]{|a_k|} = \infty$ so, for $x \neq c$,

$$\limsup_k \sqrt[k]{|a_k||x-c|^k} = |x-c| \limsup_k \sqrt[k]{|a_k|} = \infty.$$

By the root test the series cannot converge for $x \neq c$. The other cases are similarly established by the root test as in the discussion following our examples. ■

In general, a power series

$$\sum_{k=0}^{\infty} a_k x^k$$

with a finite radius of convergence R must have as its set of convergence one of the four intervals

$$(-R, R), \quad [-R, R], \quad (-R, R] \text{ or } \quad [-R, R).$$

As we have seen from the examples, each of these four cases can occur. The other possibilities are for series with radius of convergence $R = 0$, in which case the set of convergence is trivially $\{0\}$, or with radius of convergence $R = \infty$, in which case the set of convergence is the entire real line. Note too that if the series converges absolutely at $x = R$ or at $x = -R$, then it must converge absolutely on all of $[-R, R]$. It is possible, though, for the series to converge nonabsolutely at one endpoint but not at the other.

Exercises

16.2.1 Find the radius of convergence for each of the following series.

(a) $\displaystyle\sum_{k=0}^{\infty} (-1)^k x^{2k}$

(b) $\displaystyle\sum_{k=0}^{\infty} \frac{x^k}{k!}$

(c) $\displaystyle\sum_{k=0}^{\infty} k x^k$

(d) $\displaystyle\sum_{k=0}^{\infty} k! x^k$

16.2.2 If the limit

$$\lim_{k\to\infty}\left|\frac{a_k}{a_{k+1}}\right|$$

exists or equals ∞, then show that the following expression also gives the radius of convergence of a power series:

$$R=\lim_{k\to\infty}\left|\frac{a_k}{a_{k+1}}\right|.$$

SEE NOTE 253

16.2.3 For the examples

$$\sum_{k=0}^{\infty}x^k,\quad \sum_{k=1}^{\infty}\frac{x^k}{k},\qquad \text{and}\ \sum_{k=1}^{\infty}\frac{x^k}{k^2}$$

verify in each case that

$$R=\lim_{k}\left|\frac{a_k}{a_{k+1}}\right|=1.$$

16.2.4 For the series

$$\sum_{k=1}^{\infty}k^kx^k\quad \text{and}\quad \sum_{k=1}^{\infty}\frac{x^k}{k^k}$$

check that the radius of convergence is $R=0$ and ∞, respectively.

16.2.5 Give an example of a power series $\sum_0^\infty a_kx^k$ for which the radius of convergence R satisfies

$$R=\frac{1}{\lim_{k\to\infty}\sqrt[k]{|a_k|}}$$

but

$$\lim_{k\to\infty}\left|\frac{a_k}{a_{k+1}}\right|$$

does not exist.

16.2.6 Give an example of a power series $\sum_0^\infty a_k x^k$ for which the radius of convergence R satisfies

$$\liminf_k \left|\frac{a_{k+1}}{a_k}\right| < R < \limsup_k \left|\frac{a_{k+1}}{a_k}\right|.$$

16.2.7 Give an example of a power series $\sum_0^\infty a_k x^k$ with radius of convergence 1 that is nonabsolutely convergent at both endpoints 1 and -1 of the interval of convergence.

16.2.8 Give an example of a power series $\sum_0^\infty a_k x^k$ with interval of convergence exactly $[-\sqrt{2}, \sqrt{2})$.

16.2.9 If the power series $\sum_0^\infty a_k x^k$ has a radius of convergence R, what must be the radius of convergence of the series

$$\sum_{k=0}^\infty k a_k x^k \quad \text{and} \quad \sum_{k=1}^\infty k^{-1} a_k x^k?$$

16.2.10 If the coefficients $\{a_k\}$ of a power series $\sum_0^\infty a_k x^k$ form a bounded sequence show that the radius of convergence is at least 1.

16.2.11 If the power series $\sum_0^\infty a_k x^k$ has a radius of convergence R_a and the power series $\sum_0^\infty b_k x^k$ has a radius of convergence R_b and $|a_k| \leq |b_k|$ for all k sufficiently large, what relation must hold between R_a and R_b?

16.2.12 If the power series $\sum_0^\infty a_k x^k$ has a radius of convergence R, what must be the radius of convergence of the series $\sum_{k=0}^\infty a_k x^{2k}$?

16.2.13 If the power series $\sum_0^\infty a_k x^k$ has a finite positive radius of convergence show that the radius of convergence of the series $\sum_{k=0}^\infty a_k x^{k^2}$ is 1.

16.2.14 Find the radius of convergence of the series

$$\sum_{k=0}^\infty \frac{(\alpha k)!}{(k!)^\beta} x^k,$$

where α and β are positive and α is an integer.

16.2.15 Let $\{a_k\}$ be a sequence of real numbers and let $x_0 \in \mathbb{R}$. Suppose there exists $M > 0$ such that $|a_k x_0^k| \leq M$ for all $k \in \mathbb{N}$. Prove that $\sum_0^\infty a_k x^k$ converges absolutely for all x satisfying the inequality $|x| < |x_0|$. What can you say about the radius of convergence of this series?

16.3 Uniform Convergence

So far we have reached a complete understanding of the nature of the set of convergence of any power series. In order to apply many of our theorems of Chapter 14 to questions concerning term by term integration or differentiation of power series, we need to check questions related to the *uniform* convergence of power series. Our next theorem does this and also summarizes the discussion of convergence to this point.

We repeat the convergence results of Theorem 16.8 but now add a discussion of uniform convergence.

Theorem 16.9: *Let $\sum_0^\infty a_k(x-c)^k$ be a power series with radius of convergence R.*

1. *If $R=0$, then the series converges only at $x=c$.*
2. *If $R=\infty$, then the series converges absolutely and uniformly on any compact interval $[a,b]$.*
3. *If $0<R<\infty$, then the series converges absolutely and uniformly on any interval $[a,b]$ contained entirely inside the interval $(c-R,c+R)$.*

Proof. To verify (2) and (3), let us choose $0<\rho<R$ so that the interval $[a,b]$ is contained inside the interval $(c-\rho,c+\rho)$. Fix $\rho_0\in(\rho,R)$. Then

$$\limsup_k \sqrt[k]{|a_k|} = \frac{1}{R} < \frac{1}{\rho_0}.$$

Thus there exists $N\in\mathbb{N}$ such that

$$\sqrt[k]{|a_k|} < \frac{1}{\rho_0} \quad \text{for all } k\geq N. \tag{1}$$

For $k\geq N$ and $|x-c|\leq\rho$ we calculate

$$|a_k(x-c)^k| \leq |a_k|\rho^k < \left(\frac{\rho}{\rho_0}\right)^k,$$

the last inequality following from (1).

Now since $\rho/\rho_0 < 1$, it follows that
$$\sum_{k=0}^{\infty} \left(\frac{\rho}{\rho_0}\right)^k < \infty.$$
It now follows from the Weierstrass M-test (Theorem 14.16) that the series converges absolutely and uniformly on the set $\{x : |x - c| < \rho\}$ and hence also on the subset $[a, b]$. ■

If the interval of convergence of a power series is $(-R, R)$, then certainly the assertion (3) of Theorem 16.9 is the best that can be made. (See Exercise 16.3.3.) The geometric series $\sum_{n=0}^{\infty} x^n$ furnishes the clearest example of this. This series converges on $(-1, 1)$ but does not converge uniformly on the entire interval of convergence $(-1, 1)$. It does, however, converge uniformly on any $[a, b] \subset (-1, 1)$.

To improve on this we can ask the following: If R is the radius of convergence of a power series and the interval of convergence is $[-R, R]$ or $(-R, R]$ or $[-R, R)$, can uniform convergence be extended to the endpoints? If the convergence at an endpoint R (or $-R$) is absolute, then an application of the Weierstrass M-test shows immediately that the convergence is absolute and uniform on $[-R, R]$. For nonabsolute convergence a more delicate test is needed and we need to appeal to material developed in Section 14.3.3. The following theorem contains, for easy reference, a repeat of the third assertion in Theorem 16.9.

Theorem 16.10: *Suppose that the power series $\sum_0^\infty a_k(x-c)^k$ has a finite and positive radius of convergence R and an interval of convergence I.*

1. *If $I=[c-R,c+R]$, then the series converges uniformly (but not necessarily absolutely) on $[c-R,c+R]$.*
2. *If $I=(c-R,c+R]$, then the series converges uniformly (but not necessarily absolutely) on any interval $[a,c+R]$ for all*
$$c-R<a<c+R.$$
3. *If $I=[c-R,c+R)$, then the series converges uniformly (but not necessarily absolutely) on any interval $[c-R,b]$ for all*
$$c-R<b<c+R.$$
4. *If $I=(c-R,c+R)$, then the series converges uniformly and absolutely on any interval $[a,b]$ for $c-R<a<b<c+R$.*

Proof. For the purposes of the proof we can take $c=0$. Let us examine the case
$$I=(c-R,c+R]=(-R,R]$$
which is typical. Consider the intervals $[a,0]$ for $-R<a<0$ and $[0,R]$. The uniform convergence of the series on $[a,0]$ is clear since this is contained entirely inside the interval of convergence.

Now we examine uniform convergence on $[0,R]$. We consider the series
$$\sum_{k=0}^\infty a_k x^k=\sum_{k=0}^\infty A_k t^k,$$
where $A_k=a_kR^k$ and $t=(x/R)$. The series $\sum_{k=0}^\infty A_k t^k$ converges for $0\le t\le 1$ by our assumptions. Note that $\sum_{k=0}^\infty A_k$ is convergent while the sequence $\{t^k\}$ converges monotonically on the interval $[0,1]$. By a variant of Theorem 14.19 (Exercise 14.3.26) this series converges uniformly for t in the interval $[0,1]$. This translates easily to the assertion that our original series converges uniformly for $x\in[0,R]$. Thus since the series converges uniformly on $[a,0]$ and on $[0,R]$ we have obtained the uniform convergence on $[a,R]$ as required. The other cases are similarly handled. ■

Exercises

16.3.1 Characterize those power series $\sum_0^\infty a_k(x-c)^k$ that converge uniformly on $(-\infty, \infty)$.

16.3.2 Show that if $\sum_{k=0}^\infty a_k x^k$ converges absolutely at a point $x_0 > 0$, then the convergence of the series is uniform on $[-x_0, x_0]$.

16.3.3 Show that if $\sum_{k=0}^\infty a_k x^k$ converges uniformly on an interval $(-r, r)$, then it must in fact converge uniformly on $[-r, r]$. Deduce that if the interval of convergence is exactly of the form $(-R, R)$, or $[-R, R)$ or $[-R, R)$, then the series cannot converge uniformly on the entire interval of convergence.

See Note 254

16.4 Functions Represented by Power Series

Suppose now that a power series $\sum_0^\infty a_k(x-c)^k$ has positive or infinite radius of convergence R. Then this series represents a function f on (at least) the interval $(c-R, c+R)$:

$$f(x) = \sum_0^\infty a_k(x-c)^k \quad \text{for } |x-c| < R. \tag{2}$$

If the series converges at one or both endpoints, then this represents a function on $[c-R, c+R)$ or $(c-R, c+R]$ or $[c-R, c+R]$.

What can we say about the function f? In terms of the questions that have motivated us throughout Chapter 14 we can ask

1. Is the function f continuous on its domain of definition?
2. Can f be differentiated by termwise differentiation of its series?
3. Can f be integrated by termwise integration of its series?

We address each of these questions and find that generally the answer to each is yes.

16.4.1 Continuity of Power Series

A power series may represent a function on an interval. Is that function necessarily continuous?

Theorem 16.11: *A function f represented by a power series*

$$f(x) = \sum_{0}^{\infty} a_k (x-c)^k \tag{3}$$

is continuous on its interval of convergence.

Proof. This follows from Theorem 16.10. For example, if the interval of convergence is $(c-R, c+R]$, then we can show that f is continuous at each point of this interval. Since convergence is uniform on $[c, c+R]$ and since each of the functions $a_k(x-c)^k$ is continuous on $[c, c+R]$, the same is true of the function f (Corollary 14.23). For any point $x_0 \in (c-R, c)$ we can similarly prove that f is continuous at x_0 in the same way by noting that the series converges uniformly on an interval $[a, c]$, where a is chosen so that $c - R < a < x_0 < c$. ■

Example 16.12: The series

$$f(x) = \sum_{k=1}^{\infty} \frac{x^k}{k}$$

converges at every point of the interval $[-1, 1)$. Consequently, this function is continuous at every point of that interval. We shall show in the next section that the identity

$$\log(1-x) = \sum_{k=1}^{\infty} \frac{x^k}{k}$$

holds for all $x \in (-1, 1)$ (by integrating the geometric series term by term). Since we are also assured of continuity at the endpoint $x = -1$ we can conclude that

$$\log 2 = \sum_{k=1}^{\infty} \frac{(-1)^k}{k}.$$

◀

16.4.2 Integration of Power Series

If a function is represented by a power series, is it possible to integrate that function by integrating the power series term by term?

Theorem 16.13: *Let a function f be represented by a power series*

$$f(x) = \sum_0^\infty a_k(x-c)^k$$

with an interval of convergence I. Then for every point x in that interval f is integrable on $[c,x]$ (or $[x,c]$ if $x<c$) and

$$\int_c^x f(t)\,dt = \sum_{k=0}^\infty \frac{a_k}{k+1}(x-c)^{k+1}.$$

Proof. Let x be a point in the interval of convergence. The convergence of the series $\sum_0^\infty a_k(x-c)^k$ is uniform on $[c,x]$ (or on $[x,c]$ if $x<c$), so the series can be integrated term by term (Theorem 14.29). ■

Example 16.14: The geometric series

$$\frac{1}{1-x} = \sum_{k=0}^\infty x^k$$

has radius of convergence 1 and so can be integrated term by term provided we stay inside the interval $(-1,1)$. Thus

$$-\log(1-x) = \int_0^x \frac{1}{1-t}\,dt = \sum_{k=0}^\infty \frac{1}{k+1}x^{k+1}$$

for all $-1 < x < 1$. We would not be able to conclude from this theorem that the integral can be extended to the endpoints of $(-1, 1)$. The new series, however, also converges at $x = -1$ and so we can apply Theorem 16.11 to show that the identity just proved is actually valid on $[-1, 1)$. ◀

16.4.3 Differentiation of Power Series

If a function is represented by a power series, is it possible to differentiate that function by differentiating the power series term by term?

Note that for continuity and integration we were able to prove Theorems 16.11 and 16.13 immediately from general theorems on uniform convergence. To prove a theorem on term by term differentiation, we need to check uniform convergence of the series of *derivatives*. The following lemma gives us what we need.

Lemma 16.15: *Let $\sum_0^\infty a_k(x-c)^k$ have radius of convergence R. Then the series*

$$\sum_{k=1}^{\infty} k a_k (x-c)^{k-1}$$

obtained via term by term differentiation also has the same radius of convergence R.

Proof. The radius of convergence of the series is given by

$$R = \frac{1}{\limsup_k \sqrt[k]{|a_k|}}.$$

The radius of convergence of the differentiated series is given by

$$R' = \frac{1}{\limsup_k \sqrt[k]{|k a_k|}}.$$

But since $\sqrt[k]{k} \to 1$ as $k \to \infty$ we see immediately that these two expressions are equal. (They may be both zero or both infinite.) ■

Theorem 16.16: *Let $\sum_0^\infty a_k(x-c)^k$ have radius of convergence $R>0$, and let*

$$f(x) = \sum_0^\infty a_k(x-c)^k$$

whenever $|x-c|<R$. Then f is differentiable on $(c-R, c+R)$ and

$$f'(x) = \sum_{k=1}^\infty k a_k(x-c)^{k-1}$$

for each $x \in (c-R, c+R)$.

Proof. It follows from the preceding lemma that the series

$$\sum_{k=1}^\infty k a_k(x-c)^{k-1}$$

has radius of convergence R. Thus this series converges uniformly on any compact interval $[a,b]$ contained in $(c-R, c+R)$. Since each value of x in $(c-R, c+R)$ can be placed inside some such interval $[a,b]$ it now follows immediately from Corollary 14.35 that $f'(x) = \sum_{k=1}^\infty k a_k(x-c)^{k-1}$ whenever $|x-c|<R$. ■

We can apply the same argument to the differentiated series and differentiate once more. From the expansion

$$f'(x) = \sum_{k=1}^\infty k a_k(x-c)^{k-1}$$

we obtain a formula for $f''(x)$:

$$f''(x) = \sum_2^\infty k(k-1)a_k(x-c)^{k-2}.$$

Let us express explicitly the formulas of $f(x)$, $f'(x)$, and $f''(x)$.

$$\begin{aligned} f(x) &= a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \dots \\ f'(x) &= a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \dots \\ f''(x) &= 2a_2 + 3\cdot 2a_3(x-c) + \dots \end{aligned}$$

These expressions are valid in the interval $(c-R, c+R)$. For $x = c$ we obtain

$$\begin{aligned} f(c) &= a_0 \\ f'(c) &= a_1 \\ f''(c) &= 2a_2. \end{aligned}$$

If we continue in this way, we can obtain power series expansions for all the derivatives of f. This results in the following theorem. The proof (which requires mathematical induction) is left as Exercise 16.4.1.

Theorem 16.17: *Let $\sum_0^\infty a_k(x-c)^k$ have radius of convergence $R > 0$. Then the function*

$$f(x) = \sum_0^\infty a_k(x-c)^k$$

has derivatives of all orders and these derivatives can be calculated by repeated term by term differentiation. The coefficients a_k are related to the derivatives of f at $x = c$ by the formula

$$a_k = \frac{f^{(k)}(c)}{k!}.$$

Uniqueness of Power Series From Theorem 16.17 we deduce that any two power series representations of a function must be identical. Note that the centers have to be the same for this to be true.

Corollary 16.18: *Suppose two power series*

$$f(x) = \sum_{0}^{\infty} a_k (x-c)^k$$

and

$$g(x) = \sum_{0}^{\infty} b_k (x-c)^k$$

agree on some interval centered at c, that is $f(x) = g(x)$ for $|x-c| < \rho$ and some positive ρ. Then $a_k = b_k$ for all $k = 0, 1, 2, \ldots$.

Proof. It follows immediately from Theorem 16.17 that

$$a_k = \frac{f^{(k)}(c)}{k!} = \frac{g^{(k)}(c)}{k!} = b_k$$

for all $k = 0, 1, 2, \ldots$. ■

Example 16.19: The series for the exponential function

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

reveals one of the key facts about the exponential function, namely that it is its own derivative. Note simply that

$$\frac{d}{dx} e^x = \frac{d}{dx} \sum_{k=0}^{\infty} \frac{x^k}{k!} = \sum_{k=0}^{\infty} \frac{d}{dx} \frac{x^k}{k!} = \sum_{k=1}^{\infty} \frac{x^{k-1}}{k-1!} = e^x.$$

◀

Example 16.20: The material in this section can also be used to obtain the power series expansion of the exponential function. Suppose that we know that the exponential function $f(x) = e^x$ does in fact have a power series expansion

$$f(x) = \sum_{k=0}^{\infty} a_k x^k.$$

Then the coefficients must be given by the formulas we have obtained, namely

$$a_k = \frac{f^{(k)}(0)}{k!}.$$

But for $f(x) = e^x$ it is clear that $f^{(k)}(0) = 1$ for all k and so the series must be indeed be given by $a_k = 1/k!$ as we well know. But how can we be assured that the exponential function does have a power series expansion? This argument shows only that if there is a series, then that series is precisely $\sum_{k=0}^{\infty} \frac{x^k}{k!}$. There remains the possibility that there may not be a series after all. This is the subject of the next section. ◀

16.4.4 Power Series Representations

Corollary 16.18 shows that if we can obtain a power series representation for a function f by any means whatsoever, then that series must have its coefficients given by the equations $a_k = f^{(k)}(c)/k!$. In particular, a power series representation for f about a given point must be unique.

Example 16.21: For example, the formula for the sum of a geometric series can be used to show that

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - \cdots + (-1)^j x^{2j} + \dots.$$

Thus this series represents the function $f(x) = \frac{1}{1+x^2}$ on the interval $(-1, 1)$. Note that the coefficients a_k are zero if k is odd and that $a_{2j} = (-1)^j$ for $k = 2j$ even. It now follows *automatically* that for even integers $k = 2j$

$$\frac{f^{(k)}(0)}{k!} = a_k = (-1)^j$$

while all the odd derivatives are zero. Thus

$$\frac{d^k}{dx^k}\left(\frac{1}{1+x^2}\right) = 0 \quad \text{at } x = 0$$

if k is odd and, if $k = 2j$ is even,
$$\frac{d^k}{dx^k}\left(\frac{1}{1+x^2}\right) = (-1)^j (2j)! \quad \text{at } x = 0.$$
◄

Note. There is a curious fact here that should be puzzled upon. The formula
$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - \cdots + (-1)^j x^{2j} + \ldots$$
is valid precisely for $-1 < x < 1$. But the function on the left-hand side of this identity is defined for all values of x. We might have hoped for a representation valid for all x, but we do not obtain one!

Sometimes the easiest way to obtain a power series expansion formula for a function is by using the formula
$$a_k = \frac{f^{(k)}(c)}{k!}.$$
For example, this is how we obtained the power series for $f(x) = e^x$. We compute $f^{(k)}(x) = e^x$ for $k = 0, 1, 2, \ldots$, so $f^{(k)}(0) = 1$ for all k. Thus the series expansion for this function (if it has a series expansion) would have to be
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_0^\infty \frac{x^k}{k!}. \tag{4}$$
Note that the series converges for all $x \in \mathbb{R}$. In the next section we will show how to verify that the equality holds for all x.

If we had wanted a formula for $g(x) = e^{x^2}$ we might have used the same idea and determined all the derivatives $g^{(k)}(0)$. It would be simplest, however, to just substitute x^2 for x in the expansion (4), obtaining
$$e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots = \sum_0^\infty \frac{x^{2k}}{k!}. \tag{5}$$

Also, from this expansion we can readily obtain an expansion for $2xe^{x^2}$ in either of two ways: We can multiply the expansion in (5) by $2x$ giving

$$2xe^{x^2} = 2x + 2x^3 + \frac{2x^5}{2!} + \frac{2x^7}{3!} + \cdots = \sum_0^\infty \frac{2x^{2k+1}}{k!}.$$

Alternatively, we can use Theorem 16.16 and differentiate (5) term by term, giving

$$2xe^{x^2} = \frac{d}{dx}e^{x^2} = 2x + \frac{4x^3}{2!} + \frac{6x^5}{3!} + \frac{8x^7}{4!} \cdots = \sum_0^\infty \frac{2x^{2k+1}}{k!}.$$

You may wish instead to try to obtain these expansions directly by using the formula $a_k = f^{(k)}(c)/k!$.

Exercises

16.4.1 Provide the details in the proof of Theorem 16.17.

16.4.2 Obtain expansions for

$$\frac{x}{1+x^2} \quad \text{and} \quad \frac{x}{(1+x^2)^2}.$$

16.4.3 Obtain expansions for

$$\frac{1}{1+x^3} \quad \text{and} \quad \frac{x^2}{1+x^3}.$$

16.4.4 Find a power series expansion about $x = 0$ for the function

$$f(x) = \int_0^1 \frac{1 - e^{-sx}}{s}\, ds.$$

SEE NOTE 255

16.4.5 The function

$$J_0(x) = \sum_{k=0}^\infty (-1)^k \frac{x^{2k}}{(k!)^2 2^{2k}}$$

is called a Bessel function of order zero of the first kind. Show that this is defined for all values of x. The function $J_1(x) = -J_0'(x)$ is called a Bessel function of order one of the first kind. Find a series expansion for $J_1(x)$.

16.4.6 Let
$$f(x) = \sum_{k=0}^{\infty} a_k x^k$$
have a positive radius of convergence. If the function f is *even* [i.e., if it satisfies $f(-x) = f(x)$ for all x], what can you deduce about the coefficients a_k? What can you deduce if the function is *odd* (i.e., if $f(-x) = -f(x)$ for all x)?

16.4.7 Let
$$f(x) = \sum_{k=0}^{\infty} a_k x^k$$
have a positive radius of convergence. If zero is a critical point (i.e., if $a_1 = 0$) and if $a_2 > 0$, then the point $x = 0$ is a strict local minimum. Prove this and also formulate and prove a generalization of this that would allow
$$a_2 = a_3 = a_4 = \cdots = a_{N-1} = 0 \text{ and } a_N \neq 0.$$

16.5 The Taylor Series

We have seen that if a power series $\sum_0^\infty a_k(x-c)^k$ converges on an interval I, then the series represents a function f that has derivatives of all orders. In particular, the coefficients a_k are related to the derivatives of f at c:
$$a_k = \frac{f^{(k)}(c)}{k!}.$$
We then call the series the *Taylor series* for f about the point $x = c$.

Let us turn the question around:

What functions f have a Taylor series representation in their domain?

We see immediately that such a function must be infinitely differentiable in a neighborhood of c since for such a series to be valid we know that all of the derivatives $f^{(k)}(c)$ must exist. But is that enough?

If we start with a function that has derivatives of all orders on an interval I containing the point c and write the series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k,$$

we might expect that this is exactly the representation we want. Indeed *if there is a valid representation*, then this must be the one, since such representations are unique. But can we be sure the series converges to f on I? Or even that the series converges at all on I. The answer to both questions is "no."

Example 16.22: Consider, for example, the function

$$f(x) = 1/(1+x^2).$$

This function is infinitely differentiable on all of the real line. Its Taylor series about $x=0$ is, as we have seen in Example 16.21,

$$1 - x^2 + x^4 - x^6 + \cdots = \sum_{k=0}^{\infty}(-1)^k x^{2k}.$$

This series converges for $|x| < 1$ but diverges for $|x| \geq 1$. It does represent f on the interval $(-1,1)$ but not on the full domain of f. Indeed there can be no series of the form $\sum_{k=0}^{\infty} a_k x^k$ that represents f on $(-\infty,\infty)$ since that series would agree with this present series on $(-1,1)$ and so could not be any different. ◀

Worse situations are possible. For example, there are infinitely differentiable functions whose Taylor series have zero radius of convergence for every c; for these functions

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

diverges except at $x = c$ and this is true for all $c \in \mathbb{R}$.[2] For these functions the Taylor series cannot represent the function.

Another unpleasant situation occurs when a Taylor series converges to the wrong function. This possibility seems even more startling!

Example 16.23: Consider the function

$$f(x) = \begin{cases} 0, & \text{if } x = 0 \\ e^{-1/x^2}, & \text{if } x \neq 0. \end{cases}$$

Exercise 16.5.4 provides an outline for showing that f is infinitely differentiable on the real line, and that $f^{(k)}(0) = 0$ for $k = 1, 2, 3, \dots$. Thus the Taylor series for f about $x = 0$ takes the form $\sum_{k=0}^{\infty} 0x^k$ with all coefficients equal to zero. This series converges to the zero function on the real line, so it does not represent f except at the origin, even though the series converges for all x. ◀

16.5.1 Representing a Function by a Taylor Series

The preceding discussion shows that we should not automatically assume that a Taylor series for a function f represents f. It is true, however, that the developments in the earlier sections of this chapter help us see that many of the familiar Taylor series encountered in elementary calculus are valid.

Example 16.24: For example, starting with the geometric series

$$\frac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k x^k,$$

[2]See D. Morgenstern, *Math. Nach.*, **12** (1954), p. 74. We find here that in a certain sense "most" infinitely differentiable functions have this property!

we can apply Theorem 16.13 on integrating a power series term by term to obtain, for $|x| < 1$,

$$\ln(1+x) = \int_1^x \frac{1}{1+t}\,dt = \sum_{k=0}^{\infty}\int_0^x (-1)^k t^k\,dt$$
$$= \sum_{k=0}^{\infty}\frac{(-1)^k}{k+1}x^{k+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots .$$

We can notice that the integrated series converges at $x = 1$ and so the convergence is uniform on $[0, 1]$. It follows that the representation is valid for $x \in (-1, 1]$ but for no other points. In this case we obtained a valid Taylor series expansion by integrating a series expansion that we already knew to be valid. ◀

To study the convergence of a Taylor series in general, let us return to fundamentals. Let f be infinitely differentiable in a neighborhood of c. For $n = 0, 1, 2, \dots$ let

$$P_n(x) = \sum_{k=0}^{n}\frac{f^{(k)}(c)}{k!}(x-c)^k.$$

The polynomial P_n is called the *nth Taylor polynomial* of f at c. The difference $R_n(x) = f(x) - P_n(x)$ is called the *nth remainder function*. In order for the Taylor series about c to converge to f on an interval I, it is necessary and sufficient that $R_n \to 0$ pointwise on I.

Example 16.25: We know that the geometric series represents the function $f(x) = (1-x)^{-1}$ on the interval $(-1, 1)$. We could also prove this result by relying on the remainder term. For $x \neq 1$ and $n = 0, 1, 2, \dots$ we have

$$\frac{1}{1-x} = 1 + x + x^2 + \dots + x^n + \frac{x^{n+1}}{1-x}.$$

Here

$$P_n(x) = 1 + x + x + x^2 + \dots + x^n$$

and

$$R_n(x) = \frac{x^{n+1}}{1-x}.$$

For $|x| < 1$, $R_n(x) \to 0$ as $n \to \infty$. But we have

$$f(x) = P_n(x) + R_n(x)$$

and so the Taylor series for $f(x) = 1/(1-x)$ represents f on the interval $(-1, 1)$. For $|x| \geq 1$, the remainder term does not tend to zero. As before, we see that the representation is confined to the interval $(-1, 1)$. ◀

In a more general situation than this example we would not have an explicit formula for the remainder term. How should we be able to show that the remainder term tends to zero? For functions that are infinitely differentiable in a neighborhood I of c, the various expressions we obtained in Section 7.12 for the remainder functions R_n can be used. In particular, Lagrange's form of the remainder allows us to write for $n = 0, 1, 2, 3, \ldots$

$$f(x) = P_n(x) + \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1},$$

where z is between x and c. With some information on the size of the derivatives $f^{(n+1)}(z)$ we might be able to show that this remainder term tends to zero. The integral form of the remainder term, gives us

$$f(x) = P_n(x) + \frac{1}{n!}\int_c^x (x-t)^n f^{(n+1)}(t)\,dt.$$

Again information on the size of the derivatives $f^{(n+1)}(t)$ might show that this remainder term tends to zero.

Example 16.26: Let us justify the familiar Taylor series for $\sin x$:

$$\sin x = \sum_0^\infty \frac{(-1)^k}{(2k+1)!}x^{2k+1}. \tag{6}$$

The remainder term is not expressible in any simple way but can be estimated by using the Lagrange's form of the remainder. The coefficients

$$\frac{(-1)^k}{(2k+1)!}$$

are easily verified by calculating successive derivatives of $f(x) = \sin x$ and using the formulas

$$a_k = \frac{f^{(k)}(0)}{k!}.$$

To check convergence of the series, apply Lagrange's form for $R_n(x)$: For each $x \in \mathbb{R}$, there exists z such that

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}x^{n+1}.$$

Now $|f^{(n+1)}(z)|$ equals either $|\cos z|$ or $|\sin z|$ (depending on n) so, in either case, $|f^{(n+1)}(z)| \leq 1$, and

$$|R_n(x)| \leq |x|^{n+1}/(n+1)!.$$

Since $|x|^{n+1}/(n+1)! \to 0$ as $n \to \infty$ for all $x \in \mathbb{R}$, we can see that the remainder term $|R_n(x)| \to 0$ as $n \to \infty$ for all $x \in \mathbb{R}$. Thus the series representation is completely justified for all real x.

Observe that our estimate for $|R_n(x)|$,

$$|R_n(x)| \leq |x|^{n+1}/(n+1)!,$$

gives also a sense of the *rate* of convergence of the series for fixed x. For example, for $|x| \leq 1$ we find

$$|R_n(x)| \leq 1/(n+1)!.$$

Thus, if we want to calculate $\sin x$ on $(-1, 1)$ to within .01, we need take only the first five terms of the series ($n = 4$) to achieve that degree of accuracy.

Had we used the integral form for $R_n(x)$ we would have obtained a similar estimate. We leave that calculation as Exercise 16.5.1. ◄

16.5.2 Analytic Functions

The class of functions that can be represented as power series is not large. As we have remarked, the class of infinitely differentiable functions is much larger. The terminology that is commonly used for this very special class of functions is given by the following definition.

Definition 16.27: A function f whose Taylor series converges to f in a neighborhood of c is said to be *analytic at c*.

The functions commonly encountered in elementary calculus are generally analytic except at certain obviously nonanalytic points. For example, $|x|$ is not analytic at $x = 0$, and $1/(1-x)$ is not analytic at $x = 1$. These functions fail to have even a first derivative at the point in question. Similarly a function such as $f(x) = |x|^3$ cannot be analytic at $x = 0$ because, while $f'(0)$ and $f''(0)$ exist, $f^{(3)}(0)$ does not. It is not possible to write the complete Taylor series for such a function since some of the derivatives fail to exist.

Even if a function has infinitely many derivatives at a point it need not be analytic there. We would be able to write the complete Taylor expansion but, as we have already noted, the resulting series might not converge to f on any interval. In this connection, it is instructive to work Exercise 16.5.4.

In Example 16.26 we justified the Taylor expansion for $\sin x$. Part of the justification involved the fact that $\sin x$ and all of its derivatives are bounded on the real line. This suggests a general result.

Theorem 16.28: *Let f be infinitely differentiable in a neighborhood I of c. Suppose $x \in I$ and there exists $M > 0$ such that*

$$|f^{(m)}(t)| \leq M$$

for all $m \in \mathbb{N}$ and $t \in [c, x]$ (or $[x, c]$ if $x < c$). Then

$$\lim_{n\to\infty} R_n(x) = 0.$$

Thus, f is analytic at c.

Proof. We prove the theorem for $x > c$. We leave the case $x < c$ as Exercise 16.5.5.

We use the integral form of the remainder (Theorem 7.45), obtaining

$$|R_n(x)| = \left| \frac{1}{n!} \int_c^x (x-t)^n f^{(n+1)}(t)\, dt \right|. \tag{7}$$

Using our hypothesis that $|f^{(m)}(t)| \leq M$ for all $t \in [c, x]$, we infer from (7) that

$$\begin{aligned} |R_n(x)| &\leq \left| \frac{M}{n!} \int_c^x (x-t)^n \, dt \right| \\ &= \left| \frac{M}{n!} \frac{(x-t)^{n+1}}{n+1} \Big|_c^x \right| \\ &= \left| \frac{M}{(n+1)!} (x-c)^{n+1} \right|. \end{aligned}$$

For fixed x and c, $(x-c)$ is just a constant, so

$$\frac{M(x-c)^{n+1}}{(n+1)!} \to 0.$$

Thus $|R_n(x)| \to 0$ and f is analytic at c. ■

Example 16.29: Let us show that the function $f(x) = e^x$ is analytic at $x = 0$. It is infinitely differentiable, but we need to prove more. The fact that f is analytic at $x = 0$ follows from the previous theorem: We choose, say, the interval $[-1, 1]$ and note that $|f^{(n)}(x)| = |e^x| \leq e$ for all $x \in (-1, 1)$ and $n \in \mathbb{N}$. A similar observation applies to the analyticity of f at any point $c \in \mathbb{R}$. ◀

Exercise 16.5.6 provides another theorem similar to Theorem 16.28.

Exercises

16.5.1 Justify formula (6) for $\sin x$ using the integral form of the remainder $R_n(x)$.

16.5.2 Show that

$$f(x) = \sum_{k=0}^{n} \frac{f^k(0)}{k!} x^k + \frac{x^{n+1}}{n!} \int_0^1 f^{(n+1)}(sx)(1-s)^n \, ds$$

under appropriate assumptions on f.

16.5.3 Show that

$$\int_0^1 f^{(n+1)}(sb)(1-s)^n\,ds \le \frac{n!f(b)}{b^{n+1}}$$

if f and all of its derivatives exist and are nonnegative on the interval $[0,b]$.

16.5.4 Let

$$f(x) = \begin{cases} 0, & \text{if } x = 0 \\ e^{-1/x^2}, & \text{if } x \neq 0. \end{cases}$$

Prove that f is infinitely differentiable on the real line. Show that $f^{(k)}(0) = 0$ for all $k \in \mathbb{N}$. Explain why the Taylor series for f about $x = 0$ does not represent f in any neighborhood of zero. Is f analytic at $x = c$ for $c \neq 0$?

SEE NOTE 256

16.5.5 Prove Theorem 16.28 for $x < c$.

16.5.6 Prove Bernstein's Theorem: If f is infinitely differentiable on an interval I, and $f^{(n)}(x) \geq 0$ for all $n \in \mathbb{N}$ and $x \in I$, then f is analytic on I. Apply this result to $f(x) = e^x$.

16.5.7 Use the results of this section to verify that each of the following functions is analytic at $x = 0$, and write the Taylor series about $x = 0$.

(a) $\cos x^2$

(b) e^{-x^2}

16.5.8 Show that if f and g are analytic functions at each point of an interval (a,b), then so too is any linear combination $\alpha f + \beta g$.

16.6 Products of Power Series

Suppose that we have two power series representations

$$f(x) = \sum_{k=0}^{\infty} a_k (x-x_0)^k$$

and

$$g(x) = \sum_{k=0}^{\infty} b_k (x - x_0)^k$$

valid in the intervals $(-R_f, R_f)$ and $(-R_g, R_g)$, respectively. How should we obtain a power series representation for the product $f(x)g(x)$? We might merely compute all the derivatives of this function and so construct its Taylor series. But is this the easiest or most convenient method? How do we know that such a representation would be valid?

The most direct approach to this problem is to apply here our study of products of series from Section 3.8. We know when such a product would be valid. Indeed, from that theory, we know immediately that

$$f(x)g(x) = \sum_{k=0}^{\infty} c_k (x - x_0)^k$$

would hold in the interval $(-R, R)$, where $R = \min\{R_f, R_g\}$ and the coefficients are given by the formulas

$$c_k = \sum_{j=0}^{k} a_j b_{k-j}.$$

Example 16.30: The product of the series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

and the series

$$f(x) = \sum_{k=0}^{\infty} a_k x^k$$

gives the representation

$$\frac{f(x)}{1-x} = \sum_{k=0}^{\infty} (a_0 + a_1 + a_2 + \dots + a_k) x^k.$$

Where would this be valid? ◀

Example 16.31: A representation for the function $e^x \sin x$ might be most easily obtained by forming the product

$$\left(1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\dots\right)\left(x-\frac{1}{3}x^3+\frac{1}{5}x^5-\dots\right)=x+x^2+\frac{1}{6}x^3+\dots$$

and the series continued as far as is needed for the application at hand. ◀

16.6.1 Quotients of Power Series

Suppose that we have power series representations of two functions

$$f(x)=\sum_{k=0}^{\infty} a_k x^k \quad \text{and} \quad g(x)=\sum_{k=0}^{\infty} b_k x^k$$

both valid in some interval $(-r,r)$ at least. Can we find a representation of the quotient function $f(x)/g(x)$? Certainly we must demand that $g(0)\neq 0$, which amounts to asking for the leading coefficient in the series for g (the term b_0) not to be zero.

If there is a representation, say a series $\sum_{k=0}^{\infty} c_k x^k$, then, evidently, we require that

$$\frac{\sum_{k=0}^{\infty} a_k x^k}{\sum_{k=0}^{\infty} b_k x^k}=\sum_{k=0}^{\infty} c_k x^k.$$

This merely means that we want

$$\left(\sum_{k=0}^{\infty} b_k x^k\right)\left(\sum_{k=0}^{\infty} c_k x^k\right)=\sum_{k=0}^{\infty} a_k x^k.$$

The conditions for this are known to us since we have already studied how to form the product of two power series. For this to hold the coefficients $\{c_k\}$ (which, at the moment, we do not know how to determine) should satisfy the equations

$$b_0c_0=a_0$$

$$b_0c_1 + b_1c_0 = a_1$$

$$b_0c_2 + b_1c_1 + b_2c_0 = a_2$$

and, in general,

$$b_0c_k + b_1c_{k-1} + b_2c_{k-2} + \cdots + b_kc_0 = a_k.$$

Since we know all the a_k's and b_k's, we can readily solve these equations, one at a time starting from the first to obtain the coefficients for the quotient series. This algorithm (for that is what it is) for determining the c_k's is precisely "long division." Simply divide formally the expression (the denominator)

$$b_0 + b_1x + b_2x^2 + b_3x^3 + \dots$$

into the expression (the numerator)

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$$

and you will find yourself solving exactly these equations in our algorithm.

But what have we determined? We have shown that if there is a series representation for $f(x)/g(x)$, then this method will determine it. We do not, however, have any assurances in advance that there is such a series. We offer the next theorem, without proof, for those assurances. Alternatively, in any computation we could construct the quotient series (all terms!) and determine that it has a positive radius of convergence. That, too, would justify the method although it is not likely the most practical approach.

Theorem 16.32: *Suppose that there are power series representations for two functions*

$$f(x) = \sum_{k=0}^{\infty} a_kx^k \quad \textit{and} \quad g(x) = \sum_{k=0}^{\infty} b_kx^k$$

both valid in some interval $(-r, r)$ at least and that $b_0 \neq 0$. Then there is some positive δ so that the function $f(x)/g(x)$ is analytic in $(-\delta, \delta)$ and a quotient series can be found there.

The proper setting for a proof of Theorem 16.32 is complex analysis, where it is proved that a quotient of complex analytic functions is analytic if the denominator is not zero.

Exercises

16.6.1 Show that if f and g are analytic functions at each point of an interval (a,b), then so too is the product fg.

16.6.2 Under what conditions on the functions f and g on an interval (a,b) can you conclude that the quotient f/g is analytic?

SEE NOTE 257

16.6.3 Using long division, find several terms of the power series expansion of

$$\frac{x+2}{x^2+x+1}$$

centered at $x=0$. What other method would work?

16.6.4 Using long division and the power series expansions for $\sin x$ and $\cos x$, find the first few terms of the power series expansion of $\tan x$ centered at $x=0$. What other method would have given you these same numbers?

16.6.5 Find a power series expansion centered at $x=0$ for the function

$$\frac{\sin 2x}{\sin x}.$$

Did the fact that $\sin x = 0$ at $x=0$ make you modify the method here?

16.6.6 Show that if

$$\frac{1}{\sum_{k=0}^{\infty} b_k x^k} = \sum_{k=0}^{\infty} c_k x^k$$

is valid, then

$$c_k = \frac{(-1)^k}{b_0^{k+1}} \begin{vmatrix} b_1 & b_0 & 0 & 0 & \dots & 0 \\ b_2 & b_1 & b_0 & 0 & \dots & 0 \\ b_3 & b_2 & b_1 & b_0 & \dots & 0 \\ \dots & & & & \dots & \dots \\ b_k & b_{k-1} & b_{k-2} & b_{k-3} & \dots & b_1 \end{vmatrix}.$$

16.7 Composition of Power Series

Suppose that we wished to obtain a power series expansion for the function $e^{\sin x}$ using the two series expansions

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots$$

and

$$\sin x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \dots.$$

Without pausing to decide if this makes any sense let us simply insert the series for $\sin x$ in the appropriate positions in the series for e^x. Then we might hope to justify that

$$e^{\sin x} = 1 + \left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \dots\right)$$
$$+\frac{1}{2}\left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \dots\right)^2 + \frac{1}{6}\left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \dots\right)^3 + \dots$$

and expand, grouping terms in the obvious way, getting (at least for a start)

$$e^{\sin x} = 1 + x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \dots.$$

Is this method valid?

To justify this method we state (without proof) a theorem giving some conditions when this could be verified. Note that the conditions are as we should expect for a composition of functions $f(g(x))$. The series for $g(x)$ is expanded about a point x_0. That is inserted into a series expanded about the value $g(x_0)$, thus obtaining a series for $f(g(x))$ expanded about the point x_0. The proof is not difficult if approached within a course in complex variables but would be mysterious if attempted as a real variable theorem.

Theorem 16.33: *Suppose that there are power series representations for two functions*

$$g(x) = C + \sum_{k=1}^{\infty} a_k(x-x_0)^k \quad \textit{and} \quad f(x) = \sum_{k=0}^{\infty} b_k(x-C)^k$$

both valid in some nondegenerate intervals about their centers. Then there is a power series expansion for

$$f(g(x)) = \sum_{k=0}^{\infty} c_k(x-x_0)^k$$

with a positive radius of convergence whose coefficients can be obtained by inserting the series for $g(x) - C$ into the series for f, that is, by expanding

$$f(g(x)) = \sum_{k=0}^{\infty} b_k \left(\sum_{j=1}^{\infty} a_j (x-x_0)^j \right)^k$$

formally.

Exercises

16.7.1 Under what conditions on the functions f and g on an interval (a, b) can you conclude that the composition $f \circ g$ is analytic?

See Note 258

16.7.2 Find several terms in the power series expansion of $e^{\sin x}$ by a method different from that in this section.

16.7.3 Find several terms in the power series expansion of $e^{\tan x}$ using the method discussed in this section.

16.8 Trigonometric Series

Enrich.

In this section we present a short introduction to another way of representing functions, namely as trigonometric series or Fourier series. There are deep connections between power series and Fourier series so this theory does belong in this chapter (see Exercise 16.8.1).

The origins of the subject go back to the middle of the eighteenth century. Certain problems in mathematical physics seemed to require that an arbitrary function f with a fixed period (taken here as 2π) be represented in the form of a trigonometric series

$$f(t) = \tfrac{1}{2}a_0 + \sum_{j=1}^{\infty} (a_j \cos jt + b_j \sin jt), \tag{8}$$

and mathematicians such as Daniel Bernoulli, d'Alembert, Lagrange, and Euler had debated whether such a thing should be possible. Bernoulli maintained that this would always be possible, while Euler and d'Alembert argued against it.

Joseph Fourier (1768–1830) saw the utility of these representations and, although he did nothing to verify his position other than to perform some specific calculations, claimed that the representation in (8) would be available for every function f and gave the formulas

$$a_j = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos jt\,dt \quad \text{and} \quad b_j = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin jt\,dt$$

for the coefficients.

While his mathematical reasons were not very strong and much criticized at the time, his instincts were correct, and series of this form with coefficients computed in this way are now known as *Fourier series.* The a_j and b_j are called the *Fourier coefficients* of f.

16.8.1 Uniform Convergence of Trigonometric Series

For a first taste of this theory we prove an interesting theorem that justifies some of Fourier's original intuitions. We show that if a trigonometric series converges *uniformly* to a function f, then necessarily those coefficients given by Fourier are the correct ones.

Theorem 16.34: *Suppose that*

$$f(t) = \tfrac{1}{2}a_0 + \sum_{j=1}^{\infty} (a_j \cos jt + b_j \sin jt), \tag{9}$$

with uniform convergence on the interval $[-\pi, \pi]$*. Then it follows that the function* f *is continuous and the coefficients are given by Fourier's formulas:*

$$a_j = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos jt\,dt \quad \textit{and} \quad b_j = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin jt\,dt$$

Proof. Fix $j \geq 1$, choose $n > j$ and write

$$s_n(t) = \tfrac{1}{2}a_0 + \sum_{k=1}^{n} (a_k \cos kt + b_k \sin kt)$$

that is, the partial sums of the series. A straightforward, if tiresome, calculation shows that for $j \geq 1$, and for $n > j$,

$$\int_{-\pi}^{\pi} s_n(t)\cos jt\,dt = \int_{-\pi}^{\pi} a_j(\cos jt)^2\,dt = a_j\pi. \tag{10}$$

This is, remember, just a finite sum. The orthogonality relations in Exercise 16.8.3 assist in this computation.

We are assuming that $s_n \to f$ uniformly and so it follows too, since $\cos jt$ is bounded that $s_n(t)\cos jt \to f(t)\cos jt$ uniformly for $t \in [-\pi, \pi]$. It follows, since all functions here are continuous, that

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} s_n(t)\cos jt\,dt = \int_{-\pi}^{\pi} f(t)\cos jt\,dt.$$

In view of (10) this proves the formula for a_j and $j \geq 1$. The formulas for a_0 and b_j for $j \geq 1$ can be obtained by an identical method. ■

16.8.2 Fourier Series

Enrich.

Emboldened by the theorem we have just proved we make a dramatic move, the same move that Fourier made. We start with the function f (not the series) and construct a trigonometric series by using these coefficient formulas.

Note the twist in the logic. *If* there is a trigonometric series converging uniformly to a continuous function f, then it would have to be given by the formulas of Theorem 16.34. Why not start with the series even if we have no knowledge that the series will converge uniformly, even if we do not know whether it will converge uniformly to the function we started with, indeed even if the series diverges?

Definition 16.35: Let f be an absolutely integrable function[a] on the interval $[-\pi, \pi]$ and let

$$a_j = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos jt\,dt \quad \text{and} \quad b_j = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin jt\,dt.$$

Then the series

$$\tfrac{1}{2}a_0 + \sum_{j=1}^{\infty}(a_j\cos jt + b_j\sin jt) \tag{11}$$

is called the *Fourier series* of f.

[a]While this definition promotes the study of Fourier series of absolutely integrable [i.e., Lebesgue integrable functions] in fact our theorems will concern only continuous functions.

There is a mild understanding here that the series should be somehow related to f and there is a hope that the series can be used as a "representation" of f. But, in general, uniform convergence is out of the question. Indeed, even pointwise convergence is too much to hope for. To emphasize that this relation is not one of equality, we usually write

$$f(t) \sim \tfrac{1}{2}a_0 + \sum_{j=1}^{\infty}(a_j\cos jt + b_j\sin jt).$$

Exercises

16.8.1 Let $f(z) = \sum_{k=0}^{\infty} \alpha_k z^k$ be a complex power series with a radius of convergence larger than 1. By setting $z = e^{it}$ find a connection between complex power series and trigonometric series.

16.8.2 Explain why it is that for any integrable function f we can claim that the integrals defining the Fourier coefficients of f exist.

16.8.3 Check the so-called *orthogonality relations* by computing that for integers $k \neq j$ and all i

$$\int_{-\pi}^{\pi} \sin(kt)\sin(jt)\,dt = 0, \quad \int_{-\pi}^{\pi} \cos(kt)\sin(it)\,dt = 0,$$

and

$$\int_{-\pi}^{\pi} \cos(kt)\cos(jt)\,dt = 0.$$

16.8.4 Check that for integers i, $k \neq 0$,

$$\int_{-\pi}^{\pi} (\sin kt)^2\,dt = \pi \quad \text{and} \quad \int_{-\pi}^{\pi} (\cos it)^2\,dt = \pi.$$

16.8.3 Convergence of Fourier Series

✂ *Enrich.*

The theory of Fourier Series would have a much simpler, if less fascinating, development if the Fourier series of every continuous function converged uniformly to the original function. Not only is this false, but the Fourier series of a continuous function can diverge at a large set of points. This leaves us with a serious difficulty. The Fourier series of a function is expected to represent the function, but how? If it does not converge to the function, how can it be used as a representation?

There is a mistake in our reasoning. We know that if a series converges to a function in suitable ways, then the function may be integrated and differentiated by termwise integration and differentiation of the series. But it is possible that a series may be manipulated in these ways *even if the series diverges* at some points. A representation need not be a pointwise or uniform representation to be useful.

In our next theorem we show that the Cesàro sums of the Fourier series of a suitable function do converge uniformly to the function even if the series itself is divergent. You should review the topic of

Cesàro summability in Section 3.9.1. A young Hungarian mathematician, Leopold Fejér (1880–1959), obtained this theorem in 1900.

Theorem 16.36 (Fejér) *Let f be a continuous function on $[-\pi, \pi]$ for which $f(-\pi) = f(\pi)$. Then the sequence of Cesàro means of the partial sums of the Fourier series for f converges uniformly to f on $[-\pi, \pi]$.*

Proof. Throughout the proof we may consider that f is defined on all of $\mathbb{R}$ and is 2π-periodic. We write

$$s_n(x) = \tfrac{1}{2}a_0 + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx)$$

for the partial sums of the Fourier series of f (this means the coefficients a_j, b_j are determined by using Fourier's formulas). Then we write

$$\sigma_n(x) = \frac{s_0(x) + s_1(x) + s_2(x) + \cdots + s_n(x)}{n+1}$$

for the sequence of averages (Cesàro means).

Our task is to prove that $\sigma_n \to f$ uniformly. Looking back we see that each $\sigma_n(x)$ is a finite sum of terms $s_k(x)$ and each $s_k(x)$ is a finite sum of terms involving a_j, b_j. In turn, each of these terms is expressible as an integral involving f and sin's and cos's. Thus after some considerable but routine computations, we arrive at a formula

$$\sigma_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{2}\left(f(x+t) + f(x-t)\right) K_n(t)\, dt$$

or the equivalent formula

$$\sigma_n(x) = \frac{1}{\pi}\int_{0}^{\pi} \left(f(x+t) + f(x-t)\right) K_n(t)\, dt. \tag{12}$$

Here K_n is called the *Fejér kernel* and for each n,

$$K_n(t) = \frac{1}{2(n+1)}\left(\frac{\sin\left(\frac{1}{2}(n+1)t\right)}{\sin\frac{1}{2}t}\right)^2.$$

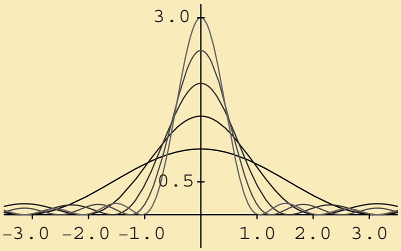

Figure 16.1. Fejér kernel $K_n(t)$ for $n = 1, 2, 3, 4$, and 5 on $[-\pi, \pi]$.

You can just accept the computations for the purposes of our short introduction to the subject.

The Fejér kernel of order n enjoys the following properties, each of which is evident from its definition:

1. Each $K_n(t)$ is a nonnegative, continuous function.

2. For each n,
$$\frac{1}{\pi}\int_{-\pi}^{\pi} K_n(t)\,dt = \frac{2}{\pi}\int_0^{\pi} K_n(t)\,dt = 1.$$

3. For each n and $0 < |t| < \pi$,
$$0 \le K_n(t) \le \frac{\pi}{(n+1)t^2}.$$

Figure 16.1 illustrates the graph of this function for $n = 1, 2, 3, 4$, and 5.

Let $\varepsilon > 0$, and choose $\delta > 0$ so that
$$|f(x+t) + f(x-t) - 2f(x)| < \varepsilon$$
for every $0 \le t \le \delta$. This uses the uniform continuity of f. We note that
$$\frac{2}{\pi}\int_0^{\pi} f(x)K_n(t)\,dt = f(x)$$

by using property 2. Thus we have

$$|\sigma_n(x) - f(x)| \le \frac{1}{\pi}\int_0^\pi |f(x+t) + f(x-t) - 2f(x)|\, K_n(t)\, dt$$
$$\le I_1 + I_2,$$

where I_1 is the integral taken over $[0, \delta]$ and I_2 is the integral taken over $[\delta, \pi]$. Since K_n is nonnegative, we did not need to keep it inside the absolute value in the integral. The part I_1 will be small (for all n) because the expression in the absolute values is small for t in the interval $[0, \delta]$. The part I_2 will be small (for large n) because of the bound on the size of K_n for t away from zero in property 3. Here are the details: For I_1 we have

$$I_1 \le \frac{\varepsilon}{\pi}\int_0^\delta K_n(t)\, dt \le \varepsilon.$$

For I_2, let

$$\kappa_n = \sup\{K_n(t) : \delta \le t \le \pi\},$$

and note that property 3 supplies us with the fact that $\kappa_n \to 0$ as $n \to \infty$. Now we have

$$I_2 \le \frac{\kappa_n \varepsilon}{\pi}\int_\delta^\pi (|f(x+t)| + |f(x-t)| + 2|f(x)|)\, dt$$

so that we can make I_2 as small as we please by choosing n large enough.

It follows, since ε and x are arbitrary, that $\lim_{n\to\infty} \sigma_n(x) = f(x)$, uniformly for $x \in [-\pi, \pi]$ as required. ■

Exercises

16.8.5 Let $s_n(x)$ be the sequence of partial sums of the Fourier series for a 2π-periodic integrable function f. Show that

$$s_n(x) = \frac{1}{\pi}\int_{-\pi}^\pi \frac{1}{2}\,(f(x+t) + f(x-t))\, D_n(t)\, dt$$

and

$$s_n(x) = \frac{1}{\pi}\int_0^\pi (f(x+t) + f(x-t))\, D_n(t)\, dt,$$

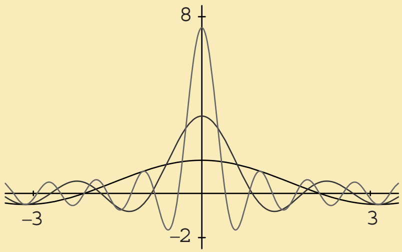

Figure 16.2. Dirichlet kernel $D_n(t)$ for $n = 1$, 3, and 7 on $[-\pi, \pi]$.

where the function $D_n(t) = \frac{1}{2} + \sum_{k=1}^{n} \cos kt$ is called the Dirichlet kernel.
Figure 16.2 illustrates the graph of this function for $n = 1$, 3, and 7. It should be contrasted with Figure 16.1.

SEE NOTE 259

16.8.6 Establish the following properties of the Dirichlet kernel $D_n(t)$ for each n:

(a) $D_n(t)$ is a continuous, 2π-periodic function.

(b) $D_n(t)$ is an even function.

(c) $\displaystyle \frac{1}{\pi}\int_{-\pi}^{\pi} D_n(t)\,dt = \frac{2}{\pi}\int_0^{\pi} D_n(t)\,dt = 1.$

(d) $\displaystyle D_n(t) = \frac{\sin\left(n+\frac{1}{2}\right)t}{2\sin\frac{1}{2}t}.$

(e) $D_n(0) = n + \frac{1}{2}$.

(f) For all t, $|D_n(t)| \le n + \frac{1}{2}$.

(g) For all $0 < |t| < \pi$, $|D_n(t)| \le \frac{\pi}{2|t|}$.

16.8.7 Let

$$K_n(t) = \frac{1}{n+1} \sum_{j=0}^{n} D_j(t),$$

where D_j are the Dirichlet kernels. Show that the formula for the averages σ_n given in the proof of Theorem 16.36 is correct.

16.8.4 Weierstrass Approximation Theorem

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Fejér's theorem allows us to prove the famous Weierstrass approximation theorem. Note that a consequence of Fejér's theorem is that continuous, 2π-periodic functions can be uniformly approximated by trigonometric polynomials. The Weierstrass theorem asserts that continuous functions on a compact interval can be uniformly approximated by ordinary polynomials.

Theorem 16.37 (Weierstrass) *Let f be a continuous function on an interval $[a, b]$, and let $\varepsilon > 0$. Then there is a polynomial*

$$g(x) = \alpha_n x^n + \alpha_{n-1} x^{n-1} + \cdots + \alpha_1 x + \alpha_0$$

so that

$$|f(x) - g(x)| < \varepsilon$$

for all $x \in [a, b]$.

Proof. It is more convenient for this proof to assume that $[a, b] = [0, 1]$. The general case can be obtained from this.

Let f be a continuous function on $[0, 1]$, let $\varepsilon > 0$, and write

$$F(t) = f(|\cos t|).$$

Then F is a continuous, 2π-periodic function and can be approximated by a trigonometric polynomial within ε. This is because, in view of Theorem 16.36, for large enough n the Cesàro means $\sigma_n(F)$ are uniformly close to F.

Since F is even [i.e., $F(t) = F(-t)$] we can figure out what form that trigonometric polynomial may take. All the coefficients b_k involving $\sin kt$ in the Fourier series for F must be zero. Thus when we form the averages of the partial sums we obtain only sums of cosines. Consequently, we can find c_0, c_1, c_2, $\dots c_n$ so that

$$\left| F(t) - \sum_{0}^{n} c_j \cos jt \right| < \varepsilon \tag{13}$$

for all t. Each $\cos jt$ can be written using elementary trigonometric identities as $T_j(\cos t)$ for some jth-order (ordinary) polynomial T_j, and so, by setting $x = \cos t$ for any $x \in [0,1]$, we have

$$\left| f(x) - \sum_{0}^{n} c_j T_j(x) \right| < \varepsilon,$$

which is exactly the polynomial approximation that we need. ■

The polynomials T_j that appear in the proof are well known as the Chebychev polynomials and are easily generated (see Exercise 16.8.9). They are named after the Russian mathematician Pafnuty Lvovich Chebychev (1821–1894).

Exercises

16.8.8 Show that once Theorem 16.37 is proved for the interval $[0,1]$ it can be deduced for any interval $[a,b]$.

16.8.9 Define the Chebychev polynomials by requiring T_j to be a polynomial so that

$$\cos jt = T_j(\cos t)$$

identically. Show that $T_0(x) = 1$, $T_1(x) = x$, and

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x).$$

Generate the first few of these polynomials.

16.8.10 Show that Theorem 16.37 can be interpreted as asserting that for any continuous function on an interval $[a,b]$ there is a sequence of polynomials p_n converging to f uniformly on $[a,b]$.

16.8.11 Does Exercise 16.8.10 also imply that there must be a power series expansion converging to f uniformly on $[a, b]$?

16.8.12 Let f be a continuous function on an interval $[a, b]$, and let $\varepsilon > 0$. Show that there must exist a polynomial p with rational coefficients so that, for all $x \in [a, b]$,

$$|f(x) - p(x)| < \varepsilon.$$

See Note 260

16.8.13 Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and let $\varepsilon > 0$. Must there exist a polynomial p so that $|f(x) - p(x)| < \varepsilon$ for all $x \in \mathbb{R}$?

See Note 261

16.8.14 Let $f : (0, 1) \to \mathbb{R}$ be a continuous function and let $\varepsilon > 0$. Must there exist a polynomial p so that $|f(x) - p(x)| < \varepsilon$ for all $x \in (0, 1)$?

See Note 262

16.8.15 Let $f : [0, 1] \to \mathbb{R}$ be a continuous function with the property that

$$\int_0^1 f(x)x^n \, dx = 0$$

for all $n = 0, 1, 2, \ldots$. What can you conclude about the function f?

See Note 263

16.8.16 Let $f : [0, 1] \to \mathbb{R}$ be a continuous function with the property that $f(0) = 0$ and

$$\int_0^1 f(x) \sin \pi n x \, dx = 0$$

for all $n = 1, 2, 3, \ldots$. What can you conclude about the function f?

See Note 264

Notes

[253]Exercise 16.2.2. This follows immediately from the inequalities

$$\liminf_k \left|\frac{a_{k+1}}{a_k}\right| \leq \liminf \sqrt[k]{|a_k|}$$

$$\leq \limsup_k \sqrt[k]{|a_k|} \leq \limsup_k \left|\frac{a_{k+1}}{a_k}\right|$$

that we obtained in Exercise 2.13.16.

[254]Exercise 16.3.3. Write out the Cauchy criterion for uniform convergence on $(-r, r)$ and deduce that the Cauchy criterion for uniform convergence on $[-r, r]$ must also hold.

[255]Exercise 16.4.4.

$$\int_0^1 \frac{1-e^{-sx}}{s}\,ds = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{k(k!)}x^k.$$

[256]Exercise 16.5.4. It is clear that $f^{(k)}$ exists for all $x \neq 0$. For $x = 0$ verify the following assertions:

1. $f^{(k)}(x)$ is of the form $R(x)e^{-1/x^2}$ for $x \neq 0$, where R is a rational function.
2. Show that
$$\lim_{x\to 0} \frac{1}{x^n}e^{-1/x^2} = 0$$
for all $n = 1, 2, \ldots$.
3. Conclude that
$$\lim_{x\to 0} f^{(k)}(x) = 0$$
for all $k = 1, 2, \ldots$.

4. Conclude that

$$f^{(k)}(0) = 0$$

for all k.

[257]Exercise 16.6.2. Just use Theorem 16.32.

[258]Exercise 16.7.1. Just use Theorem 16.33.

[259]Exercise 16.8.5. Easy, really. Just substitute

$$u = x + t$$

in the integral

$$\int_0^\pi f(x+t)D_n(t)\,dt$$

and expand the terms $\cos(ku - kx)$ using standard trigonometric identities.

[260]Exercise 16.8.12. First obtain a polynomial q so that

$$|f(x) - q(x)| < \varepsilon/2.$$

Then find a polynomial p with rational coefficients so that

$$|p(x) - q(x)| < \varepsilon/2.$$

[261]Exercise 16.8.13. Try $f(x) = e^x$.

[262]Exercise 16.8.14. Try $f(x) = 1/x$.

[263]Exercise 16.8.15. Show that f must be identically equal to zero. Use Theorem 16.37.

[264]Exercise 16.8.16. Define

$$G(t) = f(t/\pi)$$

for $t \in [0,\pi]$ and extend to $[-\pi,0]$ by

$$G(-t) = -G(t).$$

Consider the Fourier series of G and show that it contains only sin terms (no cosine terms). Show that f must be identically equal to zero. Use Theorem 16.36.

Chapter 17

LEBESGUE'S PROGRAM

Dripped Chapter[1]

Lebesgue's program is the construction of the value of the integral

$$\int_a^b f(x)\,dx$$

directly from the measure and the values of the function f in the integral. Our formal definition of the integral *appears* to do this. Since full covers are not themselves, in general, constructible from the function being integrated we cannot claim that our integral is constructed in the sense Lebesgue intends.

For his program he invented the integral as a heuristic device, imagined what properties it should possess and then went about discovering how to construct it based on this fiction. At the end he then had to take his construction as the definition itself. For us to follow the same program is much easier: we have an integral, we know many of its properties, and we can use this information to construct it.

This chapter presents an introduction to Lebesgue's methods, but backwards in a sense from conventional

[1]Note to the instructor: By adding this chapter to the basic chapters on integration, you have given the student all of rudiments of a course in Lebesgue's measure and integral, but presented in a different order. Here we *end* with the measure-theoretic construction of the integral, rather than start with that as our definition. This approach should give the student a better grounding in integration theory on the real line. That is not, of course, the goal of most early graduate courses: integration on the real line is just a particular application of general integration methods, and has no interesting special features.

presentations. We already have a formal definition of the integral, so we do not need to define an integral by Lebesgue's method. We need to show how to construct the value of an object $\int_a^b f(x)\,dx$ that we have already defined by other means.

17.1 Lebesgue measure

We define the following three versions of Lebesgue measure (similar to the three versions of null set) for a set $E \subset \mathbb{R}$:

- $\mathcal{L}(E) = \inf\{\mathcal{L}(G) :\ G \text{ open and } G \supset E\ \}$.
- $\mathcal{L}^*(E) = \inf_{\beta} \sup_{\pi \subset \beta} \sum_{([u,v],w)\in\pi} (v-u)$ where the infimum is taken over all full covers β of the set E and $\pi \subset \beta$ is an arbitrary subpartition.
- $\mathcal{L}_*(E) = \inf_{\beta} \sup_{\pi \subset \beta} \sum_{([u,v],w)\in\pi} (v-u)$ where the infimum is taken over all fine covers β of the set E and $\pi \subset \beta$ is an arbitrary subpartition.

The first of these is Lebesgue's original version of his measure. We have already (in Section 11.1.1) defined the Lebesgue measure of open sets. This definition extends that, by a simple infimum, to all sets.

The three definitions are equivalent, a fact which is proved as the Vitali covering theorem in Section 17.2 below.

17.1.1 Basic property of Lebesgue measure

Theorem 17.1: *Lebesgue measure $\mathcal{L}$ is a nonnegative real-valued set function defined for all sets of reals numbers that is a* measure[a] *on $\mathbb{R}$, i.e., it has the following properties:*

1. $\mathcal{L}(\emptyset) = 0$.
2. *For any sequence of sets E, E_1, E_2, E_3, ... for which*

$$E \subset \bigcup_{n=1}^{\infty} E_n$$

the inequality

$$\mathcal{L}(E) \leq \sum_{n=1}^{\infty} \mathcal{L}(E_n)$$

must hold.

[a]Most authors would call this an *outer measure*.

This result is often described by the following language that splits the property (b) in two parts:

Subadditivity: $\mathcal{L}\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \mathcal{L}(E_n)$.

Monotonicity: $\mathcal{L}(A) \leq \mathcal{L}(B) \quad \text{if } A \subset B$.

Since we have three representations of the Lebesgue measure, as $\mathcal{L}$, $\mathcal{L}^*$, or as $\mathcal{L}_*$ we can prove this using any one of the three. The exercises ask for all three; any one would suffice in view of the Vitali covering theorem proved in the next section.

Exercises

17.1.1 Prove that $\mathcal{L}$ is a measure in the sense of Theorem 17.1.

SEE NOTE 265

17.1.2 Prove that $\mathcal{L}^*$ is a measure in the sense of Theorem 17.1.

17.1.3 Prove that $\mathcal{L}_*$ is a measure in the sense of Theorem 17.1.

17.1.4 Prove that $\mathcal{L}(A) < t$ if and only if there is an open set G that contains all but countably many points of A and for which $\mathcal{L}(G) < t$.

17.2 Vitali covering theorem

These three measures are identical and we can use any version. The proof is just a bit more difficult than the proof of the narrower version, the mini-Vitali theorem, given in Section 11.4 where we showed that sets of measure zero were equivalent to both full null and fine null sets.

Theorem 17.2 (Vitali Covering Theorem) $\mathcal{L} = \mathcal{L}^* = \mathcal{L}_*$.

17.2.1 Classical version of Vitali's theorem

Vitali's covering theorem asserts that the measure of an arbitrary set can be determined from full and fine covers of that set. The basic computation about fine covers is the following lemma, known as the classical version of Vitali's theorem.

Lemma 17.3 (Vitali covering theorem) *Let β be a fine cover of a bounded set E and suppose that $\varepsilon > 0$. Then there must exist a subpartition $\pi \subset \beta$ for which*

$$\mathcal{L}\left(E \setminus \bigcup_{([u,v],w)\in\pi} [u,v]\right) < \varepsilon. \tag{1}$$

Proof. For the proof of this theorem we need only one simple fact (Exercise 17.1.4) about the Lebesgue measure $\mathcal{L}(E)$ of a real set A:

> $\star$ *$\mathcal{L}(A) < \varepsilon$ if and only if there is an open set G containing all but countably many points of A and for which $\mathcal{L}(G) < \varepsilon$.*

Thus the proof is really about open sets. Indeed in our proof we use only the Lebesgue measure of open sets and several covering lemmas.

The proof is just a repeated application of Lemma 11.17. Since E is bounded there is an open set U_1 containing E for which $\mathcal{L}(U_1) < \infty$. If $\mathcal{L}(U_1) < \varepsilon$ then, since $E \subset U_1$, $\mathcal{L}(E) < \varepsilon$ and there is nothing more to prove: take $\pi = \emptyset$ and the statement (1) is satisfied. If $\mathcal{L}(U_1) \geq \varepsilon$ we start our process.

We prune β by the open set U_1: define $\beta_1 = \beta(U)$. Note that this, too, is a fine cover of E. Set

$$G_1 = \bigcup_{([u,v],w)\in\beta_1} (u, v).$$

Then G_1 is an open set and, $g_1 = \mathcal{L}(G_1) < \mathcal{L}(U_1)$, is finite. We know from Lemma 11.16, that G_1 covers all of E except for a countable set. [We shall ignore countable sets in this proof, to keep the bookkeeping simple]. By Lemma 11.17 there must exist a subpartition $\pi_1 \subset \beta_1$ for which

$$U_2 = G_1 \setminus \bigcup_{([u,v],w)\in\pi} [u, v]$$

is an open subset of G_1 and

$$\mathcal{L}(U_2) \leq 5g_1/6 \leq 5\mathcal{L}(U_1)/6.$$

Define

$$E_1 = E \setminus \bigcup_{([u,v],w)\in\pi_1} [u, v].$$

If $\mathcal{L}(U_2) < \varepsilon$ then $\mathcal{L}(E_1) < \varepsilon$. This because U_2 is an open set containing all of E_1 except possibly some countable set; thus $\star$ stated above implies that $\mathcal{L}(E_1) < \varepsilon$. But if $\mathcal{L}(E_1) < \varepsilon$ the process can stop: take $\pi = \pi_1$ and the statement (1) is satisfied.

If $\mathcal{L}(U_2) \geq \varepsilon$ we continue our process. Define $\beta_2 = \beta(U_2)$ and note that this is a fine cover of E_1 (i.e., the points in E not already handled by the subpartition π_1 or the countably many points of E discarded in the first stage of our proof).

Set

$$G_2 = \bigcup_{([u,v],w)\in\beta_2} (u,v).$$

Then G_2 is an open set and $g_2 = \mathcal{L}(G_2) \leq \mathcal{L}(U_2)$. As before, we know from Lemma 11.16, that G_2 covers all of E_1 except for a countable set. [We are ignoring countable sets in this proof, throw these points away].

Again applying Lemma 11.17, we find a subpartition $\pi_2 \subset \beta_2$ for which

$$U_3 = G_2 \setminus \bigcup_{([u,v],w)\in\pi_2} [u,v]$$

is an open subset of G_2 and $\mathcal{L}(U_3) \leq 5g_2/6$. Define

$$E_2 = E_1 \setminus \bigcup_{([u,v],w)\in\pi_2} [u,v]$$

$$= E \setminus \bigcup_{([u,v],w)\in\pi_1\cup\pi_2} [u,v].$$

If $\mathcal{L}(U_3) < \varepsilon$ then $\mathcal{L}(E_2) < \varepsilon$. This because U_3 is an open set containing all of E_2 except possibly some countable set; thus $\star$ stated above implies that $\mathcal{L}(E_2) < \varepsilon$. But if $\mathcal{L}(E_2) < \varepsilon$ the process can stop: take $\pi = \pi_1 \cup \pi_2$ and the statement (1) is satisfied. [Be sure to check that the intervals from π_1 have been arranged to be disjoint from the intervals in π_2.]

This process is continued, inductively, until it stops. It certainly *must* stop since

$$\mathcal{L}(U_{k+1}) < \frac{5}{6}\mathcal{L}(U_k) \leq \cdots \leq \left(\frac{5}{6}\right)^k \mathcal{L}(U_1)$$

so that eventually $\mathcal{L}(U_{k+1}) < \varepsilon$ and $\mathcal{L}(E_k) < \varepsilon$. Take

$$\pi = \pi_1 \cup \pi_2 \cup \dots \pi_k$$

and the statement (1) is satisfied. ■

17.2.2 Proof that $\mathcal{L} = \mathcal{L}^* = \mathcal{L}_*$.

The inequality

$$\mathcal{L}_* \leq \mathcal{L}^* \leq \mathcal{L}$$

is trivial. First of all, any full cover is also a fine cover so that $\mathcal{L}_* \leq \mathcal{L}^*$ must be true. Second, if $\mathcal{L}(E) < t$ there is an open set G containing E for which it is also true that $\mathcal{L}(G) < t$. But then we can define a covering relation β to consist of all pairs $([u,v],w)$ provided $w \in [u,v] \subset G$. This is a full cover of E. Note that

$$\sum_{([u,v],w)\in\pi} (v-u) \leq \mathcal{L}(G) < t$$

whenever $\pi \subset \beta$ is an arbitrary subpartition. It follows that $\mathcal{L}^*(E) < t$. As this is true for all t,

$$\mathcal{L}^*(E) \leq \mathcal{L}(E).$$

Finally, then, Lemma 17.3 completes the proof. Let β be any fine cover of a bounded set E and suppose that $\varepsilon > 0$. Then there must exist a subpartition $\pi \subset \beta$ for which

$$\mathcal{L}\left(E \setminus \bigcup_{([u,v],w)\in\pi} [u,v]\right) < \varepsilon. \tag{2}$$

In particular, using subadditivity measure property of $\mathcal{L}$,

$$\mathcal{L}(E) \leq \mathcal{L}\left(E \setminus \bigcup_{([u,v],w)\in\pi} [u,v]\right) + \sum_{([u,v],w)\in\pi} \mathcal{L}([u,v])$$
$$< \sum_{([u,v],w)\in\pi} (v-u) + \varepsilon.$$

So, since this is true for any fine cover of E,

$$\mathcal{L}(E) \leq \mathcal{L}_*(E) + \varepsilon.$$

It follows that $\mathcal{L}(E) \leq \mathcal{L}_*(E)$ for all bounded sets E.

That establishes the identity $\mathcal{L} = \mathcal{L}^* = \mathcal{L}_*$ for all bounded sets. The extension to unbounded sets can be accomplished with the standard measure properties.

17.3 Density theorem

As an application of the Vitali covering theorem we prove the density theorem. This asserts that for an arbitrary set E almost every point is a point of density, a point x where

$$\frac{\mathcal{L}(E \cap [u,v])}{\mathcal{L}([u,v])} \to 1$$

as $[u,v]$ shrinks to x.

Theorem 17.4: *Almost every point of an arbitrary set E is a point of density.*

Proof. To define this with a bit more precision write

$$\underline{d}(E,x) = \sup_{\delta>0} \inf \left\{ \frac{\mathcal{L}(E \cap [u,v])}{\mathcal{L}([u,v])} : u \leq x \leq v,\ 0 < v-u < \delta \right\}.$$

This is called the *lower density* of E at x. The theorem asserts that

$$\underline{d}(E,x) = 1$$

at almost every point x of E.

We may assume that E is bounded. Take any $\alpha < 1$ and define

$$E_\alpha = \{x \in E : \underline{d}(E,x) < \alpha\}$$

and

$$E' = \{x \in E : \underline{d}(E,x) < 1\}.$$

We show that E_α is necessarily a set of measure zero. It follows that E' is then a set of measure zero since evidently

$$E' = \bigcup_{n=1}^{\infty} E_{\frac{n}{n+1}}.$$

Fix $\alpha < 1$ and any open set G containing E_α, and define

$$\beta = \{([u,v],w) : u \leq x \leq v,\ \mathcal{L}(E \cap [u,v]) < \alpha \mathcal{L}([u,v])\}\,.$$

This is a fine cover of E_α, and since G is an open set containing E_α, the pruned relation $\beta(G)$ is also a fine cover of E_α. Let $\varepsilon > 0$. By the Vitali covering theorem (Lemma 17.3) there must exist a subpartition $\pi \subset \beta(G)$ for which

$$\mathcal{L}\left(E_\alpha \setminus \bigcup_{([u,v],w)\in\pi} [u,v]\right) < \varepsilon. \tag{3}$$

Now we simply compute, using subadditivity, that

$$\begin{aligned}\mathcal{L}(E_\alpha) &\leq \mathcal{L}\left(E_\alpha \setminus \bigcup_{([u,v],w)\in\pi} [u,v]\right) + \sum_{([u,v],w)\in\pi} \mathcal{L}(E_\alpha \cap [u,v]) \\ &\leq \varepsilon + \sum_{([u,v],w)\in\pi} \mathcal{L}(E \cap [u,v]) \\ &\leq \varepsilon + \alpha \sum_{([u,v],w)\in\pi} \mathcal{L}([u,v]) \leq \varepsilon + \alpha\mathcal{L}(G).\end{aligned}$$

We deduce that $\mathcal{L}(E_\alpha) \leq \mathcal{L}(G)$ for all such open sets G and hence that $\mathcal{L}(E_\alpha) \leq \alpha\mathcal{L}(E_\alpha)$. This is possible only if $\mathcal{L}(E_\alpha) = 0$. ■

17.4 Additivity

Lebesgue measure is *subadditive* in general on the union of two sets E_1 and E_2. The subadditivity formula is

$$\mathcal{L}(A \cap (E_1 \cup E_2)) \leq \mathcal{L}(A \cap E_1) + \mathcal{L}(A \cap E_2)$$

We know that this same subadditivity formula holds for a sequence of sets $\{E_i\}$:

$$\mathcal{L}\left(A \cap \left(\bigcup_{i=1}^{\infty} E_i\right)\right) \leq \sum_{i=1}^{\infty} \mathcal{L}(A \cap E_i).$$

We now ask for conditions under which we can claim equality (not inequality). The additivity formula we wish to investigate is

$$\mathcal{L}\left(A \cap \left(\bigcup_{i=1}^{\infty} E_i\right)\right) = \sum_{i=1}^{\infty} \mathcal{L}(A \cap E_i)?$$

Our first observation is that this is possible if the sets $\{E_i\}$ are separated by open sets. This means merely that there exist open sets G_i and G_j that have no point in common, with $E_i \subset G_i$ and $E_j \subset G_j$. This is stronger than the requirement that E_i and E_j have no point in common. But note that two disjoint closed sets can always be separated in this fashion.

Lemma 17.5: *Let E_1 and E_2 be sets that are separated by open sets. Then, for any set A*

$$\mathcal{L}(A \cap (E_1 \cup E_2)) = \mathcal{L}(A \cap E_1) + \mathcal{L}(A \cap E_2).$$

Proof. Let us use the full version $\mathcal{L}^*$. We know that

$$\mathcal{L}^*(A \cap (E_1 \cup E_2)) \leq \mathcal{L}^*(A \cap E_1) + \mathcal{L}^*(A \cap E_2).$$

Let us prove the opposite direction. Let β be any full cover of $A \cap (E_1 \cup E_2)$. Select G_1 and G_2, disjoint open sets containing E_1 and E_2 (respectively). Then $\beta(G_1 \cup G_2)$ is necessarily a full cover of $A \cap (E_1 \cup E_2)$. Note that $\beta(G_1)$ is a full cover of $A \cap E_1$ and that $\beta(G_2)$ is a full cover of $A \cap E_2$. If $t_1 < \mathcal{L}^*(A \cap E_1)$ and $t_2 < \mathcal{L}^*(A \cap E_2)$ then there must be subpartitions $\pi_1 \subset \beta(G_1)$ and $\pi_2 \subset \beta(G_2)$ with

$$\sum_{([u,v],w) \in \pi_1} (v - u) > t_1$$

and

$$\sum_{([u,v],w) \in \pi_2} (v - u) > t_2.$$

It follows that β contains a subpartition $\pi = \pi_1 \cup \pi_2$ for which

$$\sum_{([u,v],w)\in\pi} (v-u) > t_1 + t_2.$$

From this we deduce that $\mathcal{L}^*(A \cap (E_1 \cup E_2)) > t_1 + t_2$. Then

$$\mathcal{L}^*(A \cap (E_1 \cup E_2)) \geq \mathcal{L}^*(A \cap E_1) + \mathcal{L}^*(A \cap E_2)$$

follows. ■

Corollary 17.6: *Let E_1, E_2, E_3, ... be a sequence of pairwise disjoint subsets of $\mathbb{R}$ and write*

$$E = \bigcup_{i=1}^{\infty} E_i.$$

Suppose that each pair of sets in the sequence are separated by open sets. Then, for any set A,

$$\mathcal{L}(A \cap E) = \sum_{i=1}^{\infty} \mathcal{L}(A \cap E_i).$$

Proof. We know from the usual measure properties that

$$\mathcal{L}(A \cap E) \leq \sum_{i=1}^{\infty} \mathcal{L}(A \cap E_i).$$

We also know that

$$\mathcal{L}(A \cap (E_1 \cup E_2)) = \mathcal{L}(A \cap E_1) + \mathcal{L}(A \cap E_2).$$

An inductive argument would show, too, that for any $n > 1$,

$$\mathcal{L}(A \cap (E_1 \cup E_2 \cdots \cup E_n)) = \mathcal{L}(A \cap E_1) + \mathcal{L}(A \cap E_2) + \cdots + \mathcal{L}(A \cap E_n).$$

Thus, from the monotonicity property of measures,

$$\sum_{i=1}^{n} \mathcal{L}(A \cap E_i) \leq \mathcal{L}(A \cap E) \leq \sum_{i=1}^{\infty} \mathcal{L}(A \cap E_i).$$

From this the corollary evidently follows. ■

Corollary 17.7: *Let E_1, E_2, E_3, ... be a sequence of pairwise disjoint closed subsets of $\mathbb{R}$. Then, for any set A,*

$$\mathcal{L}(A \cap E) = \sum_{i=1}^{\infty} \mathcal{L}(A \cap E_i).$$

To push the countable additivity one step further we use the previous corollary in a natural way. This looks like a highly technical lemma, but it is the basis and motivation for our definition of measurable sets and the theory is more natural than it might appear. The proof is left as an exercise; working through a proof should make it clear how and why the measurability definition in the next section works.

Lemma 17.8: *Let E_1, E_2, E_3, ... be a sequence of pairwise disjoint subsets of $\mathbb{R}$ and write*

$$E = \bigcup_{i=1}^{\infty} E_i.$$

Suppose that for every $\varepsilon > 0$ and for every n there is an open set G_n so that $E_n \setminus G_n$ is closed and so that $\mathcal{L}(G_n) < \varepsilon$. Then, for any set A,

$$\mathcal{L}(A \cap E) = \sum_{i=1}^{\infty} \mathcal{L}(A \cap E_i).$$

17.5 Measurable sets

Definition 17.9: An arbitrary subset E of $\mathbb{R}$ is *measurable*[a] if for every $\varepsilon > 0$ there is an open set G with $\mathcal{L}(G) < \varepsilon$ and so that $E \setminus G$ is closed.

[a]Most advanced courses will start with a different definition of measurable and later on show that this property used here is equivalent in certain settings. See Section 17.7.2 for the connections.

Immediately we see that closed sets are measurable and null sets are measurable. The definition is exactly designed to produce the following theorem.

Theorem 17.10: *Let E_1, E_2, E_3, ... be a sequence of pairwise disjoint measurable subsets of $\mathbb{R}$ and write*

$$E = \bigcup_{i=1}^{\infty} E_i.$$

Then, for any set A,

$$\mathcal{L}(A \cap E) = \sum_{i=1}^{\infty} \mathcal{L}(A \cap E_i).$$

Proof. This follows immediately from Lemma 17.8. ■

Theorem 17.11: *The class of all measurable subsets of $\mathbb{R}$ forms a Borel family that contains all closed sets and all null sets.*

Proof. The class of all measurable subsets of $\mathbb{R}$ forms a Borel family: it a collection of sets that is closed under the formation of unions and intersections of sequences of its members, and contains the complement of each of its members. Here are the details of the proof. Items (3), (4), and (5) are specifically the requirements that the class of measurable sets forms a Borel family.

We prove that the family of all measurable sets has the following properties:

1. Every null set is measurable.

2. Every closed set is measurable.

3. If E_1, E_2, E_3, is a sequence of measurable sets then the union $\bigcup_{n=1}^{\infty} E_n$ is also measurable.

4. If E_1, E_2, E_3, is a sequence of measurable sets then the intersection $\bigcap_{n=1}^{\infty} E_n$ is also measurable.

5. If E is measurable then the complement $\mathbb{R} \setminus E$ is also measurable.

Items (1) and (2) are easy. Let us prove (5) first. Let E be measurable and E' is its complement. Let $\varepsilon > 0$ and choose an open set G_1 so that $E \setminus G_1$ is closed and $\mathcal{L}(G_1) < \varepsilon/2$. Let O be the complement of $E \setminus G_1$; evidently O is open.

First find an open set G_2 with $\mathcal{L}(G_2) < \varepsilon/2$ so that $O \setminus G_2$ is closed. [Simply display the component intervals of O, handle the infinite components first, and then a finite number of the bounded components.] Now observe that

$$E' \setminus (G_1 \cup G_2) = O \setminus G_2$$

is a closed set while $G_1 \cup G_2$ is an open set with measure smaller than ε. This verifies that E' is measurable.

Now check (e): let $\varepsilon > 0$ and choose open sets G_n so that $\mathcal{L}(G_n) < \varepsilon 2^{-n}$ and each $E_n \setminus G_n$ is closed. Observe that the set $G = \bigcup_{n=1}^{\infty} G_n$ is an open set for which

$$\mathcal{L}(G) \leq \sum_{n=1}^{\infty} \mathcal{L}(G_n) \leq \sum_{n=1}^{\infty} \varepsilon 2^{-n} = \varepsilon.$$

Finally

$$E' = E \setminus G = \bigcap_{n=1}^{\infty} (E_n \setminus G_n)$$

is closed.

For (4), write E'_n for the complementary set to E_n. Then the complement of the set $A = \bigcup_{n=1}^{\infty} E_n$ is the set $B = \bigcap_{n=1}^{\infty} E'_n$. Each E'_n is measurable by (5) and hence B is measurable by (d). The complement of B, namely the set A, is measurable by (5) again. ■

17.6 Measurable functions

Definition 17.12: An arbitrary function $f : \mathbb{R} \to \mathbb{R}$ is *measurable* if for any real number r

$$A_r = \{x \in \mathbb{R} : f(x) < r\}$$

is a measurable set.

A function $f : [a, b] \to \mathbb{R}$ would be measurable if there is a measurable function $g : \mathbb{R} \to \mathbb{R}$ and $f(x) = g(x)$ for all $x \in [a, b]$.

Exercises

17.6.1 Let f be a measurable function. Show that each of $|f|$, $[f]^+$, and $[f]^-$ must also be measurable.

17.6.2 Show that the function $f(x) = \chi_A(x)$ is measurable if and only if the set A is a measurable set.

17.6.1 Continuous functions are measurable

Lemma 17.13: *A function $f : \mathbb{R} \to \mathbb{R}$ that is continuous everywhere is measurable.*

Proof. To prove that f is measurable we need to verify that, for any real number r,

$$A_r = \{x \in \mathbb{R} : f(x) < r\}$$

is a measurable set. But we already know that, for continuous functions, such sets are open. ■

We know too that a continuous function $f : [a, b] \to \mathbb{R}$ is also measurable by our definition since f agrees on $[a, b]$ with the continuous function g defined by $g(t) = f(t)$ for $a \leq t \leq b$, $g(t) = g(b)$ for $t > b$, and $g(t) = g(a)$ for $t < a$.

17.6.2 Derivatives and integrable functions are measurable

Suppose that $f : \mathbb{R} \to \mathbb{R}$ is almost everywhere the derivative of some function. Then f is measurable[2]. If we combine that fact with the fundamental theorem of the calculus we see that all integrable functions are measurable.

Lemma 17.14: *A function $f : \mathbb{R} \to \mathbb{R}$ that is almost everywhere the derivative of some function is measurable.*

[2] A theorem of Lusin states the converse: if f is measurable then there is a continuous function F for which $F'(x) = f(x)$ almost everywhere. This should not be confused with the fundamental theorem of the calculus.

Proof. We suppose that $F : \mathbb{R} \to \mathbb{R}$ and $F'(x) = f(x)$ almost everywhere, say everywhere in $\mathbb{R} \setminus N$ where N is a set of measure zero. Consider the set $E = \{x : \overline{D}F(x) > r\}$ for any r. Let m, n be positive integers and define β_{mn} to be the covering relation consisting of all pairs $([u, v], w)$ for which $u \le w \le v$, and for which $0 < v - u < 1/m$ and

$$\frac{F(v) - F(u)}{v - u} \ge r + 1/n.$$

Write

$$E_{mn} = \bigcup \{[u, v] : ([u, v], w) \in \beta_{mn}\}.$$

Each set E_{mn} is thus a fairly simple object: it is a union of a family of compact intervals. In Lemma 11.16 we have seen that this means it has a simple structure: it differs from an open set by a countable set. In particular each E_{mn} is an measurable set. We check that

$$E = \bigcup_{n=1}^{\infty} \bigcap_{m=1}^{\infty} E_{mn}. \tag{4}$$

To begin suppose that $x \in E$. Then $\overline{D}F(x) > r$. There must be at least one integer n with $\overline{D}F(x) > r+1/n$. Moreover, for every integer m there would have to be at least one compact interval $[u, v]$ containing x with length less than $1/m$ so that

$$\frac{F(v) - F(u)}{v - u} \ge r + 1/n.$$

Hence x is a point in the set on the right-hand side of the proposed identity. Conversely, should x belong to that set, then there is at least one n so that for all m, x belongs to E_{mn}. It would follow that $\overline{D}F(x) > r$ and so $x \in E$.

The identity (4) now exhibits E as a combination of sequences of measurable sets and so E too is an measurable set because the measurable sets form a Borel family (Theorem 17.11). Finally then

$$\{x : f(x) > r\} = \big(\{x : \overline{D}F(x) > r\} \cap [\mathbb{R} \setminus N]\big) \cup N'$$

where N' is an appropriate subset of N. This exhibits the set $\{x : f(x) > r\}$ as the union of a measurable set and a set of measure zero. Consequently that set is measurable. This is true for all r and verifies that f is a measurable function. ■

Corollary 17.15: *If $f : [a, b] \to \mathbb{R}$ is integrable then f is measurable.*

17.6.3 Simple functions

A function $f : \mathbb{R} \to \mathbb{R}$ is *simple* if there is a finite collection of measurable sets E_1, E_2, E_3, ..., E_n and real numbers r_1, r_2, r_3, ..., r_n so that

$$f(x) = \sum_{k=1}^{n} r_k \chi_{E_k}(x)$$

for all real x.

Lemma 17.16: *Any simple function is measurable.*

Proof. Suppose that

$$f(x) = \sum_{k=1}^{n} r_k \chi_{E_k}(x)$$

and s is any real number. It is easy to sort out, for any value of s, exactly what the set

$$A_s = \{x : f(x) < s\}$$

must be in terms of the sets $\{E_k\}$. In each case we see that A_s is some simple combination of measurable sets and so is itself measurable. ■

17.6.4 Series of simple functions

Theorem 17.17: *Every nonnegative, measurable function $f : \mathbb{R} \to \mathbb{R}$ can be written as the sum of a series of nonnegative simple functions by the following inductive procedure: Take $\{r_k\}$ to be any sequence of positive numbers for which $r_k \to 0$ and $\sum_{k=1}^{\infty} r_k = +\infty$. Define the sets*

$$A_k = \left\{ x : f(x) \geq r_k + \sum_{j<k} r_j \chi A_j(x) \right\}$$

inductively, starting with $A_0 = \emptyset$. Then

$$f(x) = \sum_{k=1}^{\infty} r_k \chi A_k(x)$$

at every x.

The proof is just a matter of deciding whether and why this works.

Exercises

17.6.1 Prove Theorem 17.17.

17.6.2 Show that the following procedure expresses a nonnegative, measurable function $f : \mathbb{R} \to \mathbb{R}$ as a nondecreasing limit of a sequence $\{f_k\}$ of simple functions: Fix an integer k. Subdivide $[0, k]$ into subintervals

$$[(j-1)2^{-k}, j2^{-k}] \quad (j = 1, 2, 3, \ldots, k2^k)$$

and, for all $x \in [a, b]$, define $f_k(x)$ to be $(j-1)2^{-k}$ if

$$(j-1)2^{-k} \leq f(x) < j2^{-k}$$

and to be k if $f(x) \geq k$.

17.6.3 In the preceding exercise show that, if f is bounded, then f is the *uniform* limit of the sequence of simple functions $\{f_k\}$.

17.6.5 Limits of measurable functions

Theorem 17.18: *Let $f_n : \mathbb{R} \to \mathbb{R}$ be a sequence of measurable functions. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function for which*

$$f(x) = \lim_{n\to\infty} f_n(x)$$

for almost every x. Then f is measurable.

Proof. We fix a real number r and verify that

$$\{x \in \mathbb{R} : f(x) < r\}$$

is a measurable set. We use the fact that sets of the form

$$\{x \in \mathbb{R} : f_n(x) < s\}$$

are measurable. This follows from the measurability of each function f_n.

Let N be the null set consisting of points x where we do not have

$$f(x) = \lim_{n\to\infty} f_n(x)$$

and let $E = \mathbb{R} \setminus N$. Then both E and N are measurable.

We claim the following set identity:

$$\{x \in E : f(x) < r\} = \bigcup_{k=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \{x \in E : f_n(x) < r - 1/k\}.$$

This is a matter of close interpretation. If x_0 belongs to the simple set on the left of the proposed identity, then $x_0 \in E$ and $f(x_0) < r$. There must exist a k so that $f(x_0) < r - 1/k$. Then there must exist an integer m so that $f_n(x) < r - 1/k$ for all $n \geq m$. That places x_0 in the set on the right.

In the other direction if x_0 belongs to the complicated set on the right of the proposed identity, then for some k and m, $f_n(x_0) < r - 1/k$ for all $n \geq m$. It follows that $f(x_0) \leq r - 1/k < r$. That places x_0 in the set on the left.

Each set

$$\{x \in E : f_n(x) < r - 1/k\} = E \cap \{x \in \mathbb{R} : f_n(x) < r - 1/k\}$$

thus is measurable since it is the intersection of a measurable set and an open set. As measurable sets form a Borel family the intersections and unions of these sets remain measurable.

Finally then

$$\{x \in \mathbb{R} : f(x) < r\}$$

is seen to be the union of the measurable set

$$\{x \in E : f(x) < r\}$$

and some subset of N. This checks the measurability of the function f. ∎

17.7 Construction of the integral

We now give Lebesgue's construction of the integral in a series of steps, starting with characteristic functions, then simple functions, then nonnegative measurable functions, and finally all absolutely integrable functions.

17.7.1 Characteristic functions of measurable sets

Lemma 17.19: *Let E be a subset of an interval $[a, b]$. Then χ_E is integrable on $[a, b]$ if and only if E is a measurable set, and in that case*

$$\mathcal{L}(E) = \int_a^b \chi_E(x)\, dx.$$

Proof. For any set $E \subset [a, b]$, measurable or not, we can easily establish the identity

$$\mathcal{L}^*(E) = \overline{\int_a^b} \chi_E(x)\, dx.$$

The two concepts in this identity are defined by the same process. Thus the proof of the lemma depends only on showing that integrability of $\chi_E(x)$ is equivalent to the measurability of E.

We already know that if $\chi_E(x)$ is integrable then it is a measurable function. But this can happen only if E is a measurable set. Conversely let us suppose that E is measurable and verify that χ_E is integrable on $[a,b]$. In fact we show that this function satisfies the McShane criterion on this interval (see Definition 10.7).

Since E is measurable we know that

$$\mathcal{L}(E) + \mathcal{L}([a,b]\setminus E) = b-a.$$

Let $\varepsilon > 0$. Select open sets $E \subset G_1$ and $[a,b]\setminus G_2$ so that

$$\mathcal{L}(G_1) < \mathcal{L}(E) + \varepsilon/2$$

and

$$\mathcal{L}(G_2) < \mathcal{L}([a,b]\setminus E) + \varepsilon/2.$$

Then, use the identity

$$\mathcal{L}(G_1 \cup G_2) = \mathcal{L}(G_1) + \mathcal{L}(G_2) - \mathcal{L}(G_1 \cap G_2)$$

to get

$$\begin{aligned}\mathcal{L}(G_1 \cap G_2) &= \mathcal{L}(G_1) + \mathcal{L}(G_2) - \mathcal{L}(G_1 \cup G_2)\\ &< [\mathcal{L}(E) + \varepsilon/2] + [\mathcal{L}([a,b]\setminus E) + \varepsilon/2] - (b-a) = \varepsilon.\end{aligned}$$

This will enable us to apply the McShane criterion to establish that χ_E is integrable on $[a,b]$. Define β as the collection of all pairs $([u,v],w)$ for which either $w \in E$ and $[u,v] \subset G_1$ or $w \in [a,b]\setminus E$ and $[u,v] \subset G_2$. This is a full cover of $[a,b]$. Choose any two partitions π, π' of $[a,b]$ contained in β. We compute

$$\sum_{([u,v],w)\in\pi} \ \sum_{([u',v'],w')\in\pi'} \left|\chi_E(w) - \chi_E(w')\right| \mathcal{L}([u,v]\cap[u',v']). \tag{5}$$

Note, in this sum, that terms for which both w and w' are in E or for which neither is in E vanish. Terms for which $w \in E$ and $w' \in [a,b]\setminus E$ must have $|\chi_E(w) - \chi_E(w')| = 1$, $[u,v] \subset G_1$ and $[u',v'] \subset G_2$. In particular $[u,v]\cap[u',v'] \subset (G_1 \cap G_2)$. The same is true if $w' \in E$ and $w \in [a,b]\setminus E$. Remembering that $\mathcal{L}(G_1 \cap G_2) < \varepsilon$, we see that the sum in (5) is smaller than ε. By the McShane criterion χ_E is integrable on $[a,b]$. ■

17.7.2 Characterizations of measurable sets

As corollaries we obtain a number of characterizations of measurable sets, including the original Lebesgue definition which is assertion (c). Assertion (d) is known as Carathéodory's criterion.

Corollary 17.20: *Let E be a set of real numbers. Then the following assertions are equivalent:*

1. *E is measurable.*
2. *χ_E is integrable on every compact interval $[a,b]$.*
3. *For every compact interval $[a,b]$,*
$$\mathcal{L}([a,b]\cap E)+\mathcal{L}([a,b]\setminus E)=b-a. \tag{6}$$
4. *For every set $T\subset\mathbb{R}$,*
$$\mathcal{L}(T)\geq\mathcal{L}(T\cap E)+\mathcal{L}(T\setminus E). \tag{7}$$
5. *For every $\varepsilon>0$ and every compact interval $[a,b]$, there is a full cover β of $[a,b]$ so that*
$$\sum_{([u,v],w)\in\pi}\ \sum_{([u',v'],w')\in\pi'}\mathcal{L}([u,v]\cap[u',v'])<\varepsilon$$
whenever π, π' are subpartitions of $[a,b]$ with $\pi\subset\beta[E]$ and $\pi'\subset\beta[[a,b]\setminus E]$.

Proof. First note that a set E is measurable if and only if $E\cap[a,b]$ is measurable for every compact interval $[a,b]$. In one direction this is because $[a,b]$ is a measurable set (it is closed) and the intersection of measurable sets is also measurable. In the other direction, if $E\cap[a,b]$ is measurable for every compact interval $[a,b]$, then $E=\bigcup_{n=1}^{\infty}E\cap[-n,n]$ expresses E as a measurable set.

The first three conditions (a), (b), and (c) we have explicitly shown to be equivalent in the proof of the lemma. Let us check that (d) implies (c). Observe that the inequality,

$$\mathcal{L}(T)\leq\mathcal{L}(T\cap E)+\mathcal{L}(T\setminus E)$$

holds in general, so that the condition (7) is equivalent to the assertion of equality:

$$\mathcal{L}(T) = \mathcal{L}(T \cap E) + \mathcal{L}(T \setminus E).$$

Thus (c) is a special case of (d) with $T = [a, b]$. On the other hand, (a) implies (d). Measurability of E implies that E and $\mathbb{R} \setminus E$ are disjoint measurable sets for which

$$\mathcal{L}(T) = \mathcal{L}(T \cap E) + \mathcal{L}(T \setminus E)$$

must hold for any set $T \subset \mathbb{R}$. Finally the fifth condition (e) is just a rewriting of the McShane criterion for integrability of the function χ_E on $[a, b]$. We have seen in the proof of the lemma that measurability of $E \cap [a, b]$ is equivalent to that criterion applied to χ_E on $[a, b]$ ■

17.7.3 Integral of simple functions

Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is *simple* if there is a finite collection of measurable sets E_1, E_2, E_3, ..., E_n and real numbers r_1, r_2, r_3, ..., r_n so that

$$f(x) = \sum_{k=1}^{n} r_k \chi_{E_k}(x)$$

for all real x. It follows from the integration theory (Theorem 10.12) and the integration of characteristic functions (Lemma 17.19) that such a function is necessarily integrable on any compact interval $[a, b]$ and that

$$\int_a^b f(x)\,dx = \sum_{k=1}^{n} \left(\int_a^b r_k \chi_{E_k}(x)\,dx \right) = \sum_{k=1}^{n} r_k \mathcal{L}(E_k \cap [a, b]).$$

Thus the integral of simple functions can be constructed from the values of the function in a finite number of steps using the Lebesgue measure.

17.7.4 Integral of nonnegative measurable functions

We have seen (Theorem 17.17) that every nonnegative measurable function can be represented by simple functions. Consequently the integral of such a function can be constructed.

Theorem 17.21: *Let f be a nonnegative, measurable function on an interval $[a,b]$. Then, for any representation of f as the sum of a series of nonnegative, simple functions*

$$f(x) = \sum_{k=1}^{\infty} f_n(x) \quad (a \le x \le b)$$

the identity

$$\int_a^b f(x)\,dx = \sum_{k=1}^{\infty} \left(\int_a^b f_n(x)\,dx \right)$$

must hold (finite or infinite). Moreover f is integrable on $[a,b]$ if and only if this series of integrals converges to a finite value.

Proof. This requires only an appeal to the monotone convergence theorem. ■

Corollary 17.22: *Let f be a nonnegative, measurable function on an interval $[a,b]$. Then*

$$\int_a^b f(x)\,dx$$

exists (finitely or infinitely). Moreover f is integrable on $[a,b]$ if and only if this value is finite.

Proof. This follows from the theorem. ■

17.7.5 Derivatives of functions of bounded variation

As a consequence of Lebesgue's program to this point we can prove some facts about derivatives of monotonic functions and derivatives of functions of bounded variation. These are due to Lebesgue, but our proofs are rather easier since we do not need much of the measure theory to obtain them.

Theorem 17.23: *Let $F : [a,b] \to \mathbb{R}$ be a function of bounded variation. Then $F'(x)$ exists almost everywhere in $[a,b]$ and*

$$\int_a^b |F'(x)|\,dx \leq V(F,[a,b]).$$

Proof. We know from the Lebesgue differentiation theorem that F is a.e. differentiable. Let $f(x) = |F'(x)|$ at every point at which $F'(x)$ exists and as zero elsewhere. Then f is a nonnegative function. At every point w in $[a,b]$ there is a $\delta > 0$ so that, whenever $u \leq w \leq v$ and $0 < v - u < \delta$,

$$f(w) - \varepsilon \leq \frac{|F(v) - F(u)|}{v - u}.$$

At points w where $f(w) = 0$ this is obvious, while at points w where $F'(w)$ exists this follows from the definition of the derivative.

Take β as the collection of all pairs $([u,v],w)$ subject to the requirement only that

$$|F(v) - F(u)| > [f(w) - \varepsilon](v - u)$$

if $w \in [a,b]$ and $[u,v] \subset [a,b]$. This collection β is a full cover.

Every partition $\pi \subset \beta$ of the interval $[a,b]$ satisfies

$$\sum_{([u,v],w)\in\pi} [f(w) - \varepsilon](v - u) < \sum_{([u,v],w)\in\pi} |F(v) - F(u)| \leq V(F,[a,b]).$$

It follows that

$$-\varepsilon(b-a) + \overline{\int_a^b} f(x)\,dx \leq V(F,[a,b]).$$

Since ε is an arbitrary positive number,

$$\overline{\int_a^b} f(x)\,dx \leq V(F,[a,b]).$$

Since f is almost everywhere a derivative it is necessarily measurable. Thus we may use the integral in place of the upper integral. ■

Corollary 17.24: *Let $F : [a,b] \to \mathbb{R}$ be a nondecreasing function. Then $F'(x)$ exists almost everywhere in $[a,b]$ and*

$$\int_a^b F'(x)\,dx \leq F(b) - F(a).$$

Corollary 17.25 (Lebesgue decomposition) *Let $F : [a,b] \to \mathbb{R}$ be a continuous, nondecreasing function. Then $F'(x)$ exists almost everywhere in $[a,b]$ and*

$$F(t) = \int_a^t F'(x)\,dx + S(t) \quad (a \leq t \leq b)$$

expresses F as the sum of an integral and a continuous, nondecreasing singular function.

Proof. Simply define

$$S(t) = F(t) - \int_a^t F'(x)\,dx \quad (a \leq t \leq b).$$

Check that $S'(t) = 0$ almost everywhere (trivial) and so S is singular. That S is continuous is evident since it is the difference of two continuous functions. That S is nondecreasing follows from the theorem, since

$$S(d) - S(c) = F(d) - F(c) - \int_c^d F'(x)\,dx \geq 0$$

for any $[c,d] \subset [a,b]$. ■

17.7.6 Integral of absolutely integrable functions

A function f is *absolutely integrable* on an interval $[a,b]$ if both f and $|f|$ are integrable on that interval.

Theorem 17.26: *Let $f : [a,b] \to \mathbb{R}$. Then f is absolutely integrable if and only if f is measurable and*

$$\int_a^b |f(x)|\,dx < \infty.$$

Proof. We know, from Exercise 17.6.1, that the functions $|f|$, $[f]^+$, and $[f]^-$ are also measurable. The finiteness of this integral implies (by Corollary 17.22) that each of these functions are integrable. In particular both functions $f = [f]^+ - [f]^-$ and $|f|$ are integrable. Thus f must be absolutely integrable. Conversely if f is absolutely integrable, this means that $|f|$ is integrable and consequently, by definition, it has a finite integral. ■

Our final theorem for Lebesgue's program shows that the integral is constructible by his methods for all absolutely integrable functions. We see in the next section that this is as far as one can go.

Theorem 17.27: *If f is absolutely integrable on a compact interval $[a,b]$ then f, $|f|$, $[f]^+$, and $[f]^-$ are measurable and*

$$\int_a^b |f(x)|\,dx = \int_a^b [f(x)]^+\,dx + \int_a^b [f(x)]^-\,dx$$

and

$$\int_a^b f(x)\,dx = \int_a^b [f(x)]^+\,dx - \int_a^b [f(x)]^-\,dx$$

Proof. If f is absolutely integrable then we know that f and $|f|$ are integrable. It follows that $[f]^+ = (f + |f|)/2$ and $[f]^- = (|f| - f)/2$ are both integrable. All functions are measurable since all are integrable. Since

$$|f(x)| = [f(x)]^+ + [f(x)]^-$$

and

$$f(x) = [f(x)]^+ - [f(x)]^-$$

the integration formulas are immediately available. ■

17.7.7 McShane's Criterion

Lebesgue's integral can also be characterized by the McShane criterion.

Theorem 17.28: *Let $f : [a,b] \to \mathbb{R}$. Then f is absolutely integrable if and only if it satisfies McShane's criterion on that interval.*

Proof. We already know that any function satisfying the McShane criterion is absolutely integrable, thus we need a proof in only one direction.

To simplify the notation let us write

$$S(f,\pi,\pi') = \sum_{([u,v],w)\in\pi} \; \sum_{([u',v'],w')\in\pi'} \left|f(w) - f(w')\right| \mathcal{L}([u,v]\cap[u',v']) \tag{8}$$

for any two partitions π, π' of $[a,b]$. Some preliminary computations will help. If g_1, g_2, $\dots$, g_n are functions on $[a,b]$ then,

$$S\left(\sum_{i=1}^{n} g_i, \pi, \pi'\right) \le \sum_{i=1}^{n} S(g_i,\pi,\pi'). \tag{9}$$

If

$$\overline{\int_a^b} |f(x)|\,dx < t$$

then there must exist a full cover β with the property that for any two partitions π, π' of $[a,b]$ from β,

$$S(f,\pi,\pi') < 2t. \tag{10}$$

Finally

$$S(f,\pi,\pi') \le \sup\{|f(t)| : a \le t \le b\} \cdot 2(b-a). \tag{11}$$

Each of the statements (9), (10), and (11) require only simple computations that we leave to the reader.

Now for our argument. We assume that f is absolutely integrable and verify the criterion. But f can be written as a difference of two nonnegative integrable functions. If both of these satisfy the criterion then, using (9) we deduce that so too does f. Consequently for the remainder of the proof we assume that f is nonnegative and integrable.

The first step is to observe that every characteristic function of a measurable set satisfies the McShane criterion. This is proved in Lemma 17.19. Using (9) we easily deduce, as our second step, that every nonnegative simple function also satisfies the McShane criterion.

The third step is to show that every nonnegative, bounded measurable function also satisfies this criterion. But such a function is the uniform limit of a sequence of nonnegative simple functions. It follows then, from (11), that such functions satisfy the McShane criterion. For if f is a bounded measurable function, $\varepsilon > 0$, choose a simple function g so that

$$|f(t) - g(t)| < \varepsilon/(4[b-a])$$

for all $a \le t \le b$. Now using McShane's criterion on g we can select a full cover β for which $S(g,\pi,\pi') < \varepsilon/2$ for all partitions π, π' of $[a,b]$ from β. Then

$$S(f,\pi,\pi') \le S(f-g,\pi,\pi') + S(g,\pi,\pi') \le \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

The final step requires an appeal to the monotone convergence theorem. Set $f_N(t) = \min\{N, f(t)\}$ and use the monotone convergence theorem to find an integer N large enough so that

$$\int_a^b [f(x) - f_N(x)]\,dx < \varepsilon/4.$$

Using (10) select a full cover β_1 for which $S(f - f_N,\pi,\pi') < \varepsilon/2$ for all partitions π, π' of $[a,b]$ from β_1. Select a full cover β_2 for which $S(f_N,\pi,\pi') < \varepsilon/2$ for all partitions π, π' of $[a,b]$ from β_2. Then set $\beta = \beta_1 \cap \beta_2$. This is a full cover and we can check that

$$S(f,\pi,\pi') \le S(f - f_N,\pi,\pi') + S(f_N,\pi,\pi') \le \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

for all partitions π, π' of $[a,b]$ from β. This verifies the McShane criterion for an arbitrary nonnegative integrable function f. ■

17.7.8 Nonabsolutely integrable functions

A function f is *nonabsolutely integrable* on an interval $[a,b]$ if it is integrable, but not absolutely integrable there, i.e., f is integrable $[a,b]$ but $|f|$ is not integrable. Lebesgue's program will not construct the integral of a nonabsolutely integrable function. The only method that his program offers is the hope that

$$\int_a^b f(x)\,dx = \int_a^b [f(x)]^+\,dx - \int_a^b [f(x)]^-\,dx?$$

Theorem 17.29: *If f is nonabsolutely integrable on a compact interval $[a,b]$ then*

$$\int_a^b |f(x)|\,dx = \int_a^b [f(x)]^+\,dx = \int_a^b [f(x)]^-\,dx = \infty.$$

Proof. If f is nonabsolutely integrable then it is measurable. It follows from Exercise 17.6.1 that the functions $|f|$, $[f]^+$, and $[f]^-$ are also measurable. If, for example,

$$\int_a^b [f(x)]^+\,dx < \infty,$$

contrary to what we wish to prove, then we must conclude (from Theorem 17.26) that $[f]^+$ is integrable. But if $[f]^+$ is integrable then from the identity

$$[f(x)]^- = [f(x)]^+ - f(x)$$

we could conclude that $[f]^-$ must also be integrable and consequently each of the functions f, $|f|$, $[f]^+$, and $[f]^-$ must be integrable, contradicting the hypothesis of the theorem. ■

17.8 Characterizations of the indefinite integral

Under what conditions can we be sure that a function $F : [a,b] \to \mathbb{R}$ can be written as

$$F(t) = C + \int_a^t f(t)\,dt$$

for a constant C and an integrable function f. The property and the characterization itself for absolutely integrable functions were given by Guiseppe Vitali in 1905, only shortly after the publication by Lebesgue of his integration theory.

Definition 17.30: Suppose that $F : [a,b] \to \mathbb{R}$ is a function. Then F is *Vitali continuous*[a] if for all $\varepsilon > 0$ there is a $\delta > 0$ so that

$$\sum_i |F(v_i) - F(u_i)| < \varepsilon$$

whenever $\{[u_i, v_i]\}$ are nonoverlapping subintervals of $[a,b]$ for which $\sum_i [v_i - u_i] < \delta$.

[a] Most texts call this (as did Vitali himself) "absolute continuity." We prefer to reserve this term for the "zero variation on zero measure sets" which is the preferred use of the expression in measure theory.

There are several simple consequences of this definition that we will require in order to better understand this concept.

Lemma 17.31: *Suppose that $F : [a,b] \to \mathbb{R}$ is a function that is is Vitali continuous on $[a,b]$. Then*

1. *F is continuous on $[a,b]$,*
2. *F is absolutely continuous on (a,b), and*
3. *F has bounded variation on $[a,b]$.*

Proof. The first two statements are trivial and follow easily from the definition. For the third, choose a positive number δ so that

$$\sum_i |F(v_i) - F(u_i)| < 1$$

whenever $\{[u_i, v_i]\}$ are nonoverlapping subintervals of $[a,b]$ for which

$$\sum_i [v_i - u_i] < \delta.$$

Then any partition of $[a,b]$ into subintervals smaller than δ must have

$$\sum_i |F(v_i) - F(u_i)| < N$$

where N is an integer chosen large enough so that $N\delta > b - a$. ■

17.8.1 Integral of nonnegative, integrable functions

Theorem 17.32: *A necessary and sufficient condition in order that a function $F : [a,b] \to \mathbb{R}$ can be written as*

$$F(t) = C + \int_a^t f(t)\,dt$$

for a constant C and a nonnegative *integrable function f is that F is Vitali continuous and monotonic nondecreasing.*

17.8.2 Integral of absolutely integrable functions

Theorem 17.33: *A necessary and sufficient condition in order that a function $F : [a,b] \to \mathbb{R}$ can be written as*

$$F(t) = C + \int_a^t f(t)\,dt$$

for a constant C and an absolutely integrable *function f is that F is Vitali continuous.*

Corollary 17.34: *A necessary and sufficient condition in order that a function $F : [a,b] \to \mathbb{R}$ can be written as*

$$F(t) = C + \int_a^t f(t)\,dt$$

for a constant C and an absolutely integrable *function f is that*

1. *F is continuous on $[a,b]$.*
2. *F is absolutely continuous on (a,b).*
3. *$V(F,[a,b]) < \infty$.*

17.8.3 Integral of nonabsolutely integrable functions

Theorem 17.35: *Necessary and sufficient conditions in order that a function $F:[a,b]\to\mathbb{R}$ can be written as*

$$F(t) = C + \int_a^t f(t)\,dt$$

for a constant C and a nonabsolutely integrable *function f are that*

1. *F is continuous on $[a,b]$.*
2. *F is absolutely continuous on (a,b).*
3. *$V(F,[a,b]) = \infty$.*
4. *F is differentiable[a] almost everywhere in (a,b).*

[a]It is possible to show that when F is absolutely continuous on (a,b), F *must* be almost everywhere differentiable.

17.8.4 Proofs

The necessity of the conditions in the three theorems can be addressed first. Suppose that

$$F(t) = C + \int_a^t f(t)\,dt$$

for a constant C and an integrable function f.

If f is nonnegative then F is certainly nondecreasing We check that it is also Vitali continuous.

Let $f_n(x) = \min\{f(x), n\}$ and note that f_n is measurable and nonnegative, and that $\lim_{n\to\infty} f_n(x) = f(x)$ everywhere. Then, by the monotone convergence theorem, on every subinterval $[c,d]\subset[a,b]$,

$$0 < \int_c^d f(x)\,dx - \int_c^d f_n(x)\,dx < \int_c^d [f(x) - f_n(x)]\,dx \to 0.$$

Choose N so large that

$$\int_a^b f(x)\,dx < \int_a^b f_N(x)\,dx + \varepsilon/2.$$

Choose $\delta = \varepsilon/(2N)$. Then check that, if $[c_i, d_i]$ are nonoverlapping subintervals of $[a,b]$ with $\sum_i (d_i - c_i) < \delta$, then

$$0 \leq \sum_i [F(d_i) - F(c_i)] = \sum_i \int_{c_i}^{d_i} f(x)\,dx$$

$$\leq \sum_i \int_{c_i}^{d_i} f_N(x)\,dx + \varepsilon/2$$

$$\leq \sum_i N((d_i - c_i) + \varepsilon/2 < N\delta + \varepsilon/2 < \varepsilon.$$

This verifies that F is Vitali continuous.

If we assume instead that f is absolutely integrable we can again obtain the fact that F is Vitali continuous merely by splitting f into its positive and negative parts.

Finally, if f is merely integrable, then we already know that the relation

$$F(t) = C + \int_a^t f(t)\,dt$$

requires that F is continuous everywhere, and that F is absolutely continuous. The fundamental theorem of the calculus requires $F'(x) = f(x)$ almost everywhere in $[a,b]$. Thus each of the necessity parts of the three theorems is proved.

Conversely the stated conditions in the theorems are sufficient to verify that

$$F(t) = C + \int_a^t f(t)\,dt$$

for some function f as stated and constant C. For the third theorem we already know this from the fundamental theorem of the calculus.

That same theorem shows that the proof of the first theorem is also complete provided we know that F is differentiable almost everywhere and that $F'(x) \geq 0$ almost everywhere. But we already know that nondecreasing functions are almost everywhere differentiable. Take $f(x) = F'(x)$ at points where the derivative exists and $f(x) = 0$ elsewhere and the first theorem is proved.

We complete the proof of the second theorem in the same way. The assumption that F is Vitali continuous assures us that F is continuous and has bounded variation. So again F is almost everywhere differentiable and again the same argument supplies the representation.

Exercises

17.8.1 Show that a function that is Vitali continuous on $[a, b]$ must be uniformly continuous there.

17.8.2 Give an example of a uniformly continuous on an interval $[a, b]$ that is not Vitali continuous there.

17.8.3 Show that a function that is Lipschitz on $[a, b]$ is also Vitali continuous on $[a, b]$.

17.8.4 Given an example of a function that is not Lipschitz on $[a, b]$ but is Vitali continuous on $[a, b]$.

17.8.5 Show that a function that is Vitali continuous on $[a, b]$ must have bounded variation on $[a, b]$.

17.8.6 Show that if a function is Vitali continuous on $[a, b]$ then both parts of the Jordan decomposition have the same property on $[a, b]$.

17.8.7 Show that any continuously differentiable function on an interval $[a, b]$ is Vitali continuous on $[a, b]$.

SEE NOTE 266

17.8.8 Show that a differentiable function on an interval $[a, b]$ need not be Vitali continuous on $[a, b]$ but that it must be absolutely continuous in the more general sense (zero variation on zero measure sets).

17.8.9 Show that a function may be absolutely continuous but not Vitali continuous.

SEE NOTE 267

17.8.10 Let $F : \mathbb{R} \to \mathbb{R}$ and suppose that F is Vitali continuous on every compact interval $[a, b]$. Show that F is absolutely continuous.

SEE NOTE 268

17.8.11 Suppose that F, $f : [a, b] \to \mathbb{R}$, that f is bounded and integrable and that

$$F(t) = \int_a^b f(x)\,dx \quad (a \le t \le b).$$

Show directly that F is Vitali continuous on $[a, b]$.

SEE NOTE 269

17.8.12 Suppose that $F : [a, b] \to \mathbb{R}$ is absolutely continuous in $[a, b]$. Show that F is also absolutely continuous on $[a, b]$ in the sense of Vitali if and only if F has finite total variation on $[a, b]$, i.e., $V(F, [a, b] < \infty$.

17.8.13 (Fichtenholz) Suppose that $F : [a, b] \to \mathbb{R}$ satisfies the following condition: for every $\varepsilon > 0$ there is a $\delta > 0$ so that whenever $\{[c_i, d_i]\}$ is *any* sequence of subintervals of $[a, b]$ satisfying $\sum_i (d_i - c_i) < \delta$ then necessarily $\sum_i |F(d_i) - F(c_i)| < \varepsilon$. Show that this condition is strictly stronger than Vitali continuity.

SEE NOTE 270

17.8.14 Show that every Lipschitz function satisfies the condition of the preceding exercise.

17.8.15 Show that a function that satisfies the condition of the preceding exercises *must* be a Lipschitz function.

17.9 Denjoy's program

For nonabsolutely integrable functions the integral is not constructive by any of the methods of Lebesgue. If we know in advance that $F'(x) = f(x)$ everywhere, then certainly we can "construct" the value of the integral by using the formula

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

But even if we are assured that f is a derivative of some function, but we are not provided that function itself, then there may be no constructive method of determining either the value of the integral or the antiderivative function itself. This may surprise some calculus students since much of an elementary course is devoted to various methods of *finding* antiderivatives.

After Lebesgue's constructive integral was presented there still remained this problem. All *bounded* derivatives can be handled by his methods, but there exist unbounded derivatives that are nonabsolutely integrable. What procedure (outside of our formal integration theory) would handle these?

Starting with the class of absolutely integrable functions, Arnaud Denjoy discovered in 1912 that a series of extensions of this class could be constructed that would eventually encompass all derivatives and, indeed, all nonabsolutely integrable functions. The methods are beyond the scope of this text as they require not merely an ordinary sequence of extensions, but a transfinite sequence of extensions using infinite ordinal numbers. He called his process totalization. Added to Lebesgue's methods, totalization reveals exactly how constructive our integral is. His process completely catalogues the class of nonabsolutely integrable functions. In effect the integral that is discussed in this text could be (and has been) called *the Denjoy integral.*

17.10 Challenging Problems for Chapter 17

17.10.1 A function $f : [a, b] \to \mathbb{R}$ is a *step function* if there is a finite collection of intervals sets E_1, E_2, E_3, ..., E_n and real numbers r_1, r_2, r_3, ..., r_n so that

$$f(x) = \sum_{k=1}^{n} r_k \chi_{E_k}(x)$$

for all real x. [The intervals can be of any kind, open, closed, half open/closed, or even degenerate (i.e., containing a single point).] Show directly that, if $f : [a, b] \to \mathbb{R}$ is a step function, then f is integrable. What is the value of the integral?

17.10.2 A function is said to be *regulated* if it is a uniform limit of step functions. What is the relations among the classes of step functions, simple functions, regulated functions and measurable functions?

17.10.3 Show directly that, if $f : [a, b] \to \mathbb{R}$ is a regulated function, then f is integrable. How may the value of the integral be computed?

17.10.4 Show that a regulated function on an interval $[a, b]$ must be bounded and has both left and right hand limits at every point.

17.10.5 Show that a regulated function on an interval $[a, b]$ must be continuous at all but a countable set of points.

17.10.6 Show that a function on an interval $[a,b]$ that has both left and right hand limits at every point must be regulated.

17.10.7 Let $\{f_n\}$ be a sequence of absolutely integrable functions on an interval $[a,b]$ and suppose that

$$f(x) = \lim_{n\to\infty} f_n(x)$$

uniformly on $[a,b]$. Show that

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \int_a^b f_n(x)\,dx.$$

17.10.8 (Fatou's Lemma for nonnegative functions) Let $\{f_n\}$ be a sequence of nonnegative, measurable functions on an interval $[a,b]$ and suppose that $\liminf_{n\to\infty} f_n(x)$ is finite almost everywhere. . Show that

$$\int_a^b \liminf_{n\to\infty} f_n(x)\,dx \le \liminf_{n\to\infty} \int_a^b f_n(x)\,dx.$$

17.10.9 (Fatou's Lemma) Let $\{f_n\}$ be a sequence of measurable functions on an interval $[a,b]$ for which $\liminf_{n\to\infty} f_n(x)$ is finite almost everywhere. Suppose further that there is an integrable function $g : [a,b] \to \mathbb{R}$ such that $f_n(x) \ge g(x)$ for almost every x and for every n. Show that

$$\int_a^b \liminf_{n\to\infty} f_n(x)\,dx \le \liminf_{n\to\infty} \int_a^b f_n(x)\,dx.$$

17.10.10 (Bounded convergence theorem) Let $\{f_n\}$ be a sequence of measurable functions on an interval $[a,b]$ and suppose that

$$f(x) = \lim_{n\to\infty} f_n(x)$$

exists almost everywhere on $[a,b]$. Suppose further that $|f_n(x)| \le M$ for almost every x and for every n. Show that

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \int_a^b f_n(x)\,dx.$$

17.10.11 (Dominated convergence theorem) Let $\{f_n\}$ be a sequence of measurable functions on an interval $[a,b]$ and suppose that

$$f(x) = \lim_{n\to\infty} f_n(x)$$

exists almost everywhere on $[a,b]$. Suppose further that there is an integrable function $g : [a,b] \to \mathbb{R}$ such that $|f_n(x)| \leq g(x)$ for almost every x and for every n. Show that

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\int_a^b f_n(x)\,dx.$$

17.10.12 Let $F : \mathbb{R} \to \mathbb{R}$ be a continuous function. Show that F maps any $\mathcal{F}_\sigma$ subset of $[a,b]$ to another $\mathcal{F}_\sigma$ set. (See Section 6.6.2 for the definition of this class of sets.)

17.10.13 Show that every measurable set is a union of a $\mathcal{F}_\sigma$ set and a set of measure zero.

17.10.14 Let $F : \mathbb{R} \to \mathbb{R}$ be a continuous function that satisfies Lusin's condition N. Show that F maps measurable subsets of $\mathbb{R}$ to measurable sets. (See Exercise 12.8.8.)

See Note 271

17.10.15 Let $F : \mathbb{R} \to \mathbb{R}$ be a continuous function that maps measurable subsets of $\mathbb{R}$ to measurable sets. Show that F satisfies Lusin's condition N.

See Note 272

Notes

[265]Exercise 17.1.1. Use the subadditive property of open sets expressed in Lemma 11.3.

[266]Exercise 17.8.7. First use the mean-value theorem to get an inequality of the form $|F(x) - F(y) \leq M|x-y|$ that must hold for all x, y in the interval $[a,b]$.

[267]Exercise 17.8.9. Define $F : \mathbb{R} \to \mathbb{R}$ by $F(0) = F(1/(2n-1)) = 0$ and $F(1/2n) = 1/n$ for all $n = 1,2,3,\ldots$. Extend F to be linear on each of the intervals contiguous to these points where it has so far been defined. Show that F is absolutely continuous but that Vitali's condition does not hold on the interval $[0,1]$.

[268]Exercise 17.8.10. Let E be a null set. Write $E_n = E \cap (-n, n)$ for any integer n. We show that F has zero variation on E_n.

Using the δ, ε of the Vitali definition on the interval $[-n, n]$ cover E_n with a subsequence of open intervals $\{(c_i, d_i)\}$ with total length less than δ. Let β be the collection of all pairs $([u, v], w)$ for which $w \in E_n$ and $[u, v]$ is a subset of one at least of the open intervals $\{(c_i, d_i)\}$. This collection β is a full cover of E_n.

Let π be any subpartition contained in β. It must be the case, by the way that β has been constructed, that

$$\sum_{(([u,v],w)\in\pi} (v - u) < \delta.$$

Consequently

$$\sum_{(([u,v],w)\in\pi} |F(v) - F(u)| < \varepsilon.$$

From this it follows that F has zero variation on E_n. Since E is the union of the sequence of sets $\{E_n\}$ it follows too that F has zero variation on E.

[269]Exercise 17.8.11. Establish

$$|F(d) - F(c)| \leq M(d - c)$$

for some M and all $[c, d] \subset [a, b]$.

[270]Exercise 17.8.12. Perhaps hard to spot. Note that the condition does not specify that the intervals should be nonoverlapping.

[271]Exercise 17.10.14. Use the two preceding exercises.

[272]Exercise 17.10.15. You will need this fact: if E is a set that is not of measure zero then E must contain a set that is not measurable. (This cannot be proved without an appeal to certain logical principles not mentioned in the text.)

Chapter 18

STIELTJES INTEGRALS

Dripped Chapter

Recall that the *total variation* of a function F on a compact interval is the supremum of sums of the form

$$V(F,[a,b]) = \sum_{([u,v],w)\in\pi} |F(v)-F(u)|$$

taken over all possible partitions π of $[a,b]$. This a measure of the variability of the function F on this interval.

Functions of bounded variation play a significant role in real analysis. The earliest application was to the study of arc length of curves, a subject we will discuss as well.

Our main tool in the study of this important class of functions is a slight generalization of the integral, called the Stieltjes integral.[1]

[1] *Note to the instructor*: As part of the ***drip*** program we drop the Riemann-Stieltjes integral, too, in favor of a Stieltjes integral defined using the filter of full covers. This is more general than the Riemann-Stieltjes integral, but that is not the point. It is much easier to work with for the usual reason: the filter of full covers is more convenient than the filter of uniformly full covers that defines the Riemann-Stieltjes integral.

18.1 Stieltjes integrals

The definition of the total variation $V(F,[a,b])$ contains what looks very much like one of our Riemann sums, but in place of the usual sum

$$\sum_{([u,v],w)\in\pi} f(w)(v-u)$$

we are here checking values of the sum

$$\sum_{([u,v],w)\in\pi} |F(v)-F(u)|.$$

This might suggest to us that integration methods would prove a useful tool in the study of functions of bounded variation.

Let us, accordingly, enlarge the scope of our integration theory by considering limits of Riemann sums that are more general than we have used so far. Let $f,G:[a,b]\to\mathbb{R}$ and by analogy with

$$\int_a^b f(x)\,dx \sim \sum_{([u,v],w)\in\pi} f(w)(v-u)|$$

we introduce new integrals by making only the obvious changes suggested by the following slogans:

$$\int_a^b f(x)\,dG(x) \sim \sum_{([u,v],w)\in\pi} f(w)(G(v)-G(u))$$

$$\int_a^b f(x)\,|dG(x)| \sim \sum_{([u,v],w)\in\pi} f(w)|G(v)-G(u)|$$

$$\int_a^b f(x)\,[dG(x)]^+ \sim \sum_{([u,v],w)\in\pi} f(w)[G(v)-G(u)]^+$$

$$\int_a^b f(x)\,[dG(x)]^- \sim \sum_{([u,v],w)\in\pi} f(w)[G(v)-G(u)]^-$$

as well as a few other variants we consider in later sections:

$$\int_a^b \sqrt{|dG(x)|\,dx} \sim \sum_{([u,v],w)\in\pi} \sqrt{|G(v)-G(u)|(v-u)}$$

and

$$\int_a^b \sqrt{[dG(x)]^2+[dx]^2} \sim \sum_{([u,v],w)\in\pi} \sqrt{|G(v)-G(u)|^2+(v-u)^2}.$$

We will refer to all of these as *Stieltjes integrals*, although it is only the first variant of these,

$$\int_a^b f(x)\,dG(x),$$

that the Dutch mathematician Thomas Stieltjes (1856–1894) himself used and the one that most people would mean by the terminology.

18.1.1 Definition of the Stieltjes integral

The slogans in the preceding section should be enough to lead the reader to the correct definition of the various Stieltjes integral. Even so, let us give precise definitions for the simplest case. This is just a copying exercise: take the usual definition and repeat it with the Riemann sums adjusted in the manner required.

Definition 18.1: For functions G, $f:[a,b]\to\mathbb{R}$ we define an *upper integral* by

$$\overline{\int_a^b} f(x)\,dG(x) = \inf_{\beta} \sup_{\pi\subset\beta} \sum_{([u,v],w)\in\pi} f(w)(G(v)-G(u))$$

where the supremum is taken over all partitions π of $[a,b]$ contained in β, and the infimum over all full covers β.

Similarly we define a *lower integral*, as

$$\underline{\int_a^b} f(x)\,dG(x) = \sup_{\beta} \inf_{\pi\subset\beta} \sum_{([u,v],w)\in\pi} f(w)(G(v)-G(u))$$

where, again, π is a partition of $[a,b]$ and β is a full cover.

If the upper and lower integrals are identical we say the integral is *determined* and we write the common value as

$$\int_a^b f(x)\,dG(x).$$

We are interested, mostly, in the case in which the integral is determined and finite.

Exercises

18.1.1 Let $G : [a,b] \to \mathbb{R}$. Show that

$$\int_a^b dG(x) = G(b) - G(a).$$

18.1.2 Let $G : \mathbb{R} \to \mathbb{R}$ defined so that $G(x) = 0$ for all $x \neq 0$ and $G(1) = 1$. Compute

$$\overline{\int_0^2} |dG(x)| \quad \text{and} \quad \underline{\int_0^2} |dG(x)|.$$

18.1.3 Let $G : [0,1] \to \mathbb{R}$ and let $f(x) = 0$ for all $x \neq 1/2$ with $f(1/2) = 1$. What are

$$\overline{\int_0^1} f(x)\,dG(x) \text{ and } \underline{\int_0^1} f(x)\,dG(x)?$$

18.1.4 Let $G,\ f : [0,1] \to \mathbb{R}$ and let $G(x) = 0$ for all $x \leq 1/2$ and with $G(x) = 1$ for all $x > 1/2$. What are

$$\overline{\int_0^1} f(x)\,dG(x) \text{ and } \underline{\int_0^1} f(x)\,dG(x)?$$

See Note 273

18.1.5 Let G, $f : [a,b] \to \mathbb{R}$ and let f be continuous and let G be a step function, i.e. there are points

$$a < \xi_1 < \xi_2 < \cdots < \xi_m < b$$

so that G is constant on each interval (ξ_{i-1}, ξ_i). What are possible values for

$$\overline{\int_a^b} f(x)\, dG(x) \text{ and } \underline{\int_a^b} f(x)\, dG(x)?$$

See Note 274

18.1.6 Let G, $F : [-1,1] \to \mathbb{R}$ be defined by $F(x) = 0$ for $-1 \leq x < 0$, $F(x) = 1$ for $0 \leq x \leq 1$, $G(x) = 0$ for $-1 \leq x \leq$, and $G(x) = 1$ for $0 < x \leq 1$. Discuss $\int_{-1}^{1} F(x)\, dG(x)$ and $\int_{-1}^{1} G(x)\, dF(x)$.

See Note 275

18.1.7 If $a < b < c$ is the formula

$$\int_a^b f(x)\, dG(x) + \int_b^c f(x)\, dG(x) = \int_a^c f(x)\, dG(x)$$

valid?

See Note 276

18.1.8 Show that a function f can be altered at a finite number of points where G is continuous without altering the values of the upper and lower integrals. Give an example to show that continuity may not be dropped here.

18.1.9 Show that a function f can be altered at a countable number of points where G is continuous without altering the values of the upper and lower integrals.

18.1.10 Give a Cauchy I criterion for $\int_a^b f(x)\, dG(x)$.

18.1.11 Give a Cauchy II criterion for $\int_a^b f(x)\, dG(x)$.

18.1.12 Give a McShane criterion for $\int_a^b f(x)\, dG(x)$.

18.1.13 Give a Henstock criterion for $\int_a^b f(x)\, dG(x)$.

18.1.14 For integrals of the form $\int_a^b f(x)\,|dG(x)|$ what changes have to be made in the various criteria?

SEE NOTE 277

18.1.15 For integrals of the form $\int_a^b f(x)\,[dG(x)]^+$ what changes have to be made in the various criteria?

18.1.16 Let $F:[0,2]\to\mathbb{R}$ with $F(t)=0$ for all $t\neq 1$ and $F(1)=1$. Show that

$$\underline{\int_0^2}|dF(x)| < \overline{\int_0^2}|dF(x)| = V(F,[0,2]).$$

18.1.17 Let $F:[a,b]\to\mathbb{R}$. Show that the total variation of F can be expressed as an upper integral:

$$V(F,[a,b]) = \overline{\int_a^b}|dF(x)|.$$

18.1.18 Let $F:[a,b]\to\mathbb{R}$ and suppose that one at least of the integrals

$$\overline{\int_a^b}|dF(x)|\,,\ \ \overline{\int_a^b}[dF(x)]^+ \ \text{ or } \ \overline{\int_a^b}[dF(x)]^-$$

is finite. Show that F is a function of bounded variation on $[a,b]$ and that, for all $a<t\leq b$,

$$F(t)-F(a) = \overline{\int_a^t}[dF(x)]^+ - \overline{\int_a^t}[dF(x)]^-. \tag{1}$$

The identity (1) is a representation of F as a difference of two nondecreasing functions.

18.1.19 Let $F:[a,b]\to\mathbb{R}$ be a continuous function. Show that F has bounded variation on $[a,b]$ if and only if there is a continuous, strictly increasing function $G:[a,b]\to\mathbb{R}$ for which $F(d)-F(c)<G(d)-G(c)$ for all $a\leq c<d\leq b$.

18.1.20 What basic properties of the ordinary integral $\int_a^b f(x)\,dx$ from Chapter 10 can you prove for Stieltjes integrals without any but the most obvious of changes in the proofs?

18.1.2 Henstock's zero variation criterion

Since the Stieltjes integral is defined in exactly the same way as the ordinary integral one expects almost the same properties. Indeed this integral has the same linear, additive, and monotone properties (suitably expressed). There also must be an indefinite integral. Finally, the most important of these properties that carries over, is the Henstock criterion. We give that now.

Theorem 18.2: *Let F, G, $f : [a,b] \to \mathbb{R}$. Then a necessary and sufficient condition for the existence of the Stieltjes integral and the formula*

$$\int_c^d f(x)\,dG(x) = F(d) - F(c) \quad [c,d] \subset [a,b]$$

is that

$$\int_a^b |dF(x) - f(x)\,dG(x)| = 0.$$

The proof would merely be a copying exercise of material from Section 10.2.6. Note that we are taking advantage of our general Stieltjes notation here to allow us to interpret the integral

$$\int_a^b |dF(x) - f(x)\,dG(x)|$$

as a limit of the Riemann sums

$$\sum_{([u,v],w)\in\pi} |F(v) - F(u) - f(x)[G(v) - G(u)]|\,.$$

18.2 Regulated functions

We say $F(c+)$ exists if, for all sequences of positive numbers t_n tending to zero,

$$\lim_{n\to\infty} F(c + t_n) = F(c+).$$

Similarly, we say $F(c-)$ exists if, for all sequences of positive numbers t_n tending to zero,

$$\lim_{n\to\infty} F(c-t_n) = F(c-).$$

Definition 18.3: Let $F : [a,b] \to \mathbb{R}$. Then

- F is said to be *regulated* if the one-sided limit $F(c+)$ exists and is finite for all $a \le c < b$ and the limit on the other side $F(c-)$ exists and is finite for all $a < c \le b$.
- F is said to be *naturally regulated* if F is regulated and, for all $a < c < b$, either
$$F(c+) \le F(c) \le F(c-)$$
or else
$$F(c-) \le F(c) \le F(c+).$$

Theorem 18.4: *Let $F : [a,b] \to \mathbb{R}$ be monotonic. Then F is naturally regulated.*

Proof. Simply notice that

$$F(c-) = \sup\{F(t) : a \le t < c\} \le F(c)$$
$$\le \inf\{F(t) : c < t \le b\} = F(c+).$$

for all $a < c < b$. ■

Theorem 18.5: *Let $F : [a,b] \to \mathbb{R}$ be a function of bounded variation. Then F is regulated and has at most countably many discontinuities*[a]*.*

[a] In fact it can be proved that *all* regulated functions have at most countably many discontinuities.

Proof. Suppose that $a < c \le b$ and $F(c-)$ does not exist. Then there is a positive number ε and a sequence of numbers c_n increasing to c so that, for all n,

$$F(c_n) - F(c_{n+1}) < -\varepsilon < \varepsilon < F(c_{n+2}) - F(c_{n+1}).$$

But then, for all m,

$$\infty > V(F,[a,b]) \geq \sum_{n=1}^{m} |F(c_n) - F(c_{n+1})| > m\varepsilon.$$

This is impossible. Similarly $F(c+)$ must exist for all $a \leq c < b$.

Let us show that there are only countably many points $c \in [a,b)$ for which $F(c) \neq F(c+)$. Let c_1, c_2, $\dots c_m$ denote a set of points from (a,b) for which $|F(c_m+) - F(c)| > 1/n$. Then there is a disjointed collection of intervals $[c_i, t_i]$ for which

$$|F(t_i) - F(c_i)| > 1/(2n).$$

In particular

$$\infty > V(F,[a,b]) \geq \sum_{i=1}^{m} |F(t_i) - F(c_i)| > m/(2n).$$

Thus there are only finitely many such choices of points c_1, c_2, $\dots c_m$ for which $|F(c_m+) - F(c_m)| > 1/n$. It follows that there are only countably many choices of points c_i for which $|F(c_i+) - F(c_i)| > 0$. A similar argument handles the points $c \in (a,b)]$ for which $F(c) \neq F(c-)$. It follows that the set of points of discontinuity must be countable. ■

Lemma 18.6 (Approximate additivity) *Suppose that $F : [a,b] \to \mathbb{R}$ is a function that is naturally regulated. Then at any point $a < c < b$, and for any $\varepsilon > 0$ there is $\delta > 0$ so that, for all $c - \delta < u < c < v < c + \delta$,*

$$|F(v) - F(c)| + |F(c) - F(u)| \geq |F(v) - F(u)|$$

and

$$|F(v) - F(u)| \geq |F(v) - F(c)| + |F(c) - F(u)| - \varepsilon. \tag{2}$$

Proof. Since F is naturally regulated we know that

$$|F(c+) - F(c-)| = |F(c+) - F(c)| + |F(c-) - F(c)|$$

for each $a < c < b$. At such points there is a $\delta > 0$ so that

$$|F(u) - F(c-)| < \varepsilon/4 \text{ and } |F(v) - F(c+)| < \varepsilon/4$$

for all $c - \delta < u < c < v < c + \delta$. In particular

$$|F(c+) - F(c-)| \leq |F(c+) - F(v)| + |F(v) - F(u)| + |F(u) - F(c-)|$$
$$\leq |F(v) - F(u)| + \varepsilon/2$$

and so

$$|F(v) - F(c)| + |F(c) - F(u)| \leq$$
$$|F(v) - F(c+)| + |F(c+) - F(c)| + |F(c-) - F(c)| + |F(c-) - F(u)|$$
$$\leq |F(c+) - F(c-)| + \varepsilon/2 \leq |F(v) - F(u)| + \varepsilon.$$

Thus

$$|F(v) - F(u)| \geq |F(v) - F(c)| + |F(c) - F(u)| - \varepsilon.$$

The other inequality

$$|F(v) - F(c)| + |F(c) - F(u)| \geq |F(v) - F(u)|$$

is obviously true. ■

18.3 Variation expressed as an integral

Lemma 18.7: *Suppose that $F : [a, b] \to \mathbb{R}$ is a function of bounded variation that is naturally regulated. Then*

$$V(F, [a, b]) = \int_a^b |dF(x)|.$$

Proof. It is clear that

$$V(F, [a, b]) \geq \overline{\int_a^b} |dF(x)|.$$

In fact these are equal for all functions, but we do not need that. Let $\varepsilon > 0$ and select points

$$a = s_0 < s_1 < \cdots < s_{n-1} < s_n = b$$

so that

$$\sum_{i=1}^{n} |F(s_i) - F(s_{i-1})| > V(F, [a, b]) - \varepsilon.$$

Define a covering relation β to include only those pairs $([u, v], w)$ for which either $w \neq s_1, s_2, \ldots, s_{n-1}$ and $[u, v]$ contains no point $s_1, s_2, \ldots, s_{n-1}$, or else $w = s_i$ for some $i = 1, 2, \ldots, n-1$ and

$$|F(v) - F(u)| \geq |F(v) - F(s_i)| + |F(s_i) - F(u)| - \varepsilon/n. \tag{3}$$

It is clear that β is full at every point w. For points $w \neq s_1, s_2, \ldots, s_{n-1}$ this is transparent, while for points $w = s_i$ for some $i = 1, 2, \ldots, n-1$, Lemma 18.6 may be applied.

We use a standard endpointed argument. Take any partition π of $[a, b]$ chosen from β. Scan through π looking for any elements of the form $([u, v], s_i)$ for $u < s_i < w$ and $i = 1, 2, \ldots, n-1$. Replace each one by the new elements $([u, s_i], s_i)$ and $([s_i, v], s_i)$. Call the new partition π'. Because of (3) we see that

$$\sum_{([u,v],w)\in\pi} |F(v) - F(u)| \geq \sum_{([u,v],w)\in\pi'} |F(v) - F(u)| - \varepsilon.$$

Write $\pi_i = \pi'([s_{i-1}, s_i])$ and note that, by the way we have arranged π', each π_i is a partition of the interval $[s_{i-1}, s_i]$. Consequently

$$\sum_{([u,v],w)\in\pi} |F(v) - F(u)| \geq \sum_{([u,v],w)\in\pi'} |F(v) - F(u)| - \varepsilon$$

$$\geq \sum_{i=1}^{n} \sum_{([u,v],w)\in\pi_i} |F(v) - F(u)| - \varepsilon$$

$$\geq \sum_{i=1}^{n} |F(s_i) - F(s_{i-1})| - \varepsilon > V(F, [a, b]) - 2\varepsilon.$$

We have shown that for *every* partition π of $[a,b]$ contained in β this sum is larger than $V(F,[a,b])-2\varepsilon$. It follows that

$$\underline{\int_a^b} |dF(x)| \geq V(F,[a,b]) - 2\varepsilon.$$

Since ε is arbitrary the inequality

$$V(F,[a,b]) \leq \underline{\int_a^b} |dF(x)| \leq \overline{\int_a^b} |dF(x)| \leq V(F,[a,b])$$

must hold and the theorem is proved. ■

Corollary 18.8: *Suppose that $F:[a,b]\to\mathbb{R}$ is a function of bounded variation that is naturally regulated. Then*

$$V(F,[a,b]) = \int_a^b |dF(x)| = \int_a^t [dF(x)]^+ + \int_a^t [dF(x)]^-.$$

Proof. The proof of the lemma can easily be adjusted to prove that all three of these integrals must exist. The identity is trivial: the expression

$$dF(x) = [dF(x)]^+ + [dF(x)]^-$$

integrated over $[a,b]$ produces the required identity. ■

The role of the naturally regulated assumption is exhibited in Exercise 18.1.16. It can be checked that if a function is not naturally regulated then the integral is not determined and the variation must be displayed using the upper integrals.

18.4 Representation theorems for functions of bounded variation

18.4.1 Jordan decomposition

The structure of functions of bounded variation is particularly simplified by a theorem of Jordan: every function of bounded variation is merely a linear combination of monotonic functions. We prove this for functions that are naturally regulated, by interpreting the statement as an integration assertion about certain Stieltjes integrals. The statement is true in general for all functions of bounded variation, but then the upper integrals would be needed (cf. Exercise 18.1.18).

Theorem 18.9: *Let $F : [a,b] \to \mathbb{R}$ be a function of bounded variation and suppose that F is naturally regulated. Then, for all $a < t \leq b$,*

$$F(t) - F(a) = \int_a^t [dF(x)]^+ - \int_a^t [dF(x)]^-. \tag{4}$$

The identity (4) is a representation of F as a difference of two functions, both nondecreasing, both naturally regulated.

Proof. The existence of the integrals is given in Corollary 18.8. The identity is trivial: the expression

$$dF(x) = [dF(x)]^+ - [dF(x)]^-$$

integrated over $[a,b]$ produces the required identity. ■

Corollary 18.10: *Let $F : [a,b] \to \mathbb{R}$ be a function of bounded variation and suppose that F is continuous. Then, for all $a < t \leq b$,*

$$F(t) - F(a) = \int_a^t [dF(x)]^+ - \int_a^t [dF(x)]^-. \tag{5}$$

The identity (5) is a representation of F as a difference of two functions, both continuous and nondecreasing.

18.4.2 Jordan decomposition theorem: differentiation

We know that all functions of bounded variation and all monotonic functions are almost everywhere differentiable. This and the integral representation given in Theorem 18.9 allows the following corollary.

Corollary 18.11: *Let $F : [a,b] \to \mathbb{R}$ be a function of bounded variation and suppose that F is naturally regulated. Write*

$$F_1(t) = \int_a^t [dF(x)]^+ \quad (a \leq t \leq b), \tag{6}$$

and

$$F_2(t) = \int_a^t [dF(x)]^- \quad (a \leq t \leq b), \tag{7}$$

Then

$$F(t) - F(a) = F_1(t) - F_2(t) \quad \textit{and} \quad T(t) = V(F,[a,t]) = F_1(t) + F_2(t).$$

Moreover, at almost every t in $[a,b]$,

$$F'(t) = F_1'(t) - F_2'(t), \quad F_1'(t) = \max\{F'(t), 0\}, \quad F_2'(t) = \max\{-F'(t), 0\},$$

$$T'(t) = F_1'(t) + F_2'(t) = |F'(t)| \quad \textit{and} \quad F_1'(t)F_2'(t) = 0.$$

Proof. There are three tools needed for the differentiation statements: the Lebesgue differentiation theorem (that monotonic functions have derivatives a.e.), the Henstock zero variation criterion for integrals, and the zero variation implies zero derivative a.e. rule.

We illustrate with a proof for one of the statements in the corollary. Define

$$h([u,v],w) = F_1(v) - F_1(u) - [F(v) - F(u)]^+.$$

The identity $F_1(t) = \int_a^t [dF(x)]^+$ requires that h have zero variation on (a,b). This, in term, requires that

$$\lim_{h\to 0+} \frac{F_1(t+h) - F_1(t) - \max\{F(t+h) - F(t), 0\}}{h}$$

$$= \lim_{h\to 0+} \frac{F_1(t) - F_1(t-h) - \max\{F(t) - F(t-h), 0\}}{h} = 0$$

for almost every t in (a,b). From that we deduce that $F_1'(t) = \max\{F'(t), 0\}$ must be true for almost every t in (a,b). Proofs for the other statements are similar. ■

18.4.3 Representation by saltus functions

Theorem 18.12: *Let $F : [a, b] \to \mathbb{R}$ be a monotonic nondecreasing function and let C be the set of points of continuity of F in $[a, b]$. Then, for all $a < t \le b$,*

$$F(t) - F(a) = \int_a^t \chi_C(x)\, dF(x) + \int_a^t [1 - \chi_C(x)]\, dF(x). \tag{8}$$

and

$$\int_a^t [1 - \chi_C(x)]\, dF(x) = [F(t) - F(t-)] + \sum_{s \in [a,t) \setminus C} [F(s+) - F(s-)]$$

The identity (8) is a representation of F as a sum of two functions, the first continuous and nondecreasing, the second a saltus function.

18.4.4 Representation by singular functions

Theorem 18.13: *Let $F : [a, b] \to \mathbb{R}$ be a continuous monotonic function. Let D be the set of points of differentiability of F in $[a, b]$. Then*

$$F(t) - F(a) = \int_a^t \chi_D(x)\, dF(x) + \int_a^t [1 - \chi_D(x)]\, dF(x) \tag{9}$$

and

$$\int_a^t \chi_D(x)\, dF(x) = \int_a^t F'(x)\, dx.$$

The identity (9) is a representation of F as a sum of two monotonic functions, the first Vitali continuous and the second a continuous singular function.

18.5 Reducing a Stieltjes integral to an ordinary integral

The Stieltjes integral reduces to an ordinary integral in a number of interpretations. When the integrating function G is an indefinite integral the whole theory reduces to ordinary integration. The formula is compelling since, as calculus students often learn,

$$dG(x) = G'(x)\,dx$$

can be assigned a meaning. That meaning is convenient here too and suggests that

$$\int_a^b f(x)\,dG(x) = \int_a^b f(x)G'(x)\,dx.$$

Theorem 18.14: *Suppose that G, f, $g : \mathbb{R} \to \mathbb{R}$ and that g is integrable on a compact interval $[a,b]$ with an indefinite integral*

$$G(d) - G(c) = \int_c^d g(x)\,dx \quad (a \leq c < d \leq b).$$

Then the Stieltjes integral

$$\int_a^b f(x)\,dG(x)$$

exists if and only if fg is integrable on $[a,b]$, in which case

$$\int_a^b f(x)\,dG(x) = \int_a^b f(x)g(x)\,dx.$$

Proof. The proof depends simply on the Henstock criterion. The existence of the ordinary integral

$$\int_a^b g(x)\,dx$$

with an indefinite integral G is equivalent to the zero criterion:

$$\int_a^b |dG(x) - g(x)\,dx| = 0$$

Whenever this identity holds, then one checks that, for any function f,

$$\int_a^b |f(x)dG(x) - f(x)g(x)\,dx| = 0$$

would also be true. For example, if we have a bounded f this is trivial; for unbounded one only has to split $[a,b]$ into the sequence of sets

$$\{x \in [a,b] : n-1 \leq |f(x)| < n\}$$

and argue on each of these (cf. Exercise 18.6.2).

The existence of the Stieltjes integral

$$\int_a^b f(x)\,dG(x)$$

with an indefinite integral F is equivalent to the zero criterion:

$$\int_a^b |dF(x) - f(x)\,dG(x)| = 0.$$

Together these give

$$\int_a^b |dF(x) - f(x)g(x)\,dx| \leq$$

$$\int_a^b |dF(x) - f(x)\,dG(x)| + \int_a^b |f(x)dG(x) - f(x)g(x)\,dx| = 0.$$

From this it is easy to read off the required identity. ■

18.6 Properties of the indefinite integral

Theorem 18.15: *Suppose that*

$$F(t) = \int_a^t f(x)\,dG(x) \quad (a \le t \le b).$$

Then

1. *F is continuous at every point at which G is continuous.*

2. *F is absolutely continuous in any set $E \subset (a,b)$ in which G is absolutely continuous.*

3. *F has zero variation on any set $E \subset (a,b)$ on which G has zero variation.*

4. *F has bounded variation on $[a,b]$ if f is bounded and if G has bounded variation.*

5. *If G is Vitali continuous on $[a,b]$ and if f is bounded then F is also Vitali continuous on $[a,b]$.*

6. *If G is a saltus function on $[a,b]$ and f is nonnegative then so too is the indefinite integral F. Moreover the jumps of F occur precisely at points that are jumps of G for which f does not vanish.*

7. *For almost every point x in $[a,b]$*
$$\lim_{y\to x} \frac{F(y) - F(x) - f(x)(G(y) - G(x))}{y - x} = 0.$$

8. *For almost every point x in $[a,b]$,*
$$\overline{D}F(x) = f(x)\overline{D}G(x) \quad \text{and} \quad \underline{D}F(x) = f(x)\underline{D}G(x)$$
or else
$$\overline{D}F(x) = f(x)\underline{D}G(x) \quad \text{and} \quad \underline{D}F(x) = f(x)\overline{D}G(x)$$
depending on whether $f(x) \ge 0$ or $f(x) \le 0$.

9. *In particular, $F'(x) = f(x)G'(x)$ at almost every point x at which either F or G is differentiable.*

10. *Finally, $F'(x) = 0$ at almost every point x where $f(x) = 0$.*

Proof. The proof for each of these depends simply on the Henstock criterion. The existence of the Stieltjes integral

$$\int_a^b fx)\,dG(x)$$

with an indefinite integral F is equivalent to the zero criterion:

$$\int_a^b |dF(x) - f(x)\,dG(x)| = 0$$

From the latter will flow each of the statements of the theorem. The individual proofs are left in the Exercises to the reader. ■

Exercises

18.6.1 Suppose that

$$\int_a^b |dF(x) - f(x)\,dx| = 0.$$

Show that if g is any bounded function on $[a, b]$ then

$$\int_a^b |g(x)dF(x) - f(x)g(x)\,dx| = 0.$$

18.6.2 Suppose that

$$\int_a^b |dF(x) - f(x)\,dx| = 0.$$

Show that if g is any real-valued function on $[a, b]$ then

$$\int_a^b |g(x)dF(x) - f(x)g(x)\,dx| = 0.$$

18.6.3 Suppose that

$$\int_a^b |dF(x) - f(x)\,dG(x)| = 0.$$

Show that F is continuous at any point at which G is continuous. Is the converse necessarily true?

18.6.4 Suppose that

$$\int_a^b |dF(x) - f(x)\,dG(x)| = 0.$$

Show that F has zero variation on any set on which G has zero variation. Is the converse necessarily true?

18.6.5 Suppose that

$$\int_a^b |dF(x) - f(x)\,dG(x)| = 0$$

and suppose that G has bounded variation on $[a, b]$ and that f is bounded. Show that F has bounded variation on $[a, b]$.

18.6.6 Suppose that

$$\int_a^b |dF(x) - f(x)\,dG(x)| = 0.$$

Show that

$$\lim_{y \to x} \frac{F(y) - F(x) - f(x)(G(y) - G(x))}{y - x} = 0$$

almost everywhere by using the zero variation implies zero derivative criterion.

18.6.7 Complete the remaining arguments needed to establish the parts of the theorem.

18.6.8 Suppose that

$$\int_a^b |dF(x) - f(x)\,dG(x)| = 0.$$

Show that, for every point x in $[a, b]$

$$\lim_{y \to x} \frac{F(y) - F(x)}{G(y) - G(x)} = f(x)$$

except perhaps for points x in a set N in which G has *fine* variation zero.

18.6.9 Suppose that at every point x of a compact interval $[a, b]$

$$\lim_{y\to x} \frac{F(y) - F(x) - f(x)[G(y) - G(x)]}{y - x} = 0.$$

Show that

$$\int_a^b |dF(x) - f(x)\, dG(x)| = 0.$$

18.6.10 Suppose that at every point x of a compact interval $[a, b]$

$$\lim_{y\to x} \frac{F(y) - F(x) - f(x)[G(y) - G(x)]}{y - x} = 0$$

except for points x in a set N for which both F and G have zero variation. Show that

$$\int_a^b |dF(x) - f(x)\, dG(x)| = 0.$$

18.6.11 Suppose that

$$\int_a^b |dF(x) - f(x)\, dG(x)| = 0.$$

Show that, at almost every point x,

$$\overline{D}F(x) = f(x)\overline{D}G(x) \quad \text{and} \quad \underline{D}F(x) = f(x)\underline{D}G(x)$$

if $f(x) \geq 0$ while

$$\overline{D}F(x) = f(x)\underline{D}G(x) \quad \text{and} \quad \underline{D}F(x) = f(x)\overline{D}G(x)$$

if $f(x) \leq 0$. In particular $F'(x) = 0$ at almost every point x where $f(x) = 0$.

18.6.1 Existence of the integral from derivative statements

The existence of the integral

$$\int_a^b f(x)\, dG(x)$$

can be deduced from a variety of differentiation statements. For example, using Exercise 18.6.10, we can prove the following simple version:

Theorem 18.16: *Suppose that at every point x of a compact interval $[a,b]$*

$$\lim_{y\to x}\frac{F(y)-F(x)-f(x)[G(y)-G(x)]}{y-x}=0$$

except for points x in a set N for which both F and G have zero variation. Then the Stieltjes integral exists and

$$\int_a^b f(x)\,dG(x)=F(b)-F(a).$$

18.7 Existence of the Stieltjes integral for continuous functions

Theorem 18.17: *Let f, $G:\mathbb{R}\to\mathbb{R}$ and suppose that f is continuous on a compact interval $[a,b]$ and that G is monotonic nondecreasing throughout that interval. Then the Stieltjes integral exists and*

$$\left|\int_a^b f(x)\,dG(x)\right|\le\|f\|_\infty[G(b)-G(a)].$$

where $\|f\|_\infty=\max_{t\in[a,b]}|f(t)|$.

Proof. The inequality is easy since, for any pair $([u,v],w)$ with $[u,v]\subset[a,b]$,

$$|f(w)(G(v)-G(u)|\le\|f\|_\infty[G(v)-G(u)]. \tag{10}$$

To prove that the integral exists we merely copy the same proof for the ordinary integral that uses the McShane criterion (Theorem 10.10). The details are left as an exercise. ■

Theorem 18.18: *Let f, $G:\mathbb{R}\to\mathbb{R}$ and suppose that f is continuous on a compact interval $[a,b]$ and that G has bounded variation throughout that interval. Then the Stieltjes integral exists and*

$$\left|\int_a^b f(x)\,dG(x)\right|\le\|f\|_\infty V(G,[a,b]).$$

where $\|f\|_\infty=\max_{t\in[a,b]}|f(t)|$.

Proof. Again the methods of Theorem 10.10 can be used here with only minor modifications to show that the integral exists. The inequality follows, once again, from (10). ■

18.8 Integration by parts

Integration by parts for the Stieltjes integral assumes the following form[2]:

Theorem 18.19: *Let F, $G : \mathbb{R} \to \mathbb{R}$. Then*

$$\int_a^b [F(x)\,dG(x) + G(x)\,dF(x)] = F(b)G(b) - F(a)G(a) - \int_a^b dF(x)\,dG(x)$$

in the sense that if one of the integrals exists, so too does the other with the stated identity.

Proof. First check a simple identity: that, for any u and v,

$$F(u)[G(v) - G(u)] + G(u)[F(v) - F(u)]$$
$$= F(v)G(v) - G(u)G(u) - [F(v) - F(u)][G(v) - G(u).$$

This suggests that

$$\int_a^b |F(x)\,dG(x) + G(x)\,dF(x) - dF(x)\,dG(x) - dF(x)\,dG(x)| = 0 \tag{11}$$

is simply true because of an identity. If indeed this is true then the statement in the theorem is obvious because

$$\int_a^b dF(x)dG(x) = F(b)G(b) - F(a)G(a).$$

To complete the proof we have to address just one concern here. If a partition π of the interval $[a, b]$ contains only pairs $([u, v], u)$ or $([u, v], v)$ [i.e., $([u, w], w)$ with w only at an endpoint] then our simple

[2]For the *Riemann*-Stieltjes integral the extra term $\int_a^b dF(x)\,dG(x)$ does not appear, since this would be zero whenever the integral exists in that sense. (See Corollary 18.21, which should look familiar to fans of the Riemann-Stieltjes integral.)

identity would indeed supply

$$\sum_{([u,v],w)\in\pi} [F(w)[G(v)-G(u)] + G(w)[F(v)-F(u)] - F(v)G(v) - G(u)G(u)]$$

$$= \sum_{([u,v],w)\in\pi} [F(v)-F(u)][G(v)-G(u)].$$

That surely proves (11) if we are allowed to use only such partitions. But what happens if we permit (as we must) partitions π containing a pair $([u,v],w)\in\pi$ for which $u<w<v$?

To clear this up note that we can always adjust full covers and partitions π by replacing any pair $([u,v],w)\in\pi$ for which $u<w<v$ by the two items $([u,w],w)$ and $([w,v],w)$. That does not change the sums here because, for example,

$$F(w)[G(v)-G(u)] = F(w)[G(w)-G(u)] + F(w)[G(v)-G(w)].$$

This "endpointed" argument (which we have seen before in Exercise 10.1.7) means that in these simple Stieltjes integrals the partitions used can all be restricted to ones where only elements of the form $([u,v],u)$ or $([u,v],v)$ can appear. ■

Corollary 18.20: *Let F, $G:\mathbb{R}\to\mathbb{R}$ and suppose that*

$$\int_a^b |dF(x)\,dG(x)| = 0.$$

Then

$$\int_a^b [F(x)\,dG(x) + G(x)\,dF(x)] = F(b)G(b) - F(a)G(a).$$

If, in addition one of the following two integrals exists then so too does the other and

$$\int_a^b F(x)\,dG(x) + \int_a^b G(x)\,dF(x) = F(b)G(b) - F(a)G(a).$$

Corollary 18.21: *Let F, $G : \mathbb{R} \to \mathbb{R}$ and suppose that F is continuous and G has bounded variation. Then*
$$\int_a^b F(x)\,dG(x) + \int_a^b G(x)\,dF(x) = F(b)G(b) - F(a)G(a).$$

Proof. The assumption that F is continuous and G has bounded variation requires that
$$\int_a^b |dF(x)\,dG(x)| = 0.$$
Thus Theorem 18.19 can be applied. But we know, from Theorem 18.17, that the integral $\int_a^b F(x)dG(x)$ must exist. It follows, from Corollary 18.20, that $\int_a^b G(x)dF(x)$ must also exist and that the integration by parts formula is valid. ■

18.9 Mutually singular functions

Definition 18.22: Let F, $G : [a, b] \to \mathbb{R}$ be functions of bounded variation. Then F and G are said to be *mutually singular* provided that
$$\int_a^b \sqrt{|dF(x)\,dG(x)|} = 0.$$

Lemma 18.23: *Let F, $G : [a, b] \to \mathbb{R}$ be functions of bounded variation. If F and G are mutually singular, then $F'(x)G'(x) = 0$ almost everywhere in $[a, b]$.*

Proof. This follows easily (as usual) from the zero variation implies zero derivative a.e. rule together with the fact that both $F'(x)$ and $G'(x)$ must exist a.e.. ■

Our main theorem shows that mutually singular functions grow on separate parts of the interval $[a, b]$ in a sense made precise here.

Theorem 18.24: *Let F, $G : [a,b] \to \mathbb{R}$ be functions of bounded variation. Then F and G are mutually singular on $[a,b]$ if and only for every $\varepsilon > 0$ there is a full cover β with the property that every partition π of $[a,b]$ contained in β can be split into two disjoint subpartitions $\pi = \pi' \cup \pi''$ so that*

$$\sum_{([u,v],w)\in\pi'} |F(v) - F(u)| < \varepsilon$$

and

$$\sum_{([u,v],w)\in\pi''} |G(v) - G(u)| < \varepsilon.$$

Proof. Suppose that

$$\int_a^b \sqrt{|dF(x)dG(x)|} = 0.$$

Let $\varepsilon > 0$ and select a full cover β so that

$$\sum_{([u,v],w)\in\pi} \sqrt{|[F(v) - F(u)][G(v) - G(u)]|} < \varepsilon$$

for all partitions π of $[a,b]$ contained in β. Split such a π as follows:

$$\pi' = \{([u,v],w) : |[F(v) - F(u)]| \leq |[G(v) - G(u)]|\}$$

and

$$\pi'' = \{([u,v],w) : |[F(v) - F(u)]| > |[G(v) - G(u)]|\}.$$

Verify that $\pi = \pi' \cup \pi''$ and that

$$\sum_{([u,v],w)\in\pi'} |[F(v) - F(u)]| \leq \sum_{([u,v],w)\in\pi'} \sqrt{|[F(v) - F(u)][G(v) - G(u)]|} < \varepsilon$$

and that

$$\sum_{([u,v],w)\in\pi''} |[G(v) - G(u)]| \leq \sum_{([u,v],w)\in\pi''} \sqrt{|[F(v) - F(u)][G(v) - G(u)]|} < \varepsilon.$$

This proves one direction in the theorem.

For the converse select a number $M > 0$ and a full cover β_1 so that

$$\sum_{([u,v],w)\in\pi} [|[F(v) - F(u)]| + |[G(v) - G(u)]|] < M$$

for all partitions π of $[a, b]$ from β_1. This is possible merely because the functions F and G have bounded variation. Select a full cover β_2 with the property presented in the statement of the theorem (for ε). Let $\beta = \beta_1 \cap \beta_2$. This is a full cover. Consider any partition π of $[a, b]$ contained in β. There must be, by hypothesis, a split $\pi = \pi' \cup \pi''$ so that

$$\sum_{([u,v],w)\in\pi'} |[F(v) - F(u)]| < \varepsilon$$

and

$$\sum_{([u,v],w)\in\pi''} |[G(v) - G(u)]| < \varepsilon.$$

We now compute

$$\sum_{([u,v],w)\in\pi} \sqrt{|[F(v) - F(u)][G(v) - G(u)]|} =$$

$$\sum_{([u,v],w)\in\pi'} \sqrt{|[F(v) - F(u)][G(v) - G(u)]|}$$

$$+ \sum_{([u,v],w)\in\pi''} \sqrt{|[F(v) - F(u)][G(v) - G(u)]|}$$

$$\leq \sqrt{\sum_{([u,v],w)\in\pi'} |[F(v) - F(u)]|}\sqrt{\sum_{([u,v],w)\in\pi'} |[G(v) - G(u)]|}$$

$$+ \sqrt{\sum_{([u,v],w)\in\pi''} |[F(v) - F(u)]|}\sqrt{\sum_{([u,v],w)\in\pi''} |[G(v) - G(u)]|}$$

$$\leq 2\sqrt{M\varepsilon}.$$

Here we have used the Cauchy-Schwartz inequality. Since ε is an arbitrary positive number it follows that

$$\int_a^b \sqrt{|dF(x)\,dG(x)|} = 0.$$

Consequently F and G must be mutually singular. ■

18.10 Singular functions

We have defined the notion of a singular function elsewhere and given the usual remarkable example of such a function, the Cantor function (Devil's staircase). We show that there are further characterizations of this notion, in particular one given exactly by a Stieltjes-type integral.

Theorem 18.25: *Let $F : [a, b] \to \mathbb{R}$ be a function of bounded variation. Then the following are equivalent:*

1. *F is singular.*
2. *$F'(x) = 0$ almost everywhere in $[a, b]$.*
3. $\displaystyle\int_a^b \sqrt{|dF(x)|\,dx} = 0.$

Proof. It is only the third property that we show here, since we know from elsewhere that the first two are equivalent. If the third statement is true then we can check, using the zero variation implies zero derivative a.e. rule that $F'(x) = 0$ a.e..

Conversely suppose that $F'(x) = 0$ almost everywhere. Let $\varepsilon > 0$ and choose a sequence of open intervals $\{(c_i, d_i)\}$ with total length smaller than ε so that $F'(x) = 0$ for all $x \in [a, b]$ not in one of the intervals. Define two covering relations. The first β_1 consists of all pairs $([u, v], w)$ subject only to the condition that if w is in $[a, b]$ and not covered by an open interval $\{(c_i, d_i)\}$ then

$$|F(v) - F(u)| < \varepsilon(v - u)/(b - a).$$

The second β_2 consists of all pairs $([u,v],w)$ subject only to the condition that if w is contained in one of the open intervals $\{(c_i,d_i)\}$ then so too is $[u,v]$. Then β_1, β_2, and $\beta=\beta_1\cap\beta_2$ are all full covers.

Note that if π is a subpartition contained in β_1 consisting of pairs $([u,v],w)$ not covered by an open interval from $\{(c_i,d_i)\}$ then

$$\sum_{([u,v],w)\in\pi}|F(v)-F(u)|\leq\sum_{([u,v],w)\in\pi}\varepsilon(v-u)/(b-a)\leq\varepsilon.$$

Note that if π is a subpartition contained in β_2 consisting of pairs $([u,v],w)$ that are covered by an open interval from $\{(c_i,d_i)\}$ then

$$\sum_{(I,x)\in\pi}(v-u)\leq\sum_{i=1}^{\infty}(d_i-c_i)<\varepsilon.$$

Thus any partition of $[a,b]$ chosen from β can be split into two subpartitions with these inequalities. This verifies the conditions asserted in Theorem 18.24 for F and the function $G(x)=x$. But that is exactly our third condition in the statement of the theorem. ■

18.11 Length of curves

A curve is a pair of continuous functions F, $G:[a,b]\to\mathbb{R}$. We consider that the curve is the pair of functions itself, rather than that the curve is the geometric set of points

$$\{(F(t),G(t)):t\in[a,b]\}$$

that is the object we might likely think about when contemplating a curve.

Definition 18.26: Suppose that F, $G:[a,b]\to\mathbb{R}$ is a pair of continuous functions. By the *length* of the curve given by the pair F and G we shall mean

$$\int_a^b\sqrt{[dF(x)]^2+[dG(x)]^2}.$$

That this integral is determined (but may be infinite) is pointed out in the proof of the next theorem.

Theorem 18.27: *A curve given by a pair of continuous functions $F, G : [a,b] \to \mathbb{R}$ has finite length if and only if both functions F and G have bounded variation.*

Proof. Note that as F and G are continuous, then so too is the interval function
$$h([u,v]) = \sqrt{[F(v)-F(u)]^2 + [G(v)-G(u)]^2}.$$
A simple application of the Pythagorean theorem will verify that the function h here is a continuous, subadditive interval function. The existence of the integral can be established by a repetition of the argument of Lemma 18.7.

Thus the integral
$$\int_a^b \sqrt{[dF(x)]^2 + [dG(x)]^2}$$
in the definition must necessarily be determined, although it might have an infinite value. It will have a finite value if h has bounded variation. That follows from a simple computation:
$$\max\left\{\int_a^b |dF(x)|, \int_a^b |dG(x)|\right\} \le \int_a^b \sqrt{[dF(x)]^2 + [dG(x)]^2}$$
and
$$\int_a^b \sqrt{[dF(x)]^2 + [dG(x)]^2} \le \int_a^b |dF(x)| + \int_a^b |dG(x)|.$$
■

18.11.1 Formula for the length of curves

In the elementary (computational) calculus one usually assumes that a curve is given by a pair of continuously differentiable functions (i.e., a pair F, G of continuous functions for which F' and G' are also continuous). In that case the familiar formula for length used in elementary applications is

$$\int_a^b \sqrt{[F'(x)]^2 + [G'(x)]^2}\, dx.$$

We study this now. Note that the formula is rather compelling if we think that $dF(x) = F'(x)\,dx$ and $dG(x) = G'(x)\,dx$ would be possible here.

Lemma 18.28: *For any pair of continuous functions F, $G : [a,b] \to \mathbb{R}$ of bounded variation on $[a,b]$ define the following function*

$$L(t) = \int_a^t \sqrt{[dF(x)]^2 + [dG(x)]^2} \qquad (a < t \leq b).$$

Then

$$L'(t) = \sqrt{[F'(t)]^2 + [G'(t)]^2}$$

almost everywhere in $[a,b]$.

Proof. We are now quite familiar with the zero variation implies zero derivative a.e. rule. This is all that is needed here to establish this fact, since the statement in the Lemma can be expressed, by the Henstock zero variation criterion, as

$$\int_a^b \left| dL(x) - \sqrt{[dF(x)]^2 + [dG(x)]^2} \right| = 0.$$

. ■

Lemma 18.29: *The function L in the lemma is Vitali continuous if and only if both F and G are Vitali continuous.*

Proof. This follows easily from the inequalities of Lemma 18.27. ■

The length of the curve is now available as a familiar formula precisely in the case where the two functions defining the curve are absolutely continuous.

Lemma 18.30: *For any pair of continuous functions F, $G : [a,b] \to \mathbb{R}$ of bounded variation on $[a,b]$,*

$$\int_a^b \sqrt{[dF(x)]^2 + [dG(x)]^2} \geq \int_a^b \sqrt{[F'(x)]^2 + [G'(x)]^2}\, dx.$$

The two expressions are equal if and only if both F and G are Vitali continuous on $[a,b]$.

Proof. Using the function L introduced above we see that this assertion is easily deduced from the fact that

$$L(t) \geq \int_a^t L'(x)\, dx$$

with equality precisely when L is Vitali continuous. ■

18.12 Challenging Problems for Chapter 18

18.12.1 For any continuous function $F : [a,b] \to \mathbb{R}$ define the length of the graph of F to mean

$$\int_a^b \sqrt{[dx]^2 + [dF(x)]^2}.$$

Show that the graph has finite length if and only if F has bounded variation. Discuss the availability of the familiar formula for length used in elementary applications:

$$\int_a^b \sqrt{1 + [F'(x)]^2}\, dx.$$

18.12.2 Let F, $G : [a,b] \to \mathbb{R}$ where $[a,b]$ is a compact interval. Suppose that the *Hellinger integral*[3]

$$H(t) = \int_a^t \frac{dF(x)\, dG(x)}{dx} \qquad (a < t \leq b)$$

exists. Show that $H'(t) = F'(t)G'(t)$ at almost every point t in $[a,b]$ at which both F and G are differentiable.

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[3]Named after Ernst Hellinger (1883–1950).

18.12.3 (Reduction theorem) Let F, $G : [a,b] \to \mathbb{R}$ where $[a,b]$ is a compact interval. Suppose that F is Vitali continuous on $[a,b]$ and that G is a Lipschitz function. Show that

$$\int_a^t \frac{dF(x)\,dG(x)}{dx} = \int_a^b F'(x)dG(x) = \int_a^b F'(x)G'(x)\,dx.$$

18.12.4 Let F, $G : [a,b] \to \mathbb{R}$ where $[a,b]$ is a compact interval. Suppose that F is Vitali continuous on $[a,b]$ and that G is the indefinite integral of a function of bounded variation. Show that

$$\int_a^t \frac{dF(x)\,dG(x)}{dx} = \int_a^b F'(x)dG(x) = \int_a^b F'(x)G'(x)\,dx.$$

Notes

[273]Exercise 18.1.4. Take as a full cover β the collection of pairs $([u,v],w)$ for which $w \in [u,v]$ but $[u,v]$ never overlaps both of the intervals $[0,1/2]$ or $[1/2,1]$ unless $w = 1/2$. Then all partitions π of $[a,b]$ from β can be split neatly at the point $1/2$.

[274]Exercise 18.1.5. Take as a full cover β the collection of pairs $([u,v],w)$ for which $w \in [u,v]$ but $[u,v]$ never overlaps two of the intervals $[\xi_{i-1},\xi_i]$ unless w is one of the points $\{\xi_i\}$. Then all partitions π of $[a,b]$ from β can be split neatly at the points ξ_i.

[275]Exercise 18.1.6. Both integrals exist but have different values, which you can check. If you were schooled in the Riemann-Stieltjes integral then you might recall this example was used to illustrate *non-existence* of the Riemann-Stieltjes integral. These differences in the two theories are mostly irrelevant since most applications will assume that one function is continuous and the other has bounded variation.

[276]Exercise 18.1.7. Warning: If you were schooled in the Riemann-Stieltjes integral before learning *this* Stieltjes integral you may think not. Otherwise just check that the existence of the integral (finitely that is) on $[a,b]$ and $[b,c]$ is equivalent to the existence of the integral on $[a,c]$.

[277]Exercise 18.1.14. Hint: $|dG(x)|$ is subadditive whereas $dG(x)$ is additive.

[278]Exercise 18.12.2. Develop the Henstock zero variation criterion for this integral and check that the usual zero derivative procedure will supply this fact.

Appendix A

BACKGROUND

A.1 Should I Read This Chapter?

This background chapter is not meant for the instructor but for the student. It is a mostly informal account of ideas that you need to survive an elementary course in analysis. The chapters in the text itself are more formal and contain actual mathematics. This chapter is *about* mathematics and should be an easier read.

You may skip around and select those topics that you feel you really need to read. For example, you may look through the section on notation (Section A.2) to be sure that you are familiar with the normal way of writing up many mathematical ideas, such as sets and functions.

The sections on proofs (Sections A.4, A.5, A.6, A.7, and A.8) should be read if you have never taken any courses that required an ability to write up a proof. For many students this course on real analysis is the first exposure to these ideas, and you may find these sections helpful.

A.2 Notation

If you are about to embark on a reading of the text without any further preliminaries, then there is some notation that we should review.

A.2.1 Set Notation

Sets are just collections of objects. In the beginning we are mostly interested in sets of real numbers. If the word "set" becomes too often repeated, you might find that words such as *collection*, *family*, or *class* are used. Thus a set of sets might become a family of sets. (We find such variations in ordinary language, such as flock of sheep, gaggle of geese, pride of lions.)

The statement $x \in A$ means that x is one of those numbers belonging to A. The statement $x \notin A$ means that x is *not* one of those numbers belonging to A. (The stroke through the symbol $\in$ here is a familiar device, even on road signs or no smoking signs.) Here are some familiar sets and notation.

(The Empty Set) $\emptyset$ to represent the set that contains no elements, the empty set.

(The Natural Numbers) $\mathbb{N}$ to represent the set of natural numbers (positive integers) 1, 2, 3, 4, etc.

(The Integers) $\mathbb{Z}$ to represent the set of integers (positive integers, negative integers, and zero).

(The Rational Numbers) $\mathbb{Q}$ to represent the set of rational numbers, that is, of all fractions m/n where m and n are integers (and $n \neq 0$).

(The Real Numbers) $\mathbb{R}$ to represent all the real numbers.

(Closed Intervals) $[a, b]$ to represent the set of all numbers between a and b, including a and b. We assume that $a < b$. This is called the closed interval with endpoints a and b. (Some authors allow the possibility that $a = b$, in which case $[a, b]$ must be interpreted as the set containing just the one point a. This would then be referred to as a *degenerate* interval. We have avoided this usage.)

(Open Intervals) (a, b) to represent the set of all numbers between a and b excluding a and b. This is called the open interval with endpoints a and b.

(Infinite Intervals) (a, ∞) to represent the set of all numbers that are strictly greater than a. The symbol ∞ is not interpreted as a number. [It might have been better for most students if the notation had been $(a, \rightarrow)$ since that conveys the same meaning and the beginning student would not have presumed that there is some infinite number called "$\rightarrow$" at the extreme right hand "end" of the real line.]

The other infinite intervals are

$$(-\infty, a),\ [a, \infty),\ (-\infty, a],\ \text{ and }\ (-\infty, \infty) = \mathbb{R}.$$

(Sets as a List) $\{1, -3, \sqrt{7}, 9\}$ to represent the set containing precisely the four real numbers 1, -3, $\sqrt{7}$, and 9. This is a useful way of describing a set (when possible): Just list the elements that belong. Note that order does not matter in the world of sets, so the list can be given in any order that we wish.

(Set-Builder Notation) $\{x : x^2 + x < 0\}$ to represent the set of all numbers x satisfying the inequality $x^2 + x < 0$. It may take some time [see Exercise A.2.1], but if you are adept at inequalities and quadratic equations you can recognize that this set is exactly the open interval $(-1, 0)$.) This is another useful way of describing a set (when possible): Just describe, by an equation or an inequality, the elements that belong. In general, if $C(x)$ is some kind of assertion about an object x, then $\{x : C(x)\}$ is the set of all objects x for which $C(x)$ happens to be true. Other formulations can be used. For example,

$$\{x \in A : C(x)\}$$

describes the set of elements x *that belong to the set* A and for which $C(x)$ is true. The example $\{1/n : n \in \mathbb{N}\}$ illustrates that a set can be obtained by performing computations on the members of another set.

Subsets, Unions, Intersection, and Differences The language of sets requires some special notation that is, doubtless, familiar. If you find you need some review, take the time to learn this notation well as it will be used in all of your subsequent mathematics courses.

1. $A \subset B$ (A is a subset of B) if every element of A is also an element of B.
2. $A \cap B$ (the intersection of A and B) is the set consisting of elements of both sets.
3. $A \cup B$ (the union of the sets A and B) is the set consisting of elements of either set.
4. $A \setminus B$ (the difference[1] of the sets A and B) is the set consisting of elements belonging to A but not to B.

In the text we will need also to form unions and intersections of large families of sets, not just of two sets. See the exercises for a development of such ideas.

[1] Don't use $A - B$ for set difference since it suggests subtraction, which is something else.

De Morgan's Laws Many manipulations of sets require two or more operations to be performed together. The simplest cases that should perhaps be memorized are

$$A \setminus (B_1 \cup B_2) = (A \setminus B_1) \cap (A \setminus B_2)$$

and a symmetrical version

$$A \setminus (B_1 \cap B_2) = (A \setminus B_1) \cup (A \setminus B_2).$$

If you sketch some pictures these two rules become evident. There is nothing special that requires these "laws" to be restricted to two sets B_1 and B_2. Indeed any family of sets $\{B_i : i \in I\}$ taken over any indexing set I must obey the same laws:

$$A \setminus \left(\bigcup_{i\in I} B_i\right) = \bigcap_{i\in I} (A \setminus B_i)$$

and

$$A \setminus \left(\bigcap_{i\in I} B_i\right) = \bigcup_{i\in I} (A \setminus B_i)\,.$$

Here $\bigcup_{i\in I} B_i$ is just the set formed by combining all the elements of the sets B_i into one big set (i.e., forming a large union). Similarly, $\bigcap_{i\in I} B_i$ is the set of points that are in all of the sets B_i, that is, their common intersection.

Augustus De Morgan (1806–1871), after whom these laws are named, had a respectable career as a Professor in London, although he is not remembered for any deep work. He was the originator in 1838 of the expression "mathematical induction" and the first to give a rigorous account of it. He has one interesting claim to fame, in addition to his "laws:" He was the tutor of Lady Ada Lovelace, who some say is the world's first computer programmer. A puzzle of his survives: He claims that he "was x years old in the year x^2."

Ordered Pairs Given two sets A and B, we often need to discuss pairs of objects (a, b) with $a \in A$ and $b \in B$. The first item of the pair is from the first set and the second item from the second. Since order matters here these are called *ordered pairs*. The set of all ordered pairs (a, b) with $a \in A$ and $b \in B$ is denoted

$$A \times B$$

and this set is called the *Cartesian product* of A and B.

Relations Often in mathematics we need to define a relation on a set S. Elements of S could be related by sharing some common feature or could be related by a fact of one being "larger" than another. For example, the statement $A \subset B$ is a relation on families of sets and $a < b$ a relation on a set of numbers. Fractions p/q and a/b are related if they define the same number; thus we could define a relation on the collection of all fractions by $p/q \sim a/b$ if $pb = qa$.

A relation R on a set S then would be some way of deciding whether the statement xRy (read as x is related to y) is true. If we look closely at the form of this we see it is completely described by constructing the set

$$R = \{(x, y) : \; x \text{ is related to } y\}$$

of ordered pairs. Thus a relation on a set is not a new concept: It is merely a collection of ordered pairs. Let R be any set of ordered pairs of elements of S. Then $(x, y) \in R$ and xRy and "x is related to y" can be given the same meaning. This reduces relations to ordered pairs. In practice we usually view the relation from whatever perspective is most intuitive. [For example, the order relation on the real line $x < y$ is technically the same as the set of ordered pairs $\{(x, y) : x < y\}$ but hardly anyone thinks about the relation this way.]

A.2.2 Function Notation

Analysis (indeed most of mathematics) is about functions. Do you recall that in elementary calculus courses you would often discuss some function such as $f(x) = x^2 + x + 1$ in the context of maxima and minima problems, or derivatives or integrals? The most important way of understanding a function in calculus was by means of the graph: For this function the graph is the set of all pairs $(x, x^2 + x + 1)$ for real numbers x, and often this graph was sketched as a set of points in two-dimensional space.

Definition of a Function What is a function really? Calculus students usually comprehend a formula $f(x) = x^2 + x + 1$ as defining a function, but begin to be confused when the term is used less concretely. For example, what is the distinction between the function $f(x) = x^2 + x + 1$ here and a statement such as $f(x^2 + x + 1)$?

Definition A.1: A function (or sometimes *map*) f from a set A into a set B is a rule that assigns a *value* $f(a) \in B$ to each element $a \in A$. The input set A is called the *domain of the function.* Note that f is the function, while $f(x)$ (which is not the function) is the value assigned by the function at the element $x \in A$. The set of all output values is written as

$$f(A) = \{b \in B : f(a) = b \text{ for some } a \in A\}$$

and is called the *range* of the function.

Thus the calculus example above really asserts that we are given a function named f, whose domain is the set of all real numbers, and which assigns to any number a the value $f(a) = a^2 + a + 1$. The range is not transparent from the definition and would need to be computed if it is required. (It is a simple exercise to determine that the range is the interval $[3/4, \infty)$.)

Mathematicians noted long ago that the graph of a function carried all the information needed to describe the function. Indeed, since the graph is just a set of ordered pairs $(x, f(x))$, the concept of a function can be explained entirely within the language of sets without any need to invent a new concept. Thus the function *is* the graph and the graph is a set. Thus you can expect to see the more formal version of this definition of a function given as follows.

Definition A.2: Let A and B be nonempty sets. A set f of ordered pairs (a, b) with $a \in A$ and $b \in B$ is called a *function* from A to B, written symbolically as

$$f : A \to B,$$

provided that to every $a \in A$ there is precisely one pair (a, b) in f.

The notation $(a, b) \in f$ is often used in advanced mathematics but is awkward in expressing ideas in calculus and analysis. Instead we use the familiar expression $f(a) = b$. Also, when we wish to think of a function as a graph we normally remind you by using the word "graph." Thus an analysis or calculus student would expect to see a question posed like this:

Find a point on the graph of the function $f(x) = x^2 + x + 1$ where the tangent line is horizontal.

rather than the technically correct, but awkward looking

Let f be the function

$$f = \{(x, x^2 + x + 1) : x \in \mathbb{R}\}.$$

Find a point in f where the tangent line is horizontal.

Domain of a Function The set of points A in the definition is called *the domain* of the function. It is an essential ingredient of the definition of any function. It should be considered incorrect to write

Let the function f be defined by $f(x) = \sqrt{x}$.

Instead we should say

Let the function f be defined with domain $[0, \infty)$ by $f(x) = \sqrt{x}$.

The first assertion is sloppy; it requires you to guess at the domain of the function. Calculus courses, however, often make this requirement, leaving it to you to figure out from a formula what domain should be assigned to the function. Often we, too, will require that you do this.

Range of a Function The set of points B in the definition is sometimes called the *range* or *co-domain* of the function. Most writers do not like the term "range" for this and prefer to use the term "range" for the set

$$f(A) = \{f(x) : x \in A\} \subset B$$

that consists of the actual output values of the function f, not some larger set that merely contains all these values.

One-To-One and Onto Function If to each element b in the range of f there is precisely one element a in the domain so that $f(a) = b$, then f is said to be *one-to-one* or *injective*. We sometimes say, about the range $f(A)$ of a function, that f maps A *onto* $f(A)$. If $f : A \to B$, then f would be said to be *onto* B if B is the range of f, that is, if for every $b \in B$ there is some $a \in A$ so that $f(a) = b$. A function that is onto is sometimes said to be *surjective*. A function that is both one-to-one and onto is sometimes said to be *bijective*.

Inverse of a Function Some functions allow an *inverse.* If $f : A \to B$ is a function, there is, sometimes, a function $f^{-1} : B \to A$ that is the reverse of f in the sense that

$$f^{-1}(f(a)) = a \text{ for every } a \in A$$

and

$$f(f^{-1}(b)) = b \text{ for every } b \in B.$$

Thus f carries a to $f(a)$ and f^{-1} carries $f(a)$ back to a while f^{-1} carries b to $f^{-1}(b)$ and f carries $f^{-1}(b)$ back to b. This can happen only if f is one-to-one and onto B. See the exercises for some practice on these concepts.

Characteristic Function of a Set Let $E \subset \mathbb{R}$. Then a convenient function for discussing properties of the set E is the function χ_E defined to be 1 on E and to be 0 at every other point. This is called the *characteristic function* of E or, sometimes, *indicator function.*

Composition of Functions Suppose that f and g are two functions. For some values of x it is possible that the application of the two functions one after another

$$f(g(x))$$

has a meaning. If so this new value is denoted $f \circ g(x)$ or $(f \circ g)(x)$ and the function is called the *composition* of f and g. The domain of $f \circ g$ is the set of all values of x for which $g(x)$ has a meaning and for which then also $f(g(x))$ has a meaning; that is, the domain of $f \circ g$ is

$$\{x : x \in \text{dom}(g) \text{ and } g(x) \in \text{dom}(f)\}.$$

Note that the order matters here so $f \circ g$ and $g \circ f$ have, usually, radically different meanings. This is likely one of the earliest appearances of an operation in elementary mathematics that is not commutative and that requires some care.

Exercises

A.2.1 This exercise introduces the idea of set equality. The identity $X = Y$ for sets means that they have identical elements. To prove such an assertion assume first that $x \in X$ is any element. Now show that $x \in Y$. Then assume that $y \in Y$ is any element. Now show that $y \in X$.

(a) Show that $A \cup B = B$ if and only if $A \subset B$.

(b) Show that $A \cap B = A$ if and only if $A \subset B$.
(c) Show that $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.
(d) Show that $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$.
(e) Show that $(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)$.
(f) Show that $(A \cap B) \setminus C = (A \setminus C) \cap (B \setminus C)$.
(g) Show that $\{x \in \mathbb{R} : x^2 + x < 0\} = (-1, 0)$.

A.2.2 This exercise introduces the notations $\bigcup_{n=1}^{N} A_i$ and $\bigcap_{n=1}^{N} A_i$ for the union and intersection of the sets A_1, A_2, ..., A_N:

(a) Describe the sets

$$\bigcup_{n=1}^{N}(-1/n, 1/n) \text{ and } \bigcap_{n=1}^{N}(-1/n, 1/n).$$

(b) Describe the sets

$$\bigcup_{n=1}^{N}(-n, n) \text{ and } \bigcap_{n=1}^{N}(-n, n).$$

(c) Describe the sets

$$\bigcup_{n=1}^{N}[n, n+1] \text{ and } \bigcap_{n=1}^{N}[n, n+1].$$

A.2.3 This exercise introduces the notations $\bigcup_{n=1}^{\infty} A_i$ and $\bigcap_{n=1}^{\infty} A_i$ for the union and intersection of the sets A_1, A_2, ...

(a) Describe the sets

$$\bigcup_{n=1}^{\infty}(-1/n, 1/n) \text{ and } \bigcap_{n=1}^{\infty}(-1/n, 1/n).$$

(b) Describe the sets

$$\bigcup_{n=1}^{\infty}(-n, n) \text{ and } \bigcap_{n=1}^{\infty}(-n, n).$$

(c) Describe the sets

$$\bigcup_{n=1}^{\infty}[n,n+1] \text{ and } \bigcap_{n=1}^{\infty}[n,n+1].$$

A.2.4 Do you accept any of the following as an adequate definition of the function f? (The domain is not specified but it is assumed that you will try to find a domain that might work.)

(a) $f(x) = 1/\sqrt{1-x}$.

(b) $f(x) = x$ if x is rational and $f(x) = -x$ if x is irrational.

(c) $f(x) = 1$ if x contains a 9 in its decimal expansion and $f(x) = 0$ if not.

(d) $f(x) = 1$ if x contains a 7 in its decimal expansion and $f(x) = 0$ if not.

(e) $f(x) = 1$ if x is a prime number and $f(x) = 0$ if it is not.

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A.2.5 This exercise promotes the use of the term *mapping* in the study of functions.
If $f : X \to Y$ and $E \subset X$, then

$$f(E) = \{y : f(x) = y \text{ for some } x \in E\} \subset Y$$

is called the *image* of E under f and we say f *maps* E to the set $f(E)$.

(a) Let $f : \mathbb{R} \to \mathbb{R}$. Give an example of sets A, B so that

$$f(A \cap B) \neq f(A) \cap f(B).$$

(b) Would $f(A \cup B) = f(A) \cup f(B)$ be true in general?

(c) Find a function $f : \mathbb{R} \to \mathbb{R}$ so that $f([0,1]) = \{1,2\}$.

A.2.6 This exercise concerns the notion of one-to-one function (i.e., injective function):

(a) Show that $f : \mathbb{R} \to \mathbb{R}$ is one-to-one if and only if

$$f(A \cap B) = f(A) \cap f(B)$$

for all sets A, B.

(b) Show that $f : \mathbb{R} \to \mathbb{R}$ is one-to-one if and only if $f(A) \cap f(B) = \emptyset$ for all sets A, B with $A \cap B = \emptyset$.

A.2.7 This exercise concerns the notion of preimage. If $f : X \to Y$ and $E \subset Y$, then

$$f^{-1}(E) = \{x : f(x) = y \text{ for some } y \in E\ \} \subset X$$

is called the preimage of E under f. [There may or may not be an inverse function here; $f^{-1}(E)$ has a meaning even if there is no inverse function.]

(a) Show that $f(f^{-1}(E)) \subset E$ for every set $E \subset \mathbb{R}$.

(b) Show that $f^{-1}(f(E)) \supset E$ for every set $E \subset \mathbb{R}$.

(c) Can you simplify $f^{-1}(A \cup B)$ and $f^{-1}(A \cap B)$?

(d) Show that $f : \mathbb{R} \to \mathbb{R}$ is one-to-one if and only if $f^{-1}(\{b\})$ contains at most a single point for any $b \in \mathbb{R}$.

(e) Show that $f : \mathbb{R} \to \mathbb{R}$ is onto, that is, the range of f is all of $\mathbb{R}$ if and only if $f(f^{-1}(E)) = E$ for every set $E \subset \mathbb{R}$.

A.2.8 This exercise concerns the notion of composition of functions:

(a) Give examples to show that $f \circ g$ and $g \circ f$ are distinct.

(b) Give an example in which $f \circ g$ and $g \circ f$ are not distinct.

(c) While composition is not commutative, is it associative, that is, is it true that

$$(f \circ g) \circ h = f \circ (g \circ h)?$$

(d) Give several examples of functions f for which $f \circ f = f$.

A.2.9 This exercise concerns the notion of onto function (i.e., surjective function): Which of the following functions map $[0, 1]$ onto $[0, 1]$?

(a) $f(x) = x$

(b) $f(x) = x^2$

(c) $f(x) = x^3$

(d) $f(x) = 2|x - \frac{1}{2}|$

(e) $f(x) = \sin \pi x$

(f) $f(x) = \sin x$

A.2.10 This exercise concerns the notion of one-to-one and onto function (i.e., bijective function):

(a) Which of the functions of Exercise A.2.9 is a bijection of $[0,1]$ to $[0,1]$?
(b) Is the function $f(x) = x^2$ a bijection of $[-1,1]$ to $[0,1]$?
(c) Find a linear bijection of $[0,1]$ onto the interval $[3,6]$.
(d) Find a bijection of $[0,1]$ onto the interval $[3,6]$ that is not linear.
(e) Find a bijection of $\mathbb{N}$ onto $\mathbb{Z}$.

A.2.11 This exercise concerns the notion of inverse functions: For each of the functions of Exercise A.2.9, select an interval $[a,b]$ on which that function has an inverse and find an explicit formula for the inverse function. Be sure to state the domain of the inverse function.

A.2.12 This exercise concerns the notion of an equivalence relation. A relation $x \sim y$ on a set S is said to be an equivalence relation if

(a) $x \sim x$ for all $x \in S$.
(b) $x \sim y$ implies that $y \sim x$.
(c) $x \sim y$ and $y \sim z$ imply that $x \sim z$.

(a) Show that the relation $p/q \sim a/b$ if $pb = qa$ defined in the text on the collection of fractions is an equivalence relation.
(b) Define a relation on the collection of fractions that satisfies two of the requirements of an equivalence relation but is not an equivalence relation.
(c) Define nontrivial equivalence relations on the sets $\mathbb{N}$ and $\mathbb{Z}$.

A.2.13 Set builder notation can be used to "describe" some curious sets. For example,

$$S_1 = \{S : S \text{ is a set}\}.$$

This has the peculiar property that $S_1 \in S_1$. (That is similar to joining a club where you find the club appearing on the membership list as a member of itself!) Worse yet is

$$S_2 = \{S : S \text{ is a set and } S \notin S\}.$$

This has the paradoxical property that if $S_2 \in S_2$, then $S_2 \notin S_2$, while if $S_2 \notin S_2$, then $S_2 \in S_2$. Any thoughts?

SEE NOTE 280

A.3 What Is Analysis?

The term "analysis" now covers large parts of mathematics. You almost need to be a professional mathematician to understand what it might mean.

For a course at this level, though, "real analysis" mostly refers to the subject matter that you have already learned in your calculus courses: limits, continuity, derivatives, integrals, sequences, and series. Calculus as a subject can be thought of as an eighteenth century development, analysis as a nineteenth-century creation. None of the ideas of calculus rested on very firm foundation, and the lack of foundations proved a barrier to further progress. There was much criticism by mathematicians and philosophers of the fundamental ideas of calculus (limits especially), and often when new and controversial methods were proposed (such as Fourier series) the mathematicians of the time could not agree on whether they were valid.

In the first decades of the nineteenth century the foundations of the subject were reworked, most notably by Cauchy (whose name will appear frequently in this text) and new and powerful methods developed. It is this that we are studying here.

We will look once again at notions of sequence limit, function limit, etc. that we have seen before in our calculus classes, but now from a more rigorous point of view. We want to know precisely what they mean and how to prove the validity of the techniques of the subject.

At first sight you might wonder about this. Are we just reviewing our calculus but now we do not get to skip over the details of proofs? If, however, you persist you will see that we are entering instead a new and different world. By looking closely at the details of why certain things work we gain a new insight. More than that we can do new things, things that could not have been imagined at a mere calculus level.

A.4 Why Proofs?

Can't we just do mathematics without proofs? Certainly there are many applications of mathematics carried on by people unable or unwilling to attempt proofs. But at the very heart and soul of mathematics is the proof, the careful argument that shows that a statement is true.

Compare this with the natural sciences. The advancement of knowledge in those subjects rests on the experiment. No scientist considers seriously whether students can skip over experimental work and just learn the result. At the core of all scientific discovery is the experimental method. It is too central to the discipline to be removed. It is the reason for the monumental success of the subject.

Mathematicians feel the same way about proofs. We can, with imagination and insight, make reasonable conjectures. But we can't be sure a conjecture is true until we prove it. The history of mathematics is filled with plausible (but false) statements made by mathematicians, even famous ones.

Proofs are an essential part of the subject. If you can master the art of reading and writing proofs, you enter properly into the subject. If not, you remain forever on the periphery looking in, a spectator able to learn some superficial facts about mathematics but unable to *do* mathematics.

What Is a Proof? Mathematicians are always prepared to define exactly what everything in their subject means. Certainly it is possible to define exactly what constitutes a proof. But that is best left to a course in logic.

For a course in analysis just understand that a proof is a short or long sequence of arguments meant to convince us that some statement is true. You will understand what a proof is after you have read some proofs and find that you do in fact follow the argument.

A proof is always intended for a specific audience. Proofs in this text are intended for readers who have some experience in calculus and good reasoning skills, but little experience in analysis. Proofs in more advanced texts would be much shorter and have less motivation. Proofs in professional research journals, intended for other professional mathematicians, can be terse and mysterious indeed.

Traditionally courses in analysis do not start with much of a discussion of proofs even though the students will be expected to produce proofs of their own, perhaps for the first time in their career. The best advice may be merely to jump in. Start studying the proofs in the text, the proofs given in lectures, the proofs attempted by your fellow students. Try to write them yourself. Read a proof, understand its main ideas, and then attempt to write the argument up in your own words.

How to Read a Proof While a proof may look like a short story, it is often much harder to read than one. Usually some of the computations will not seem clear and you will have to figure out how they were done. Some of the arguments (this is true and hence that is true) will not be immediate but will require some thinking. Many of the steps will appear completely strange, and it will seem that the proof is going off in a weird direction that is entirely mysterious. Basically you must unravel the proof. Find out what the main ideas are and the various steps of the proof.

One important piece of advice while reading a proof: Try to remember what it is that has to be proved. Before reading the proof decide what it is that must be proved exactly. Ask yourself, "What would I have to show to prove that?"

How to Write a Proof Practice! We learn to write proofs by writing proofs. Start by just copying nearly word for word a proof in a text that you find interesting. Vary the wording to use your own phrases. Write out the proof using more steps and more details than you found in the original. Try to find a different proof of the same statement and write out your new proof. Try to change the order of the argument if it is possible. If it is not possible you'll soon see why.

We all have learned the art of proof by imitation at first.

A.5 Indirect Proof

Many proofs in analysis are achieved as indirect proofs. This refers to a specific method.

The method argues as follows. I wish to prove a statement **P** is true. Either **P** is true or else **P** is false, not both. If I suppose **P** is false perhaps I can prove that then something entirely unbelievable must be true. Since that unbelievable something is not true, it follows that it cannot be the case that **P** is false. Therefore, **P** is true.

The method appears in the classical subject of rhetoric under the label *reductio ad absurdum* (I reduce to the absurd).

> Ladies and gentlemen my worthy opponent claims **P** but I claim the opposite, namely **Q** . Suppose his claim were valid. Then ...and then ...and that would mean .... But that's ridiculous so his claim is false and my claim must be true.

The pattern of all indirect proofs (also known as "proofs by contradiction") follows this structure: We wish to prove statement **P** is true. Suppose, in order to obtain a contradiction, that **P** is false. This would imply the following statements. (Statements follow.) But this is impossible. It follows that **P** is true as we were required to prove.

Here is a simple example. Suppose we wish to prove that

> *For all positive numbers x, the fraction $1/x$ is also positive.*

An indirect proof would go like this.

Proof. Suppose the statement is false. Then there is a positive number x and yet $1/x$ is not positive. This means

$$\frac{1}{x} \leq 0.$$

Since x is positive we can multiply both sides of the inequality by x and the inequality sign is preserved (this is a property of inequalities that we learned in elementary school and so we need not explain it). Thus

$$x \times \frac{1}{x} \leq x \times 0$$

or

$$1 \leq 0.$$

This is impossible. From this contradiction it follows that the statement must be true. ■

Indirect proofs are wonderfully useful and will be found throughout analysis. In some ways, however, they can be unsatisfying. After the statement "suppose not" the proof enters a fantasy world where all manipulations work toward producing a contradiction. None of the statements that you make along the way to this contradiction is necessarily of much interest because it is based on a false premise. In a direct proof, on the other hand, every statement you make is true and may be interesting on its own, not just as a tool to prove the theorem you are working on.

Also, indirect proofs reside inside a logical system where any statement not true is false and any statement not false is true. Some people have argued that we might wish to live in a mathematical world where, even though you have proved that something is not false, you have still not succeeded in proving that it is true.

Exercises

A.5.1 Show that $\sqrt{2}$ is irrational by giving an indirect proof.

SEE NOTE 281

A.5.2 Show that there are infinitely many prime numbers.

SEE NOTE 282

A.6 Contraposition

The most common mathematical assertions that we wish to prove can be written symbolically as

$$\mathbf{P} \Rightarrow \mathbf{Q},$$

which we read aloud as "Statement **P** implies statement **Q** ." The real meaning attached to this is simply that if statement **P** is true, then statement **Q** is true.

A moment's reflection about the meaning shows that the two versions

> If **P** is true, then **Q** must be true

and

> If **Q** is false, then **P** must be false

are identical in meaning. These are called contrapositives of each other. Any statement

$$\mathbf{P} \Rightarrow \mathbf{Q}$$

has a contrapositive

$$\text{not } \mathbf{Q} \Rightarrow \text{not } \mathbf{P}$$

that is equivalent. To prove a statement it is sometimes better not to prove it directly, but instead to prove the contrapositive.

Here is a simple example. Suppose that as calculus students we were required to prove that

> Suppose that $\int_0^1 f(x)\,dx \neq 0$. Then there must be a point $\xi \in [0,1]$ such that $f(\xi) \neq 0$.

At first sight it might seem hard to think of how we are going to find that point $\xi \in [0,1]$ from such little information. But let us instead prove the contrapositive. The contrapositive would say that if there is no point $\xi \in [0,1]$ such that $f(\xi) \neq 0$, then it would not be true that $\int_0^1 f(x)\,dx \neq 0$. Let's get rid of the double negatives. Restating this, now, we see that the contrapositive says that if $f(\xi) = 0$ for every $\xi \in [0,1]$, then $\int_0^1 f(x)\,dx = 0$. Even the C- students (none of whom are reading this book) would have now been able to proceed.

Exercises

A.6.1 Prove the following assertion by contraposition: If x is irrational, then $x + r$ is irrational for all rational numbers r.
SEE NOTE 283

A.7 Counterexamples

The polynomial

$$p(x) = x^2 + x + 17$$

has an interesting feature: It generates prime numbers for some time. For example, $p(1) = 19$, $p(2) = 23$, $p(3) = 29$, $p(4) = 37$ are all prime. More examples can be checked. After many more computations we would be tempted to make the claim

For every integer $n = 1, 2, 3, \ldots$ the value $n^2 + n + 17$ is prime.

To prove that this is true (if indeed it is true) we would be required to show for any n, no matter what, that the value $n^2 + n + 17$ is prime. What would it take to disprove the statement, that is, to show that it is false?

All it would take is one instance where the statement fails. Only one! In fact there are many instances. It is enough to give one of them. Take $n = 17$ and observe that

$$17^2 + 17 + 17 = 17(17 + 1 + 1) = 17 \cdot 19,$$

which is certainly not prime. This one example is enough to prove that the statement is false. We refer to this as a *proof by counterexample*.

The Converse In analysis we shall often need to invent counterexamples. One frequent situation that occurs is the following. Suppose that we have just completed, successfully, the proof of a theorem expressed symbolically as

$$\mathbf{P} \Rightarrow \mathbf{Q}.$$

A natural question is whether the converse is also true. The *converse* is the opposite implication

$$\mathbf{Q} \Rightarrow \mathbf{P}.$$

Indeed once we have proved any theorem it is nearly routine to ask if the converse is true. Many converses are false, and a proof usually consists in looking for a counterexample.

For example, in calculus courses (and here too in analysis courses) it is shown that every differentiable function is continuous. Expressed as an implication it looks like this:

$$f \text{ is differentiable } \Rightarrow f \text{ is continuous}$$

and, hence, the converse statement is

$$f \text{ is continuous } \Rightarrow f \text{ is differentiable.}$$

Is the converse true? If it is then it, too, should be proved. If it is false, then a counterexample must be found. To prove it false we need supply just one function that is continuous and yet not differentiable. You may remember that the function $f(x) = |x|$ is continuous and yet not differentiable since at the point 0 there is no derivative.

Exercises

A.7.1 Disprove this statement: For any natural number n the equation

$$4x^2 + x - n = 0$$

has no rational root.

A.7.2 Every prime greater than two is odd. Is the converse true?

A.7.3 State both the converse and the contrapositive of the assertion "Every differentiable function is continuous." Is there a difference between them? Are they both true?

A.8 Induction

There is a convenient formula for the sum of the first n natural numbers:

$$1 + 2 + 3 + \cdots + (n-1) + n = \frac{n(n+1)}{2}.$$

An easy direct proof of this would go as follows. Let S be the sum so that

$$S = 1 + 2 + 3 + \cdots + (n-1) + n$$

or, expressed in the other order,

$$S = n + (n-1) + (n-2) + \cdots + 2 + 1.$$

Adding these two equations gives

$$2S = (n+1) + (n+1) + (n+1) + \cdots + (n+1) + (n+1)$$

and hence

$$2S = n(n+1)$$

or

$$S = \frac{n(n+1)}{2},$$

which is the formula we require.

Suppose instead that we had been unable to construct this proof. Lacking any better ideas we could just test it out for $n = 1$, $n = 2$, $n = 3$, ... for as long as we had the patience. Eventually we might run into a counterexample (proving the theorem is false) or have an inspiration as to why it is true. Indeed we find

$$1 = \frac{1(1+1)}{2}$$

$$1 + 2 = \frac{2(2+1)}{2}$$

$$1 + 2 + 3 = \frac{3(3+1)}{2}$$

and we could go on for some time. On a computer we could rapidly check for several million values, each time finding that the formula is valid.

Is this a proof? If a formula works this well for untold millions of values of n, how can we conceive that it is false? We would certainly have strong emotional reasons for believing the formula if we have checked it for this many different values, but this would not be a mathematical proof.

Instead, here is a proof that, at first sight, seems to be just a matter of checking many times. Suppose that the formula does fail for some value of n. Then there must be a first occurrence of the failure, say for some integer N. We know $N \neq 1$ (since we already checked that) and so the previous integer $N - 1$ does allow a valid formula. It is the next one N that fails. But if we can show that this never happens (i.e., there is never a situation with $N - 1$ valid and N invalid), then we will have proved our formula.

For example, if the formula

$$1 + 2 + 3 + \cdots + M = \frac{M(M+1)}{2}$$

is valid, then

$$1 + 2 + 3 + \cdots + M + (M+1) = \frac{M(M+1)}{2} + (M+1)$$

$$= \frac{M(M+1) + 2(M+1)}{2} = \frac{(M+1)(M+2)}{2},$$

which is indeed the correct formula for $n = M + 1$. Thus there never can be a situation in which the formula is correct at some stage and fails at the next stage. It follows that the formula is always true. This is a proof by induction.

This may be used to try to prove any statement about an integer n. Here are the steps:

Step 1 Verify the statement for $n = 1$.

Step 2 (The induction step) Show that whenever the statement is true for any positive integer m it is necessarily also true for the next integer $m + 1$.

Step 3 Claim that the formula holds for all n by the principle of induction.

In the exercises you are asked for induction proofs of various statements. You might try too to give direct (noninductive) proofs. Which method do you prefer?

Exercises

A.8.1 Prove by induction that for every $n = 1, 2, 3, \ldots,$

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

SEE NOTE 284

A.8.2 Compute for $n = 1$, 2, 3, 4 and 5 the value of

$$1 + 3 + 5 + \cdots + (2n - 1).$$

This should be enough values to suggest a correct formula. Verify it by induction.

A.8.3 Prove by induction for every $n = 1, 2, 3, \ldots$ that the number

$$7^n - 4^n$$

is divisible by 3.

A.8.4 Prove by induction that for every $n = 1, 2, 3, \ldots$

$$(1 + x)^n \geq 1 + nx$$

for any $x > 0$.

A.8.5 Prove by induction that for every $n = 1, 2, 3, \ldots$

$$1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}$$

for any real number $r \neq 1$.

A.8.6 Prove by induction for every $n = 1, 2, 3, \ldots$ that

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2.$$

A.8.7 Prove by induction that for every $n = 1, 2, 3, \ldots$

$$\frac{d^n}{dx^n} e^{2x} = e^{2x + n \log 2}.$$

A.8.8 Show that the following two principles are equivalent (i.e., assuming the validity of either one of them, prove the other).

> **(Principle of Induction)** Let $S \subset \mathbb{N}$ such that $1 \in S$ and for all integers n if $n \in S$, then so also is $n + 1$. Then $S = \mathbb{N}$.

and

> **(Well Ordering of $\mathbb{N}$)** If $S \subset \mathbb{N}$ and $S \neq \emptyset$, then S has a first element (i.e., a minimal element).

well ordering of $\mathbb{N}$

A.8.9 Criticize the following "proof."

(Birds of a feather flock together) Any collection of n birds must be all of the same species.

Proof This is certainly true if $n = 1$. Suppose it is true for some value n. Take a collection of $n + 1$ birds. Remove one bird and keep him in your hand. The remaining birds are all of the same species. What about the one in your hand? Take a different one out and replace the one in your hand. Since he now is in a collection of n birds he must be the same species too. Thus all birds in the collection of $n + 1$ birds are of the same species. The statement is now proved by induction.

See Note 285

A.9 Quantifiers

In all of mathematics and certainly in all of analysis you will encounter two phrases used repeatedly:

For all . . . it is true that . . .

and

There exists a . . . so that it is true that . . .

For example, the formula

$$(x + 1)^2 = x^2 + 2x + 1$$

is true *for all* real numbers x. *There is a* real number x such that

$$x^2 + 2x + 1 = 0$$

(indeed $x = -1$).

It is extremely useful to have a symbolic way of writing this. It is universal for mathematicians of all languages to use the symbol $\forall$ to indicate "for all" or "for every" and to use $\exists$ to indicate "there exists." Originally these were chosen since it was easy enough for typesetters to turn the characters "A" and "E" around or upside down. These are called by the logicians *quantifiers* since they answer (vaguely) the question "how many?" For how many x is it true that

$$(x + 1)^2 = x^2 + 2x + 1?$$

The answer is "For all real x." In symbols,

$$\forall\, x \in \mathbb{R},\ (x+1)^2 = x^2 + 2x + 1.$$

For how many x is it true that $x^2 + 2x + 1 = 0$? Not many, but there do exist numbers x for which this is true. In symbols,

$$\exists\, x \in \mathbb{R},\ x^2 + 2x + 1 = 0.$$

It is important to become familiar with statements involving one or more quantifiers whether symbolically expressed using $\forall$ and $\exists$ or merely using the phrases "for all" and "there exists." The exercises give some practice. You will certainly gain more familiarity by the time you are deeply into an analysis course in any case.

Negations of Quantified Statements Here is a tip that helps in forming negatives of assertions involving quantifiers. The two quantifiers $\forall$ and $\exists$ are complementary in a certain sense. The negation of the statement "All birds fly" would be (in conventional language) "Some bird does not fly." More formally, the negation of

For all birds b, b flies

would be

There exists a bird b, b does not fly.

In symbols let B be the set of all birds. Then the form here is

$\forall b \in B$ "statement about b" is true

and the negation of this is

$\exists b \in B$ "statement about b" is *not* true.

This allows a simple device for forming negatives. The negation of a statement with $\forall$ is a statement with $\exists$ replacing it, and the negation of a statement with $\exists$ is a statement with $\forall$ replacing it. For a complicated example, what is the negation of the statement

$\exists a \in A,\ \forall b \in B,\ \forall c \in C$
"statement about a, b and c" is true

even without assigning any meaning? It would be

$\forall a \in A,\ \exists b \in B,\ \exists c \in C,$
"statement about a, b and c" is not true.

Exercises

A.9.1 Let $\mathbb{R}$ be as usual the set of all real numbers. Express in words what these statements mean and determine whether they are true or not. Do not give proofs; just decide on the meaning and whether you think they are valid or not.

(a) $\forall x \in \mathbb{R}, x \geq 0$

(b) $\exists x \in \mathbb{R}, x \geq 0$

(c) $\forall x \in \mathbb{R}, x^2 \geq 0$

(d) $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x + y = 1$

(e) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x + y = 1$

(f) $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, x + y = 1$

(g) $\exists x \in \mathbb{R}, \exists y \in \mathbb{R}, x + y = 1$

A.9.2 Form the negations of each of the statements in the preceding exercise. If you decided that a statement was true (false) before, you should naturally now agree that the negative is false (true).

A.9.3 Explain what must be done in order to prove an assertion of the following form:

(a) $\forall s \in S$ "statement about s" is true.

(b) $\exists s \in S$ "statement about s" is true.

Now explain what must be done in order to disprove such assertions.

A.9.4 In the preceding exercise suppose that $S = \emptyset$. Could either statement be true? Must either statement be true?

Notes

[279]Exercise A.2.4. For (c) and (d): All numbers do not have a unique decimal expansion; for example, $1/2$ can be written as $0.5000000\ldots$ or as $0.499999999\ldots$. For (e): take the domain as the set $\mathbb{N}$. Are you troubled (some people might be) by the fact that nobody knows how to determine if x is a prime number when x is very large?

[280]Exercise A.2.13. As a project, research the topic of Russell's paradox [named after Bertrand Russell (1872-1969], who discovered this in the early days of set theory and caused a crisis thereby].

[281]Exercise A.5.1. Suppose not. Then $\sqrt{2}$ is rational. This means $\sqrt{2} = m/n$ where m and n are not both even. Square both sides to obtain $2n^2 = m^2$. Continue arguing until you can show that both m and n are even. That is your contradiction and the proof is complete.

[282]Exercise A.5.2. Suppose not. Then it is possible to list all the primes

$$2, 3, 5, 7, 11, 13, \ldots P$$

where P is the last of the primes. Consider the number

$$1 + (2 \times 3 \times 5 \times 7 \times 11 \times \cdots \times P).$$

From this obtain your contradiction and the proof is complete. (To be completely accurate here we need to know the prime factorization theorem: Every number can be written as a product of primes.) This is a famous proof known in ancient Greece.

[283]Exercise A.6.1. The contrapositive statement reads "if $x + r$ is not irrational for all rational numbers r, then x is not irrational." Translate this to "if $x + r$ is rational for some rational number r, then x is rational." Now this statement is easy enough to prove.

[284]Exercise A.8.1. Check for $n = 1$. Assume that

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

is true for some fixed value of n. Using this assumption (called the induction hypothesis in this kind of proof), try to find an expression for

$$1^2 + 2^2 + 3^2 + \cdots + n^2 + (n+1)^2.$$

It should turn out to be exactly the correct formula for the sum of the first $n+1$ squares. Then claim the formula is now proved for all n by induction.

[285]Exercise A.8.9. The induction step requires us to show that if the statement for n is true, then so is the statement for $n+1$. This step must be true if $n=1$ and if $n=2$ and if $n=3$ ..., in short, for all n. Check the induction step for $n=3$ and you will find that it does work; there is no flaw. Does it work for all n?

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