Approximating Functions with Taylor Series

Suppose we wish to approximate a function $f(x)$ in the vicinity of the point $x = a$. Assume that $f(x)$ and the first $N = 1$ derivatives of $f(x)$ [denoted below by $f^{(n)}(x)$, $n = 1, 2, \ldots, N-1$] are continuous and finite in some interval $|x-a| < R$. Then, for all x inside the interval $|x-a| < R$,

$$\boxed{f(x) = f(a) + \sum_{n=1}^{N} \frac{1}{n!} f^{(n)}(a)(x-a)^n + R_N(x)\,, \quad \text{where} \quad R_N(x) \equiv \int_a^x \frac{(x-t)^N}{N!} f^{(N+1)}(t)\,dt} \tag{1}$$

If $\lim_{N\to\infty} R_N(x) = 0$ for all $|x-a| < R$, then, we say that $f(x)$ possesses a convergent Taylor series,

$$\boxed{f(x) = f(a) + \sum_{n=1}^{\infty} \frac{1}{n!} f^{(n)}(a)(x-a)^n\,, \qquad \text{for } \; |x-a| < R} \tag{2}$$

In this note, we are more interested in the first result [eq. (1) above]. That is, we wish to approximate a function by the first $N+1$ terms of its Taylor series in the interval $|x-a| < R$. Eq. (2) ensures that the approximation becomes better and better the more terms we include in the sum. But in practice, we often want to fix N and determine how good the approximation is when x is restricted to some interval within the radius of convergence.

This last question can be answered in principle by computing an upper bound for $R_N(x)$. However, in certain special cases, the error can be bounded in a very simple way. Although this error bound may not be optimal, it still can provide valuable insight into the accuracy of the approximation.

This simple method is summarized by Theorem 14.4 on p. 35 of Boas, which states that if $S \equiv \sum_{n=0}^{\infty} a_n x^n$ converges for $|x| < 1$ and if $|a_{n+1}| < |a_n|$ for all $n > N$, then

$$\boxed{\left| S - \sum_{n=0}^{N} a_n x^n \right| \; < \; \frac{|a_{N+1}|}{1-|x|}\,, \qquad |x| < 1} \tag{3}$$

The proof of this theorem is simple. First, we note that

$$\left| S - \sum_{n=0}^{N} a_n x^n \right| = \left| \sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{N} a_n x^n \right| = \left| \sum_{n=N+1}^{\infty} a_n x^n \right| .$$

If we invoke the following two results,[*]

$$|x+y| \leq |x| + |y|\,, \qquad\qquad |xy| = |x||y|\,,$$

which hold for any two real (or complex) numbers x and y, then it follows that:

$$\left| S - \sum_{n=0}^{N} a_n x^n \right| = \left| \sum_{n=N+1}^{\infty} a_n x^n \right| \;\leq\; \sum_{n=N+1}^{\infty} |a_n||x|^n\,. \tag{4}$$

Using the fact that $|a_{n+1}| < |a_n|$ for all $n > N$, the following inequality must then hold:

$$\sum_{n=N+1}^{\infty} |a_n||x|^n \;<\; |a_{N+1}| \sum_{n=N+1}^{\infty} |x|^n\,.$$

We recognize the last sum as a geometric series which can be summed:

$$\sum_{n=N+1}^{\infty} |x|^n = |x|^{N+1}\left[1 + |x| + |x|^2 + |x|^3 + \cdots\right] = \frac{|x|^{N+1}}{1-|x|}\,.$$

Hence,

$$\sum_{n=N+1}^{\infty} |a_n||x|^n \;<\; \frac{|a_{N+1}x^{N+1}|}{1-|x|}\,.$$

Inserting this result back into eq. (4) establishes eq. (3).

As a simple application, consider the approximation:

$$\frac{1}{\sqrt{1-x}} \simeq 1 + \tfrac{1}{2}x\,. \tag{5}$$

These are the first two terms of the Taylor series expansion. Including the third term, $(1-x)^{-1/2} = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \cdots$. Suppose we wish to know the maximal error in the approximation of eq. (5) in the range $|x| < \frac{1}{4}$. Using eq. (3),[†]

$$\left| \frac{1}{\sqrt{1-x}} - \left(1 + \tfrac{1}{2}x\right) \right| \;<\; \frac{3x^2}{8(1-|x|)}\,, \qquad |x| < 1\,.$$

Inserting $x = \frac{1}{4}$ into the equation above, we conclude that:

$$\left| \frac{1}{\sqrt{1-x}} - \left(1 + \tfrac{1}{2}x\right) \right| \;<\; \tfrac{1}{32}\,, \qquad |x| < \tfrac{1}{4}\,. \tag{6}$$

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[*]The relation $|x+y| \leq |x| + |y|$ is called the *triangle inequality*. This name originates from the fact that if x and y are complex numbers, then $|x+y| \leq |x| + |y|$ can be interpreted geometrically in the complex plane as the statement that the length of the third side of any triangle must be less than or equal to the sum of the lengths of the other two sides. The equality holds only when the triangle is degenerate, i.e. when the complex numbers x and y are parallel in the complex plane.

[†]One can verify that in this example, we can directly bound the remainder term $R_1(x)$ of the Taylor series. In particular, eq. (1) yields,

$$\frac{1}{\sqrt{1-x}} - \left(1 + \tfrac{1}{2}x\right) = R_1(x) \equiv \tfrac{3}{4}\int_0^x \frac{x-t}{(1-t)^{5/2}}\,dt\,, \quad \text{and} \quad |R_1(x)| < \frac{3}{4(1-|x|)^{5/2}}\int_0^x (x-t)dt = \frac{3x^2}{8(1-|x|)^{5/2}}\,.$$

For $x = \frac{1}{4}$, the above inequality yields $|R_1(x)| < \frac{1}{12\sqrt{3}}$ which is not quite as good as eq. (6).