INSTRUCTIONS: This is a three-hour exam. During the exam, you may refer to the textbook, the class handouts, or your own personal notes. Collaboration with your neighbor is strictly forbidden. In answering the questions, it is not sufficient to simply write the final result. You must provide the intermediate steps needed to arrive at the solution in order to get full credit.

The exam consists of six problems with a total of 18 parts (and one extra credit question). Each part is worth ten points, for a total of 180 points. The extra credit question [problem 5(f)] is worth 20 points.

1. My integral table lists the following definite integral:

$$\int_0^1 \frac{x^2\,dx}{(a+bx)^2} = \frac{1}{b^3}\left[a+b-\frac{a^2}{a+b}-2a\ln\left(\frac{a+b}{a}\right)\right],$$

where a and $a+b$ are positive real numbers. But, my integral table is prone to typographical errors, so I need an independent method for checking the veracity of this result.

(a) Help me out by computing $\lim_{x\to 0} f(x)$, where

$$f(x) \equiv \frac{1}{x^3}\left[a+x-\frac{a^2}{a+x}-2a\ln\left(\frac{a+x}{a}\right)\right].$$

Show that your result for $\lim_{x\to 0} f(x)$ agrees with setting $b=0$ inside the integral.

(b) With only a little more effort, you can now find the *behavior* of $f(x)$ as $x\to 0$. Determine the behavior of $f(x)$ as $x\to 0$, keeping terms of $\mathcal{O}(x)$ but dropping terms of $\mathcal{O}(x^2)$ or higher. Check that your result agrees with the behavior of the integral as $b\to 0$.

2. Consider the complex number $z\equiv 1+e^{i\theta}$, where θ is a real number (such that $-\pi<\theta\leq\pi$).

(a) Locate the point z in the complex plane and exhibit graphically.

(b) Describe the curve in the complex plane described by the equation:

$$z = 1+e^{i\theta}$$

as θ varies over its allowed range. You may prove your assertion either geometrically [using the results of part (a)] or algebraically.

(c) Evaluate $\text{Im}[\text{Ln}(1+e^{i\theta})]$ as a function of θ, where $\text{Ln}\,z$ is the principal value of the complex logarithm.

3. In statistical mechanics, the following integral arises in the study of a non-relativistic ideal Bose gas:

$$g_n(x) \equiv \frac{1}{\Gamma(n)} \int_0^\infty \frac{t^{n-1}\, dt}{x^{-1}e^t - 1} \, .$$

(a) Derive a power series expansion for $g_n(x)$ about $x = 0$.

HINT: It is probably simplest to multiply the numerator and denominator by a factor of xe^{-t} before applying Taylor's theorem. Then, integrate term by term to get the final result.

(b) Identify $g_n(1)$ as one of the special functions treated in class.

4. Consider the matrices:

$$G = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} , \qquad K = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} .$$

(a) Show that these are rotation matrices. Are the corresponding rotations proper or improper?

(b) Describe precisely the nature of the rotations produced when G and K act on a vector.

(c) Do the rotations represented by G and K commute? Compare the rotations produced by GK and KG.

5. Consider the 3×3 matrix $A = [a_{ij}]$ whose matrix elements are given by $a_{ij} = x_i y_j$, where the numbers x_1, x_2, x_3 and y_1, y_2 and y_3 are real and nonzero.

(a) What is the rank of A?

(b) Evaluate $\det A$. [*HINT:* You should be able to deduce the correct result without an explicit computation.]

(c) Find the possible solutions to the following system of three equations

$$A\vec{\boldsymbol{v}} = 0 \, , \tag{1}$$

where $\vec{\boldsymbol{v}} = (v_1\, , v_2\, , v_3)^{\mathsf{T}}$, and solve for the three unknowns v_1, v_2 and v_3. How many linearly independent vectors $\vec{\boldsymbol{v}}$ are there that are solutions to eq. (1)?

(d) Based on the results of part (c), determine the eigenvalues of A.

HINT: To completely avoid some messy algebra, think carefully about what the results of part (c) tell you about the eigenvalues of A. Then, using the value of Tr A, you should be able to determine all the eigenvalues of A.

(e) Assuming that A possesses at least one nonzero eigenvalue, is A diagonalizable? Justify your answer.

(f) [*EXTRA CREDIT:*] Determine the rank, determinant, eigenvalues and eigenvectors of an $n \times n$ matrix $A = [a_{ij}]$, where $a_{ij} = x_i y_j$. Assuming that A possesses at least one nonzero eigenvalue, is A diagonalizable?

6. In 1967, Steven Weinberg proposed a unified theory of the electromagnetic and weak interactions. His theory predicted the existence of a massive neutral spin-one boson called the Z^0. In this theory, the squared-masses of the photon and the Z^0 were predicted to be the eigenvalues of the following 2×2 matrix:

$$A = \frac{v^2}{4} \begin{pmatrix} g_1^2 & -g_1 g_2 \\ -g_1 g_2 & g_2^2 \end{pmatrix} , \tag{2}$$

where v, g_1 and g_2 are real numbers. Prior to the discovery of the Z^0, the parameters v, g_1 and g_2 were measured in other experiments. The following (approximate) numerical values for these parameters are

$$v = 262 m_p \,, \qquad g_1 = 0.355 \,, \qquad g_2 = 0.65 \,, \tag{3}$$

where m_p is the mass of the proton.

(a) Compute the eigenvalues of A in terms of the parameters v, g_1 and g_2. Check your results by computing $\det A$ and $\mathrm{Tr}\ A$.

(b) Compute the matrix that diagonalizes A:

$$S = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} .$$

That is, find an expression for θ in terms of g_1 and g_2. The angle θ is called the weak (or Weinberg) mixing angle.

HINT: It is initially easier to evaluate $\sin 2\theta$ and $\cos 2\theta$. From these expressions, use the well known identities:

$$\sin 2\theta = 2 \sin\theta \cos\theta \,, \qquad \cos 2\theta = \cos^2\theta - \sin^2\theta \,,$$

in order to deduce an expression for $\sin\theta$.

(c) Is Weinberg's theory consistent with a massless photon? Using the numerical values for the parameters given in eq. (3), determine the theoretically predicted value of the ratio of m_Z/m_p (keeping in mind that the larger of the two eigenvalues of A is equal to m_Z^2). Using your result from part (b), compute the numerical value of $\sin^2\theta$ and compare with the experimentally measured value of $\sin^2\theta = 0.23$.