Here is a collection of practice problems suitable for the first midterm exam. The exam will cover material from Chapters 1 and 2 of Boas and the first three homework sets.

1. Evaluate the following limits:

(a) $\displaystyle\lim_{x\to 0}\left(\frac{1+x}{x}-\frac{1}{\sin x}\right)$,

(b) $\displaystyle\lim_{n\to\infty}\sqrt{n^2+3n}-n$,

(c) $\displaystyle\lim_{x\to\infty}\frac{\ln x}{\sqrt{x}}$.

2. Find the radius of convergence of the following three series:

$$\text{(a)}\ \sum_{n=1}^{\infty}\frac{x^n}{\ln(n+1)}\,,\qquad \text{(b)}\ \sum_{n=0}^{\infty}\frac{(n!)^2x^n}{(2n)!}\,,\qquad \text{(c)}\ \sum_{n=0}^{\infty}\frac{n^2(x-5)^n}{5^n(n^2+1)}\,.$$

3. The Taylor series about $x=0$ for some function is given by:

$$f(x)=\frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n\,x^{2n+1}}{(2n+1)n!}\,.$$

(a) What is the radius of convergence of this series?

(b) Evaluate $\displaystyle\frac{d^3}{dx^3}f(x)\Big|_{x=0}$.

(c) Find a series for df/dx.

(d) Find a closed-form expression for $f(x)$ by summing the series above.

4. Determine whether the following series are absolutely convergent, conditionally convergent or divergent.

$$\text{(a)}\ \sum_{n=1}^{\infty}(-1)^n\frac{n}{n+1}\,,\qquad \text{(b)}\ \sum_{n=1}^{\infty}\frac{(-1)^n}{2^{\ln n}}\,,\qquad \text{(c)}\ \sum_{n=2}^{\infty}\frac{1}{n\ln n}\,.$$

5. What is the *behavior* of the function:

$$f(x) = -1 + \frac{1}{x^2}\left[\frac{1}{(1+x^2)^{3/2}} - \frac{1}{(1+x^2)^{5/2}}\right],$$

as $x \to 0$? (Obtaining the limit as $x \to 0$ is not sufficient.)

6. Evaluate $f(x) = \ln\sqrt{(1+x)/(1-x)} - \tan x$ at $x = 0.0015$ without a calculator. Determine the numerical accuracy of your result. Is your calculator a useful tool for this problem? (Try it!)

7. For each expression find all possible values and express your result both in the form $x + iy$ and in polar form $re^{i\theta}$, where θ is the principal value of the argument.

(a) $i^{77} + i^{202}$

(b) $\dfrac{3+i}{2+i}$

(c) $\sqrt{-2+2i\sqrt{3}}$

(d) $\left(\dfrac{1+i}{1-i}\right)^4$

(e) $\sqrt[4]{16}$

8. Let $z = 1 - i$. Express each of the following in the form of $x + iy$. For any multi-valued function, you should indicate all possible values of the result.

(a) $\cos(1/z)$

(b) z^z

(c) $\tan(z-1)$

(d) $\text{Ln } z$

(e) $\arg z$

9. Solve for all possible values of the real numbers x and y in the following equations:

(a) $x + iy = y + ix$.

(b) $\dfrac{x+iy}{x-iy} = -i$.

10. Find the disk of convergence of the following complex power series:

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}\, z^n$, (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(z-i)^n}{n}$.

11. Evaluate the integral

$$\int_0^{\pi} \sin 3x \cos 4x \, dx \,.$$

HINT: Rewrite the trigonometric functions in exponential form.

12. Consider the multi-valued function $w = \ln z$.

(a) Show that in general

$$\ln z^2 \neq 2 \ln z \,, \tag{1}$$

by demonstrating that for a given z, some of the possible values of $\ln z^2$ can never be written in the form of $2 \ln z$.

(b) Show that

$$\ln z^2 = \ln z + \ln z \,. \tag{2}$$

How do you explain the consistency of eqs. (1) and (2)?

(c) If n is a positive integer, prove that

$$\ln(z^{1/n}) = \frac{1}{n} \ln z \,, \tag{3}$$

by showing that as multi-valued functions, the set of possible values on both sides of eq. (3) coincide.