INSTRUCTIONS: This is a one-hour exam. During the exam, you may refer to the textbook, the class handouts (including solution sets to homework and practice problems) or your own personal notes. Collaboration with your neighbor is strictly forbidden.

In answering the questions, it is not sufficient to simply give a final result. You must provide the intermediate steps needed to arrive at the final solution in order to get full credit. The point value of each problem is indicated in brackets. The total number of points available is 100, plus a bonus of 10 points for the extra credit if you answer part (c) of problem 1. However, do not spend too much valuable time on the extra credit at the expense of the other problems on this exam.

1. [20] Consider the function

$$f(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)}\, x^{\alpha-1} e^{-x/\beta}\,, \tag{1}$$

where $0 < x < \infty$, and α and β are positive constants.

Define the family of integrals

$$I_n \equiv \int_0^\infty x^n f(x)\, dx\,,$$

where n is a non-negative integer.*

(a) Evaluate the integral I_n for arbitrary n. Your answer should be expressed in terms of functions that depend on α, β and n.

(b) For the cases of I_1 and I_2, simplify the corresponding result obtained in part (a). Then, compute the quantity $I_2 - (I_1)^2$. Your final answers should be expressed in terms of simple elementary functions of α and β.

(c) [*EXTRA CREDIT*] Suppose that the integration variable x has dimensions of length. What are the dimension of α and β required for the consistency of dimensional analysis? Determine the dimension of I_n as a function of n. Then, check that your results for I_1 and I_2 in part (b) exhibit the correct dimensions of length.

*In statistics, eq. (1) is known as the gamma distribution, which is sometimes employed to describe the probability that a random variable x lies within an interval on the positive real axis. In this case, I_1 is the mean of the distribution and $I_2 - (I_1)^2$ is the variance.

2. [30] If n is a positive integer, then the double factorial is defined as:

$$(2n)!! \equiv 2{\cdot}4{\cdot}6\cdots(2n)\,, \qquad\qquad (2n-1)!! \equiv 1{\cdot}3{\cdot}5\cdots(2n-1)\,.$$

(a) Express $(2n)!!$ in terms of the gamma function. [*HINT:* Factor out a 2 from each term in the definition of $(2n)!!$]

(b) Express $(2n-1)!!$ in terms of a ratio of two gamma functions. [*HINT:* First evaluate the product $(2n)!!(2n-1)!!$, and then use the result of part (a)]

(c) Find the leading *behavior* of the ratio

$$r_n \equiv \frac{(2n-1)!!}{(2n)!!}\,, \qquad \text{as} \quad n \to \infty\,.$$

3. [20] Consider the matrix

$$M = \begin{pmatrix} x & 1 & 0 & 1 \\ 1 & x & 1 & 0 \\ 0 & 1 & x & 1 \\ 1 & 0 & 1 & x \end{pmatrix}.$$

(a) Find all values of x for which the inverse of M does *not* exist. [*HINT:* You do not have to compute M^{-1} to answer this question.]

(b) Determine the rank of M as a function of x. [*HINT:* Consider the explicit form of M for the values of x found in part (a).]

4. [30] In the following problem, c is some number (which may be complex). Note that parts (b) and (c) of this problem are independent of part (a).

(a) For what values of c (if any) will the following equations,

$$x + y = cx\,,$$

$$-x + y = cy\,,$$

have non-trivial (i.e. non-zero) solutions?

(b) For what values of c (if any) will the following equations,

$$x + y + z = 6\,,$$

$$x + cy + cz = 2\,,$$

have either a unique solution, an infinite number of solutions, or no solutions.

(c) Determine the values of x, y and z that solve the equations of part (b) as a function of c, assuming solutions exist. If an infinite number of solutions exist, write the solution set for (x, y, z) that indicates all the possible solutions.