The Riemann Zeta-function $\zeta(s)$ *: generalities*

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1 Definition

The *Zeta function* was first introduced by Euler and is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{1}$$

The series is convergent when s is a complex number with $\Re(s) > 1$. Some special values of $\zeta(s)$ are well known, for example the values $\zeta(2) = \pi^2/6$, $\zeta(4) = \pi^4/90$, were obtained by Euler.

In 1859, Riemann had the idea to define $\zeta(s)$ for all complex number s by analytic continuation. This continuation is very important in number theory and plays a central role in the study of the distribution of prime numbers. Several techniques permit to extend the domain of definition of the Zeta function (the continuation is independant of the technique used because of uniqueness of analytic continuation). One can for example start from the *Zeta alternating series* (also called the *Dirichlet eta function*)

$$\eta(s) \equiv \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s},$$

defining an analytic function for $\Re(s) > 0$. When the complex number s satisfy $\Re(s) > 1$, we have

$$\eta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} - \sum_{n=1}^{\infty} \frac{2}{(2n)^s} = \zeta(s) - \frac{2}{2^s}\zeta(s).$$

In other words, we have

$$\zeta(s) = \frac{\eta(s)}{1 - 2^{1-s}}, \qquad \Re(s) > 1. \tag{2}$$

Since $\eta(s)$ is defined for $\Re(s) > 0$, this identity (2) permits to define the Zeta function for all complex number s with positive real part, except for $s = 1$ for which we have a pole.

The extension of the Zeta function to the domain $\Re(s) \leq 0$ can also be done (a different technique should be used).

[1]This pages are from *//numbers.computation.free.fr/Constants/constants.html*

2 The behaviour of $\zeta(s)$ near $s = 1$

Starting from the formula

$$\frac{1}{n^s} = s\int_n^\infty \frac{dt}{t^{s+1}} = s\sum_{k=n}^{\infty}\int_k^{k+1}\frac{dt}{t^{s+1}},$$

a reordering of the summations gives, for $\Re(s) > 1$,

$$\zeta(s) = s\sum_{n\geq 1}\sum_{k\geq n}\int_k^{k+1}\frac{dt}{t^{s+1}} = s\sum_{k\geq 1}\left(\sum_{n\leq k}\int_k^{k+1}\frac{dt}{t^{s+1}}\right) = s\sum_{k\geq 1}k\int_k^{k+1}\frac{dt}{t^{s+1}}.$$

The last summation writes in the form

$$\zeta(s) = s\int_1^\infty \frac{[t]}{t^{s+1}}\,dt = \frac{s}{s-1} - s\int_1^\infty \frac{\{t\}}{t^{s+1}}\,dt, \tag{3}$$

where $[t]$ denotes the integer part of t and $\{t\} = t - [t]$ its fractional part. Notice that the formula (3) is an alternative way to obtain the analytic continuation of $\zeta(s)$ in the half plane $\Re(s) > 0$.

When $s = 1$, the last integral in (3) is equal to

$$\int_1^\infty \frac{\{t\}}{t^2}\,dt = \lim_{N\to\infty}\sum_{n=1}^{N}\int_n^{n+1}\frac{t-n}{t^2}\,dt = \lim_{N\to\infty}\int_1^N \frac{dt}{t} - \sum_{n=1}^{N}\frac{1}{n+1} = 1-\gamma,$$

where γ is the *Euler constant.*

Finally, formula (3) yields the following asymptotic expansion

$$\zeta(s) = \frac{1}{s-1} + \gamma + o(1), \qquad (s\to 1). \tag{4}$$

This expansion yields interesting results if one computes the expansion obtained by (2) :

$$\begin{aligned}\zeta(s) = \frac{\eta(s)}{1-2^{1-s}} &= \frac{\eta(1) + (s-1)\eta'(1)}{(s-1)\log(2) - (s-1)^2\log^2(2)/2} + o(1)\\ &= \frac{\eta(1)}{\log(2)(s-1)} + \left(\frac{\eta'(1)}{\log(2)} + \frac{\eta(1)}{2}\right) + o(1).\end{aligned}$$

By comparison with (4), we obtain $\eta(1)/\log(2) = 1$ and $\eta'(1)/\log(2) + \eta(1)/2 = \gamma$. In other words, we have obtained the classical result

$$\eta(1) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n} = \log(2)$$

and the relation $\eta'(1) = \log(2)(\gamma - \eta(1)/2)$ yields the beautiful series

$$\sum_{n=1}^{\infty}(-1)^n\frac{\log(n)}{n} = \log(2)\left(\gamma - \frac{\log(2)}{2}\right).$$

Generalized Euler constants

The expansion (4) can be continued by writing

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\gamma_n (s-1)^n.$$

The constants γ_n can be proved to satisfy

$$\gamma_n = \lim_{m\to\infty} \sum_{k=1}^{m} \frac{\log^n k}{k} - \frac{\log^{n+1} m}{n+1}.$$

These formula generalize the Euler constant definition (corresponding to the case $n = 0$) and for that reason, the constants γ_n are often called the *generalized Euler constants*. They were also called *Stieljes constants* as they were studied by Stieljes. General informations about these constants can be found on *Eric Weissten's world of mathematics* site.

3 More on the analytic continuation of zeta

We have seen two different ways to extend the analytic continuation of $\zeta(s)$ to the domain $\Re(s) > 0$ (see (2) and (3)). In fact, the Riemann Zeta function can be analytically continued to the whole complex plane. A possible way (see [2] for example) to obtain this fundamental property is to generalize the approach of (3) by using Euler-Maclaurin sum formula applied to the function $f(x) = x^{-s}$ with order q, which yields

$$\begin{aligned}\zeta(s) &= \frac{1}{s-1} + \frac{1}{2} + \sum_{r=2}^{q} \frac{B_r}{r!} s(s+1)\cdots(s+r-2) \\ &\quad - \frac{1}{q!} s(s+1)\cdots(s+q-1)\int_1^{+\infty} B_q(\{x\}) x^{-s-q}\, dx, \qquad (5)\end{aligned}$$

where the B_r are the Bernoulli numbers and $B_q(\{x\})$ the Bernoulli polynomials evaluated at the fractionnal value $\{x\} = x - [x]$. The integral does not only converge for $\Re(s) > 1$, but also for $\Re(s) > 1 - q$. Thus formula (5) provides the analytic continuation of $\zeta(s)$ to the half plane $\Re(s) > 1 - q$. Since q can be choosen arbitrarily large, we conclude that $\zeta(s)$ is analytic in the whole complex plane with a simple pole at $s = 1$.

Formula (5) also permits to easily obtain values of $\zeta(s)$ at some special points. For example, when $s = 0$, only the two first terms of the formula do not vanish, which yields

$$\zeta(0) = \frac{1}{2}.$$

When $s = -k$ with k a positive integer, choosing $q = k + 1$ in formula (5) leads to

$$\zeta(-k) \quad = \quad -\frac{1}{k+1} + \frac{1}{2} - \sum_{r=2}^{q} \frac{B_r}{r!} k(k-1)\cdots(k-r+2)$$

$$= \quad -\frac{1}{k+1}\sum_{r=0}^{q}\binom{k+1}{r}B_r = -\frac{B_{k+1}(1)}{k+1} = -\frac{B_{k+1}}{k+1}.$$

We have used the fact that the Bernoulli numbers B_k vanish when k is odd except for $k = 1$ for which $B_1 = -1/2$, and the representation of Bernoulli polynomials in terms of Bernoulli numbers. We conclude that $\zeta(s)$ vanishes for even negative integers (the corresponding zeros are called the "trivial zeros" of $\zeta(s)$) and that for positive integers m,

$$\zeta(-2m+1) = -\frac{B_{2m}}{2m}. \tag{6}$$

4 The functional equation

One of the most striking property of the zeta function, discovered by Riemann himself, is the *functional equation* :

$$\zeta(s) = \chi(s)\zeta(1-s), \qquad \chi(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s). \tag{7}$$

The $\Gamma(s)$ function is the *Euler function.*

From the continuation of $\zeta(s)$ in the half plane $\Re(s) > 0$, notice that the functional equation is another way to obtain the analytic continuation of $\zeta(s)$ to the whole complex plane.

The complement formula of the Gamma function (see *The Gamma function* $\Gamma(x)$) entails the formula

$$\chi(s)\chi(1-s) = 1,$$

which gives a symetry of the functional equation with respect to the line $\Re(s) = 1/2$.

A proof of the functional equation

The functional equation (7) is fascinating and mathematicians afforded numerous different proofs of it. Riemann himself gave several methods ; later, mathematicians like Hardy, Siegel and others, enriched the list of proofs (see for example [4], where seven different methods are presented). We present here briefly one proof of the functional equation, extracted from [2] where we will see that the functional equation is strongly related to Fourier series. The starting point is formula (5) applied with $q = 2$

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} + \frac{B_2}{2}s - \frac{s(s+1)(s+2)}{6}\int_1^{+\infty} B_3(\{x\})x^{-s-3}\,dx. \tag{8}$$

The formula is valid for $\Re(s) > -2$. The Bernoulli polynomial $B_3(x) = x(x-1/2)(x-1)$ has a factor x, thus the integral $\int_0^1 B_3(x)x^{-s-3}$ is convergent for

$\Re(s) < -1$. The value of this integral is easily obtained thanks to repeted integration by part, and plugging it into (8) leads to the simple relation

$$\zeta(s) = -\frac{s(s+1)(s+2)}{6} \int_0^{+\infty} B_3(\{x\}) x^{-s-3}\, dx \qquad (-2 < \Re(s) < -1). \quad (9)$$

We now use the Fourier series

$$B_3(\{x\}) = 12 \sum_{n=1}^{\infty} \frac{\sin(2n\pi x)}{(2n\pi)^3}$$

and plug it into (9), which gives

$$\zeta(s) = -2s(s+1)(s+2) \sum_{n=1}^{\infty} \frac{1}{(2n\pi)^3} \int_0^{\infty} \frac{\sin(2n\pi x)}{x^{s+3}}\, dx, \qquad (-2 < \Re(s) < -1).$$

We omit the justification of term-by-term integration, which can be found in [2]. The latest formula rewrites as

$$\zeta(s) = -2s(s+1)(s+2) \sum_{n=1}^{\infty} \frac{1}{(2n\pi)^{1-s}} \int_0^{\infty} \frac{\sin(y)}{y^{s+3}}\, dy, \qquad (-2 < \Re(s) < -1).$$

The integral $\int_0^\infty \sin(y) y^{-s-3}\, dy$ is classic and its value is $\Gamma(-s-2)\sin(\pi(-s-2)/2)$ (see [2] for example), thus

$$\begin{aligned} \zeta(s) &= 2s(s+1)(s+2)\Gamma(-s-2) \sin\frac{\pi(s+2)}{2} \frac{1}{(2\pi)^{1-s}} \sum_{n=1}^{\infty} \frac{1}{n^{1-s}}, \\ &= 2^s \pi^{s-1} \Gamma(1-s) \sin\frac{\pi s}{2}\, \zeta(1-s) \qquad (-2 < \Re(s) < -1). \end{aligned}$$

This is the functional equation, proved for complex values of s in the strip $-2 < \Re(s) < -1$, and it is valid in the whole complex plane by analytic continuation.

5 Values of $\zeta(s)$ at positive integers

The functional equation (7) together with identity (6) yields the other identity, valid for positive integers m

$$\zeta(2m) = \frac{4^m (-1)^{m-1} B_{2m} \pi^{2m}}{2(2m)!} \qquad (10)$$

Notice that this famous formula can be obtained independantly of the functional equation (from Fourier series of Bernoulli polynomials for example) and permits with (6) to check that the functional equation is fulfilled for even positive integers values of s. This formula applied with the first values of m give the famous special values

$$\zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}, \quad \zeta(6) = \frac{\pi^6}{945}, \quad \zeta(8) = \frac{\pi^8}{9450}.$$

Values of $\zeta(s)$ at odd positive integers

No equivalent formula of $\zeta(m)$ are known for odd positive values of m, and it is strongly expected that they do not exist. Numerical experiments show for example that if $\zeta(3)$ have the form $\pi^3 p/q$, then the denominator q has a very large number of digits.

The first historical result obtained on values of $\zeta(s)$ at odd positive integers is due to Apery who proved in 1978 that $\zeta(3)$ is irrational (for that reason, $\zeta(3)$ is now sometimes called the *Apery constant*). It is not known if $\zeta(3)$ is transcendantal. Apery's proof for $\zeta(3)$ does not generalize for $\zeta(5)$, $\zeta(7)$, $\ldots$, and it is not known if any of these constants are irrational or not. However, in 2000, Rivoal (see [3]) stated that an infinity of the values $\zeta(2m+1)$ for positive integer m are irrational (without precising which of those values are irrational). Extending the method of Rivoal, other more precise results have been found. The most striking is due to Zudilin (see [5]) who proved that one of the numbers $\zeta(5)$, $\zeta(7)$, $\zeta(9)$, $\zeta(11)$ is irrational. Other results like "for every odd integer b, one of the numbers $\zeta(b+2)$, $\zeta(b+4)$, $\ldots$, $\zeta(8b-3)$, $\zeta(8b-1)$ is irrational" can be found in [6] for example.

6 Relation with series of primes

Let p_n denote the n-th prime number ($p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $\ldots$). We have

$$\prod_{i=1}^{N} \left(1 + \frac{1}{p_i^s} + \frac{1}{p_i^{2s}} + \cdots\right) = 1 + \frac{1}{n_1^s} + \frac{1}{n_2^s} + \cdots$$

where n_1, n_2, $\ldots$ are those integers none of whose prime factors exceed $P = p_N$. Since all integers up to P are of this form, it follows that

$$\left|\zeta(s) - \prod_{i=1}^{N}\left(1 - \frac{1}{p_i^s}\right)^{-1}\right| = \left|\zeta(s) - 1 - \frac{1}{n_1^s} - \frac{1}{n_2^s} - \cdots\right| \le \frac{1}{(P+1)^{\Re(s)}} + \frac{1}{(P+2)^{\Re(s)}} + \cdots$$

Letting $N \to \infty$, we finally obtain the beautiful *Euler's product*

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}.$$

Euler's product makes the Riemann zeta function interesting in the theory of prime numbers. Combining this identity with properties of $\zeta(s)$ gives interesting information about the series of primes. The most famous result of this kind is due to Hadamard and De La Vallée Poussin, who independantly proved in 1896 that

$$\pi(x) \sim \frac{x}{\log(x)},$$

where $\pi(x)$ denote the number of primes not exceeding x. This result is known as the *prime number theorem*.

7 The Riemann hypothesis

It was conjectured by Riemann that all the non trivial complex zeros s of $\zeta(s)$ lie on the *critical line* $\Re(s) = 1/2$. This conjecture is known as the *Riemann hypothesis* has never been proved or disproved. It is undoubtely the most celebrated problem in mathematics, not only because it has gone unsolved for so long (more than one century) but also because it has some important consequences in the distribution of primes.

7.1 Consequences of the Riemann hypothesis

The most important consequence of the Riemann hypothesis lies in the estimation of $\pi(x)$, the number of primes not exceeding x. If Riemann hypothesis is true, then we have

$$\pi(x) = \int_2^x \frac{dt}{\log(t)} + O(x^{1/2}\log x), \tag{11}$$

whereas the error bound on the approximation of $\pi(x)$ by $\int_2^x dt/\log(t)$ obtained without the Riemann hypothesis is

$$\pi(x) = \int_2^x \frac{dt}{\log(t)} + O\left(x \exp\left(-\frac{A(\log x)^{3/5}}{(\log\log x)^{1/5}}\right)\right). \tag{12}$$

Estimate (11) is much better than (12) because it states that the order of approximation of $\pi(x)$ by $\int_2^x dt/\log(t)$ is better than

$O(x^{1/2+\epsilon})$ whereas the second one is worst than $O(x^{1-\epsilon})$ for all $\epsilon > 0$. In fact, the quality of the approximation is directly related to the "size" of the free zero region near the line $\Re(s) = 1$. The estimate (12) have been obtained in 1958 when Vinogradov and Korobov enlarged the known free zero region of $\zeta(s)$ near $\Re(s) = 1$.

Among the most famous other consequences of the Riemann hypothesis (RH), we have :

- The *Lindelöf hypothesis* : if the RH is true, then when the positive real number t goes to infinity, we have $\zeta(1/2 + it) = O(t^\epsilon)$ for all $\epsilon > 0$ when $t \to \infty$. The best estimate known today without the Riemann hypothesis is $\zeta(1/2 + it) = O(t^{A+\epsilon})$ with $A = 139/858 = 0.162004\ldots$ (Kolesnik).

- The *Möbius function mean value estimate* : if the RH is true, then $M(x) = O(x^{1/2+\epsilon})$ for all $\epsilon > 0$, where $M(x) = \sum_{n \le x} \mu(n)$ with $\mu(n)$ the Möbius function, defined by $\mu(n) = (-1)^k$ if n is the product of k distinct primes, and $\mu(n) = 0$ if n can be divided by the square of some prime.

- *Bounds on the values of Zeta on the vertical line* $\sigma = 1$. Under the RH one has

$$\zeta(1 + it) = O(\log\log t), \qquad \frac{1}{\zeta(1 + it)} = O(\log\log t).$$

Without the RH, the best bound known today are $\zeta(1+it) = O(\log t/\log\log t)$ and $1/\zeta(1+it) = O(\log t/\log\log t)$.

7.2 Attempts to prove the RH

Numerous mathematicians tried to prove or disprove the RH. Thousands of (wrong) proofs have been afforded for more than one century. On the other hand, some results are known about the zeros of the Zeta-function, the most classical are discussed in *Distribution of the zeros of the Riemann Zeta function*.

Another approach consists in numerical computations, which could disprove the Riemann hypothesis by exhibiting a zero off the critical line. For example, it has been numerically checked that the first 10^{12} zeros are on the critical line. In addition to the pure RH verification, Odlyzko in [1] computed statitics on the distribution of zeros, that could orientate possible proofs. More about numerical computations on the zeros of zeta is discussed in *Numerical computations about the zeros of the zeta function*.

In 2000, Clay Mathematics Institute offered a one million dollars prize for proof of the Riemann hypothesis. Interestingly, disproof of the Riemann hypothesis (e.g., by using a computer to actually find a zero off the critical line), does not earn the one million dollars award.

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