

# Evaluating $\zeta(2)$

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I list several proofs of the celebrated identity:

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}. \quad (1)$$

As it is clear that

$$\frac{3}{4}\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{m=1}^{\infty} \frac{1}{(2m)^2} = \sum_{r=0}^{\infty} \frac{1}{(2r+1)^2},$$

(1) is equivalent to

$$\sum_{r=0}^{\infty} \frac{1}{(2r+1)^2} = \frac{\pi^2}{8}. \quad (2)$$

Many of the proofs establish this latter identity first.

**None** of these proofs is original; most are well known, but some are not as familiar as they might be. I shall try to assign credit the best I can, and I would be grateful to anyone who could shed light on the origin of any of these methods. I would like to thank Tony Lezard, José Carlos Santos and Ralph Krause, who spotted errors in earlier versions, and Richard Carr for pointing out an egregious solecism.

**Proof 1:** Note that

$$\frac{1}{n^2} = \int_0^1 \int_0^1 x^{n-1} y^{n-1} dx dy$$

and by the monotone convergence theorem we get

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n^2} &= \int_0^1 \int_0^1 \left( \sum_{n=1}^{\infty} (xy)^{n-1} \right) dx dy \\ &= \int_0^1 \int_0^1 \frac{dx dy}{1 - xy}.\end{aligned}$$

We change variables in this by putting  $(u, v) = ((x+y)/2, (y-x)/2)$ , so that  $(x, y) = (u-v, u+v)$ . Hence

$$\zeta(2) = 2 \iint_S \frac{du dv}{1 - u^2 + v^2}$$

where  $S$  is the square with vertices  $(0, 0)$ ,  $(1/2, -1/2)$ ,  $(1, 0)$  and  $(1/2, 1/2)$ . Exploiting the symmetry of the square we get

$$\begin{aligned}\zeta(2) &= 4 \int_0^{1/2} \int_0^u \frac{dv du}{1 - u^2 + v^2} + 4 \int_{1/2}^1 \int_0^{1-u} \frac{dv du}{1 - u^2 + v^2} \\ &= 4 \int_0^{1/2} \frac{1}{\sqrt{1 - u^2}} \tan^{-1} \left( \frac{u}{\sqrt{1 - u^2}} \right) du \\ &\quad + 4 \int_{1/2}^1 \frac{1}{\sqrt{1 - u^2}} \tan^{-1} \left( \frac{1 - u}{\sqrt{1 - u^2}} \right) du.\end{aligned}$$

Now  $\tan^{-1}(u/(\sqrt{1 - u^2})) = \sin^{-1} u$ , and if  $\theta = \tan^{-1}((1 - u)/(\sqrt{1 - u^2}))$  then  $\tan^2 \theta = (1 - u)/(1 + u)$  and  $\sec^2 \theta = 2/(1 + u)$ . It follows that  $u = 2 \cos^2 \theta - 1 = \cos 2\theta$  and so  $\theta = \frac{1}{2} \cos^{-1} u = \frac{\pi}{4} - \frac{1}{2} \sin^{-1} u$ . Hence

$$\begin{aligned}\zeta(2) &= 4 \int_0^{1/2} \frac{\sin^{-1} u}{\sqrt{1 - u^2}} du + 4 \int_{1/2}^1 \frac{1}{\sqrt{1 - u^2}} \left( \frac{\pi}{4} - \frac{\sin^{-1} u}{2} \right) du \\ &= [2(\sin^{-1} u)^2]_0^{1/2} + [\pi \sin^{-1} u - (\sin^{-1} u)^2]_{1/2}^1 \\ &= \frac{\pi^2}{18} + \frac{\pi^2}{2} - \frac{\pi^2}{4} - \frac{\pi^2}{6} + \frac{\pi^2}{36} \\ &= \frac{\pi^2}{6}\end{aligned}$$

as required.

This is taken from an article in the *Mathematical Intelligencer* by Apostol in 1983.

**Proof 2:** We start in a similar fashion to Proof 1, but we use (2). We get

$$\sum_{r=0}^{\infty} \frac{1}{(2r+1)^2} = \int_0^1 \int_0^1 \frac{dx dy}{1 - x^2 y^2}.$$

We make the substitution

$$(u, v) = \left( \tan^{-1} x \sqrt{\frac{1-y^2}{1-x^2}}, \tan^{-1} y \sqrt{\frac{1-x^2}{1-y^2}} \right)$$

so that

$$(x, y) = \left( \frac{\sin u}{\cos v}, \frac{\sin v}{\cos u} \right).$$

The Jacobian matrix is

$$\begin{aligned} \frac{\partial(x, y)}{\partial(u, v)} &= \begin{vmatrix} \cos u / \cos v & \sin u \sin v / \cos^2 v \\ \sin u \sin v / \cos^2 u & \cos v / \cos u \end{vmatrix} \\ &= 1 - \frac{\sin^2 u \sin^2 v}{\cos^2 u \cos^2 v} \\ &= 1 - x^2 y^2. \end{aligned}$$

Hence

$$\frac{3}{4} \zeta(2) = \iint_A du dv$$

where

$$A = \{(u, v) : u > 0, v > 0, u + v < \pi/2\}$$

has area  $\pi^2/8$ , and again we get  $\zeta(2) = \pi^2/6$ .

This is due to Calabi, Beukers and Kock.

**Proof 3:** We use the power series for the inverse sine function:

$$\sin^{-1} x = \sum_{n=0}^{\infty} \frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots (2n)} \frac{x^{2n+1}}{2n+1}$$

valid for  $|x| \leq 1$ . Putting  $x = \sin t$  we get

$$t = \sum_{n=0}^{\infty} \frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots 2n} \frac{\sin^{2n+1} t}{2n+1}$$

for  $|t| \leq \frac{\pi}{2}$ . Integrating from 0 to  $\frac{\pi}{2}$  and using the formula

$$\int_0^{\pi/2} \sin^{2n+1} x dx = \frac{2 \cdot 4 \cdots (2n)}{3 \cdot 5 \cdots (2n+1)}$$

gives us

$$\frac{\pi^2}{8} = \int_0^{\pi/2} t dt = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$$

which is (2).

This comes from a note by Boo Rim Choe in the *American Mathematical Monthly* in 1987.

**Proof 4:** We use the  $L^2$ -completeness of the trigonometric functions. Let  $e_n(x) = \exp(2\pi inx)$  where  $n \in \mathbf{Z}$ . The  $e_n$  form a complete orthonormal set in  $L^2[0, 1]$ . If we denote the inner product in  $L^2[0, 1]$  by  $\langle \cdot, \cdot \rangle$ , then Parseval's formula states that

$$\langle f, f \rangle = \sum_{n=-\infty}^{\infty} |\langle f, e_n \rangle|^2$$

for all  $f \in L^2[0, 1]$ . We apply this to  $f(x) = x$ . We easily compute  $\langle f, f \rangle = \frac{1}{3}$ ,  $\langle f, e_0 \rangle = \frac{1}{2}$  and  $\langle f, e_n \rangle = \frac{1}{2\pi in}$  for  $n \neq 0$ . Hence Parseval gives us

$$\frac{1}{3} = \frac{1}{4} + \sum_{n \in \mathbf{Z}, n \neq 0} \frac{1}{4\pi^2 n^2}$$

and so  $\zeta(2) = \pi^2/6$ .

Alternatively we can apply Parseval to  $g = \chi_{[0, 1/2]}$ . We get  $\langle g, g \rangle = \frac{1}{2}$ ,  $\langle g, e_0 \rangle = \frac{1}{2}$  and  $\langle g, e_n \rangle = ((-1)^n - 1)/2\pi in$  for  $n \neq 0$ . Hence Parseval gives us

$$\frac{1}{2} = \frac{1}{4} + 2 \sum_{r=0}^{\infty} \frac{1}{\pi^2 (2r+1)^2}$$

and using (2) we again get  $\zeta(2) = \pi^2/6$ .

This is a textbook proof, found in many books on Fourier analysis.

**Proof 5:** We use the fact that if  $f$  is continuous, of bounded variation on  $[0, 1]$  and  $f(0) = f(1)$ , then the Fourier series of  $f$  converges to  $f$  pointwise. Applying this to  $f(x) = x(1-x)$  gives

$$x(1-x) = \frac{1}{6} - \sum_{n=1}^{\infty} \frac{\cos 2\pi nx}{\pi^2 n^2},$$

and putting  $x = 0$  we get  $\zeta(2) = \pi^2/6$ . Alternatively putting  $x = 1/2$  gives

$$\frac{\pi^2}{12} = - \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

which again is equivalent to  $\zeta(2) = \pi^2/6$ .

Another textbook proof.

**Proof 6:** Consider the series

$$f(t) = \sum_{n=1}^{\infty} \frac{\cos nt}{n^2}.$$

This is uniformly convergent on the real line. Now if  $\epsilon > 0$ , then for  $t \in [\epsilon, 2\pi - \epsilon]$  we have

$$\begin{aligned} \sum_{n=1}^N \sin nt &= \sum_{n=1}^N \frac{e^{int} - e^{-int}}{2i} \\ &= \frac{e^{it} - e^{i(N+1)t}}{2i(1 - e^{it})} - \frac{e^{-it} - e^{-i(N+1)t}}{2i(1 - e^{-it})} \\ &= \frac{e^{it} - e^{i(N+1)t}}{2i(1 - e^{it})} + \frac{1 - e^{-iNt}}{2i(1 - e^{it})} \end{aligned}$$

and so this sum is bounded above in absolute value by

$$\frac{2}{|1 - e^{it}|} = \frac{1}{\sin t/2}.$$

Hence these sums are uniformly bounded on  $[\epsilon, 2\pi - \epsilon]$  and by Dirichlet's test the sum

$$\sum_{n=1}^{\infty} \frac{\sin nt}{n}$$

is uniformly convergent on  $[\epsilon, 2\pi - \epsilon]$ . It follows that for  $t \in (0, 2\pi)$

$$\begin{aligned} f'(t) &= -\sum_{n=1}^{\infty} \frac{\sin nt}{n} \\ &= -\operatorname{Im} \left( \sum_{n=1}^{\infty} \frac{e^{int}}{n} \right) \\ &= \operatorname{Im}(\log(1 - e^{it})) \\ &= \arg(1 - e^{it}) \\ &= \frac{t - \pi}{2}. \end{aligned}$$

By the fundamental theorem of calculus we have

$$f(\pi) - f(0) = \int_0^{\pi} \frac{t - \pi}{2} dt = -\frac{\pi^2}{4}.$$

But  $f(0) = \zeta(2)$  and  $f(\pi) = \sum_{n=1}^{\infty} (-1)^n/n^2 = -\zeta(2)/2$ . Hence  $\zeta(2) = \pi^2/6$ .

Alternatively we can put

$$D(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2},$$

the *dilogarithm* function. This is uniformly convergent on the closed unit disc, and satisfies  $D'(z) = -(\log(1-z))/z$  on the open unit disc. Note that  $f(t) = \operatorname{Re} D(e^{2\pi it})$ . We may now use arguments from complex variable theory to justify the above formula for  $f'(t)$ .

This is just the previous proof with the Fourier theory eliminated.

**Proof 7:** We use the infinite product

$$\sin \pi x = \pi x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right)$$

for the sine function. Comparing coefficients of  $x^3$  in the MacLaurin series of sides immediately gives  $\zeta(2) = \pi^2/6$ . An essentially equivalent proof comes from considering the coefficient of  $x$  in the formula

$$\pi \cot \pi x = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{2x}{x^2 - n^2}.$$

The original proof of Euler!

**Proof 8:** We use the calculus of residues. Let  $f(z) = \pi z^{-2} \cot \pi z$ . Then  $f$  has poles at precisely the integers; the pole at zero has residue  $-\pi^2/3$ , and that at a non-zero integer  $n$  has residue  $1/n^2$ . Let  $N$  be a natural number and let  $C_N$  be the square contour with vertices  $(\pm 1 \pm i)(N + 1/2)$ . By the calculus of residues

$$-\frac{\pi^2}{3} + 2 \sum_{n=1}^N \frac{1}{n^2} = \frac{1}{2\pi i} \int_{C_N} f(z) dz = I_N$$

say. Now if  $\pi z = x + iy$  a straightforward calculation yields

$$|\cot \pi z|^2 = \frac{\cos^2 x + \sinh^2 y}{\sin^2 x + \sinh^2 y}.$$

It follows that if  $z$  lies on the vertical edges of  $C_n$  then

$$|\cot \pi z|^2 = \frac{\sinh^2 y}{1 + \sinh^2 y} < 1$$

and if  $z$  lies on the horizontal edges of  $C_n$

$$|\cot \pi z|^2 \leq \frac{1 + \sinh^2 \pi(N + 1/2)}{\sinh^2 \pi(N + 1/2)} = \coth^2 \pi(N + 1/2) \leq \coth^2 \pi/2.$$

Hence  $|\cot \pi z| \leq K = \coth \frac{\pi}{2}$  on  $C_N$ , and so  $|f(z)| \leq \pi K / (N + 1/2)^2$  on  $C_N$ . This estimate shows that

$$|I_n| \leq \frac{1}{2\pi} \frac{\pi K}{(N + 1/2)^2} 8(N + 1/2)$$

and so  $I_N \rightarrow 0$  as  $N \rightarrow \infty$ . Again we get  $\zeta(2) = \pi^2/6$ .

Another textbook proof, found in many books on complex analysis.

**Proof 9:** We first note that if  $0 < x < \frac{\pi}{2}$  then  $\sin x < x < \tan x$  and so  $\cot^2 x < x^{-2} < 1 + \cot^2 x$ . If  $n$  and  $N$  are natural numbers with  $1 \leq n \leq N$  this implies that

$$\cot^2 \frac{n\pi}{(2N+1)} < \frac{(2N+1)^2}{n^2\pi^2} < 1 + \cot^2 \frac{n\pi}{(2N+1)}$$

and so

$$\begin{aligned} & \frac{\pi^2}{(2N+1)^2} \sum_{n=1}^N \cot^2 \frac{n\pi}{(2N+1)} \\ & < \sum_{n=1}^N \frac{1}{n^2} \\ & < \frac{N\pi^2}{(2N+1)^2} + \frac{\pi^2}{(2N+1)^2} \sum_{n=1}^N \cot^2 \frac{n\pi}{(2N+1)}. \end{aligned}$$

If

$$A_N = \sum_{n=1}^N \cot^2 \frac{n\pi}{(2N+1)}$$

it suffices to show that  $\lim_{N \rightarrow \infty} A_N / N^2 = \frac{2}{3}$ .

If  $1 \leq n \leq N$  and  $\theta = n\pi / (2N + 1)$ , then  $\sin(2N + 1)\theta = 0$  but  $\sin \theta \neq 0$ . Now  $\sin(2N + 1)\theta$  is the imaginary part of  $(\cos \theta + i \sin \theta)^{2N+1}$ , and so

$$\begin{aligned} \frac{\sin(2N+1)\theta}{\sin^{2N+1}\theta} &= \frac{1}{\sin^{2N+1}\theta} \sum_{k=0}^N (-1)^k \binom{2N+1}{2N-2k} \cos^{2(N-k)}\theta \sin^{2k+1}\theta \\ &= \sum_{k=0}^N (-1)^k \binom{2N+1}{2N-2k} \cot^{2(N-k)}\theta \\ &= f(\cot^2\theta) \end{aligned}$$

say, where  $f(x) = (2N+1)x^N - \binom{2N+1}{3}x^{N-1} + \dots$ . Hence the roots of  $f(x) = 0$  are  $\cot^2(n\pi/(2N+1))$  where  $1 \leq n \leq N$  and so  $A_N = N(2N-1)/3$ . Thus  $A_N/N^2 \rightarrow \frac{2}{3}$ , as required.

This is an exercise in Apostol's *Mathematical Analysis* (Addison-Wesley, 1974).

**Proof 10:** Given an odd integer  $n = 2m + 1$  it is well known that  $\sin nx = F_n(\sin x)$  where  $F_n$  is a polynomial of degree  $n$ . Since the zeros of  $F_n(y)$  are the values  $\sin(j\pi/n)$  ( $-m \leq j \leq m$ ) and  $\lim_{y \rightarrow 0}(F_n(y)/y) = n$  then

$$F_n(y) = ny \prod_{j=1}^m \left(1 - \frac{y^2}{\sin^2(j\pi/n)}\right)$$

and so

$$\sin nx = n \sin x \prod_{j=1}^m \left(1 - \frac{\sin^2 x}{\sin^2(j\pi/n)}\right).$$

Comparing the coefficients of  $x^3$  in the MacLaurin expansion of both sides gives

$$-\frac{n^3}{6} = -\frac{n}{6} - n \sum_{j=1}^m \frac{1}{\sin^2(j\pi/n)}$$

and so

$$\frac{1}{6} - \sum_{j=1}^m \frac{1}{n^2 \sin^2(j\pi/n)} = \frac{1}{6n^2}.$$

Fix an integer  $M$  and let  $m > M$ . Then

$$\frac{1}{6} - \sum_{j=1}^M \frac{1}{n^2 \sin^2(j\pi/n)} = \frac{1}{6n^2} + \sum_{j=M+1}^m \frac{1}{n^2 \sin^2(j\pi/n)}$$

and using the inequality  $\sin x > \frac{2}{\pi}x$  for  $0 < x < \frac{\pi}{2}$ , we get

$$0 < \frac{1}{6} - \sum_{j=1}^M \frac{1}{n^2 \sin^2(j\pi/n)} < \frac{1}{6n^2} + \sum_{j=M+1}^m \frac{1}{4j^2}.$$

Letting  $m$  tend to infinity now gives

$$0 \leq \frac{1}{6} - \sum_{j=1}^M \frac{1}{\pi^2 j^2} \leq \sum_{j=M+1}^{\infty} \frac{1}{4j^2}.$$



Hence

$$\sum_{j=1}^{\infty} \frac{1}{\pi^2 j^2} = \frac{1}{6}.$$

This comes from a note by Kortram in *Mathematics Magazine* in 1996.

**Proof 11:** Consider the integrals

$$I_n = \int_0^{\pi/2} \cos^{2n} x \, dx \quad \text{and} \quad J_n = \int_0^{\pi/2} x^2 \cos^{2n} x \, dx.$$

By a well-known reduction formula

$$I_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1) \pi}{2 \cdot 4 \cdot 6 \cdots 2n} \frac{\pi}{2} = \frac{(2n)! \pi}{4^n n!^2 2}.$$

If  $n > 0$  then integration by parts gives

$$\begin{aligned} I_n &= [x \cos^{2n} x]_0^{\pi/2} + 2n \int_0^{\pi/2} x \sin x \cos^{2n-1} x \, dx \\ &= n [x^2 \sin x \cos^{2n-1} x]_0^{\pi/2} \\ &\quad - n \int_0^{\pi/2} x^2 (\cos^{2n} x - (2n-1) \sin^2 x \cos^{2n-2} x) \, dx \\ &= n(2n-1)J_{n-1} - 2n^2 J_n. \end{aligned}$$

Hence

$$\frac{(2n)! \pi}{4^n n!^2 2} = n(2n-1)J_{n-1} - 2n^2 J_n$$

and so

$$\frac{\pi}{4n^2} = \frac{4^{n-1}(n-1)!^2}{(2n-2)!} J_{n-1} - \frac{4^n n!^2}{(2n)!} J_n.$$

Adding this up from  $n = 1$  to  $N$  gives

$$\frac{\pi}{4} \sum_{n=1}^N \frac{1}{n^2} = J_0 - \frac{4^N N!^2}{(2N)!} J_N.$$

Since  $J_0 = \pi^3/24$  it suffices to show that  $\lim_{N \rightarrow \infty} 4^N N!^2 J_N / (2N)! = 0$ . But the inequality  $x < \frac{\pi}{2} \sin x$  for  $0 < x < \frac{\pi}{2}$  gives

$$J_N < \frac{\pi^2}{4} \int_0^{\pi/2} \sin^2 x \cos^{2N} x \, dx = \frac{\pi^2}{4} (I_N - I_{N+1}) = \frac{\pi^2 I_N}{8(N+1)}$$

and so

$$0 < \frac{4^N N!}{(2N)!} J_N < \frac{\pi^3}{16(N+1)}.$$

This completes the proof.

This proof is due to Matsuoka (*American Mathematical Monthly*, 1961).

**Proof 12:** Consider the well-known identity for the Fejér kernel:

$$\left( \frac{\sin nx/2}{\sin x/2} \right)^2 = \sum_{k=-n}^n (n - |k|) e^{ikx} = n + 2 \sum_{k=1}^n (n - k) \cos kx.$$

Hence

$$\begin{aligned} \int_0^\pi x \left( \frac{\sin nx/2}{\sin x/2} \right)^2 dx &= \frac{n\pi^2}{2} + 2 \sum_{k=1}^n (n - k) \int_0^\pi x \cos kx dx \\ &= \frac{n\pi^2}{2} - 2 \sum_{k=1}^n (n - k) \frac{1 - (-1)^k}{k^2} \\ &= \frac{n\pi^2}{2} - 4n \sum_{1 \leq k \leq n, 2 \nmid k} \frac{1}{k^2} + 4 \sum_{1 \leq k \leq n, 2 \mid k} \frac{1}{k} \end{aligned}$$

If we let  $n = 2N$  with  $N$  an integer then

$$\int_0^\pi \frac{x}{8N} \left( \frac{\sin Nx}{\sin x/2} \right)^2 dx = \frac{\pi^2}{8} - \sum_{r=0}^{N-1} \frac{1}{(2r+1)^2} + O\left(\frac{\log N}{N}\right).$$

But since  $\sin \frac{x}{2} > \frac{x}{\pi}$  for  $0 < x < \pi$  then

$$\begin{aligned} \int_0^\pi \frac{x}{8N} \left( \frac{\sin Nx}{\sin x/2} \right)^2 dx &< \frac{\pi^2}{8N} \int_0^\pi \sin^2 Nx \frac{dx}{x} \\ &= \frac{\pi^2}{8N} \int_0^{N\pi} \sin^2 y \frac{dy}{y} = O\left(\frac{\log N}{N}\right). \end{aligned}$$

Taking limits as  $N \rightarrow \infty$  gives

$$\frac{\pi^2}{8} = \sum_{r=0}^{\infty} \frac{1}{(2r+1)^2}.$$

This proof is due to Stark (*American Mathematical Monthly*, 1969).

**Proof 13:** We carefully square Gregory's formula

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}.$$

We can rewrite this as  $\lim_{N \rightarrow \infty} a_N = \frac{\pi}{2}$  where

$$a_N = \sum_{n=-N}^N \frac{(-1)^n}{2n+1}.$$

Let

$$b_N = \sum_{n=-N}^N \frac{1}{(2n+1)^2}.$$

By (2) it suffices to show that  $\lim_{N \rightarrow \infty} b_N = \pi^2/4$ , so we shall show that  $\lim_{N \rightarrow \infty} (a_N^2 - b_N) = 0$ .

If  $n \neq m$  then

$$\frac{1}{(2n+1)(2m+1)} = \frac{1}{2(m-n)} \left( \frac{1}{2n+1} - \frac{1}{2m+1} \right)$$

and so

$$\begin{aligned} a_N^2 - b_N &= \sum_{n=-N}^N \sum_{m=-N}^N \prime \frac{(-1)^{m+n}}{2(m-n)} \left( \frac{1}{2n+1} - \frac{1}{2m+1} \right) \\ &= \sum_{n=-N}^N \sum_{m=-N}^N \prime \frac{(-1)^{m+n}}{(2n+1)(m-n)} \\ &= \sum_{n=-N}^N \frac{(-1)^n c_{n,N}}{2n+1} \end{aligned}$$

where the dash on the summations means that terms with zero denominators are omitted, and

$$c_{n,N} = \sum_{m=-N}^N \prime \frac{(-1)^m}{(m-n)}.$$

It is easy to see that  $c_{-n,N} = -c_{n,N}$  and so  $c_{0,N} = 0$ . If  $n > 0$  then

$$c_{n,N} = (-1)^{n+1} \sum_{j=N-n+1}^{N+n} \frac{(-1)^j}{j}$$

and so  $|c_{n,N}| \leq 1/(N-n+1)$  as the magnitude of this alternating sum is not more than that of its first term. Thus

$$|a_N^2 - b_N| \leq \sum_{n=1}^N \left( \frac{1}{(2n-1)(N-n+1)} + \frac{1}{(2n+1)(N-n+1)} \right)$$

$$\begin{aligned}
&= \sum_{n=1}^N \frac{1}{2N+1} \left( \frac{2}{2n-1} + \frac{1}{N-n+1} \right) \\
&\quad + \sum_{n=1}^N \frac{1}{2N+3} \left( \frac{2}{2n+1} + \frac{1}{N-n+1} \right) \\
&\leq \frac{1}{2N+1} (2 + 4 \log(2N+1) + 2 + 2 \log(N+1))
\end{aligned}$$

and so  $a_N^2 - b_N \rightarrow 0$  as  $N \rightarrow \infty$  as required.

This is an exercise in Borwein & Borwein's *Pi and the AGM* (Wiley, 1987).

**Proof 14:** This depends on the formula for the number of representations of a positive integer as a sum of four squares. Let  $r(n)$  be the number of quadruples  $(x, y, z, t)$  of integers such that  $n = x^2 + y^2 + z^2 + t^2$ . Trivially  $r(0) = 1$  and it is well known that

$$r(n) = 8 \sum_{m|n, 4 \nmid m} m$$

for  $n > 0$ . Let  $R(N) = \sum_{n=0}^N r(n)$ . It is easy to see that  $R(N)$  is asymptotic to the volume of the 4-dimensional ball of radius  $\sqrt{N}$ , i.e.,  $R(N) \sim \frac{\pi^2}{2} N^2$ . But

$$R(N) = 1 + 8 \sum_{n=1}^N \sum_{m|n, 4 \nmid m} m = 1 + 8 \sum_{m \leq N, 4 \nmid m} m \left\lfloor \frac{N}{m} \right\rfloor = 1 + 8(\theta(N) - 4\theta(N/4))$$

where

$$\theta(x) = \sum_{m \leq x} m \left\lfloor \frac{x}{m} \right\rfloor.$$

But

$$\begin{aligned}
\theta(x) &= \sum_{mr \leq x} m \\
&= \sum_{r \leq x} \sum_{m=1}^{\lfloor x/r \rfloor} m \\
&= \frac{1}{2} \sum_{r \leq x} \left( \left\lfloor \frac{x}{r} \right\rfloor^2 + \left\lfloor \frac{x}{r} \right\rfloor \right) \\
&= \frac{1}{2} \sum_{r \leq x} \left( \frac{x^2}{r^2} + O\left(\frac{x}{r}\right) \right)
\end{aligned}$$

$$\begin{aligned} &= \frac{x^2}{2}(\zeta(2) + O(1/x)) + O(x \log x) \\ &= \frac{\zeta(2)x^2}{2} + O(x \log x) \end{aligned}$$

as  $x \rightarrow \infty$ . Hence

$$R(N) \sim \frac{\pi^2}{2}N^2 \sim 4\zeta(2) \left( N^2 - \frac{N^2}{4} \right)$$

and so  $\zeta(2) = \pi^2/6$ .

This is an exercise in Hua's textbook on number theory.