The Big-Oh Symbol and the behavior of $f(x)$

Suppose we wish to study the behavior of a function $f(x)$ for values of x in the neighborhood of $x = a$, where a is a real (or complex) number. The first step is to compute

$$\lim_{x\to a} f(x) = f(a)\,.$$

This provides a rather limited piece of information. It would be more informative if we knew how $f(x)$ approaches $f(a)$ as $x \to a$. This is what is meant by the *behavior* of $f(x)$ as $x \to a$. To be more specific, we first must introduce the concept of the big-Oh (or order) symbol.

1. The big-Oh (or order) symbol

We say that

$$f(x) = \mathcal{O}(g(x))\,, \quad \text{as } x \to a\,,$$

if there exist a finite positive real constant M such that $|f(x)| \leq M|g(x)|$ for all values of x in the neighborhood of $x = a$. This definition is more general than what is required in these notes. In practice, we will choose $g(x) = (x - a)^n$ in the case of $x \to a$ and $g(x) = x^{-n}$ in the case of $a = \infty$, where n is a non-negative integer. Note that the meaning of the big-Oh symbol depends on the value of a. In most applications, the value of a can be ascertained from the context. The most common case is $a = 0$, in which case,

$$f(x) = \mathcal{O}(x^n)\,, \quad \text{as } x \to 0 \quad \Longleftrightarrow \quad \lim_{x\to 0} |x^{-n} f(x)| = K\,, \tag{1}$$

where K is a finite non-negative constant. As an example, $f(x) = \mathcal{O}(1)$ (corresponding to $n = 0$) means that $|f(0)|$ is a finite non-negative number.

Our primary application of the big-Oh symbol is in specifying the size of the omitted terms in a Taylor series or more generally in an asymptotic series. For example, consider a Maclaurin series (which is defined to be a Taylor series about $x = 0$). Then,

$$f(x) = \sum_{n=0}^{N} a_n x^n + \mathcal{O}(x^{N+1})\,, \qquad a_n = \frac{1}{n!} f^{(n)}(0)\,, \tag{2}$$

where $f^{(n)}(0) \equiv (d^n f/dx^n)_{x=0}$. The $\mathcal{O}(x^{N+1})$ term above represents higher order terms in the series that start with $a_{N+1}x^{N+1}$. The sum of the omitted terms is called the *remainder term*, $R_N(x)$, which satisfies $\lim_{x\to 0} x^{-(N+1)} R_N(x) = K$, which is consistent with eq. (1). In this case, the constant is simply $K = a_{N+1}$.

The following properties of the order symbol are noteworthy. Let m and n be non-negative integers. First, if c is any finite constant (either positive or negative), then

$$c\,\mathcal{O}(x^n) = \mathcal{O}(x^n)\,. \tag{3}$$

That is, multiplication by a non-zero constant does not alter the order. This follows from the basic definitions above which do not specify a specific value for the constant, $K = \lim_{x\to 0} |x^{-n} f(x)|$. Second,

$$\mathcal{O}(\mathcal{O}(x^n)) = \mathcal{O}(x^n)\,, \tag{4}$$

which is a consequence of eq. (3). Third, under multiplication,

$$\mathcal{O}(x^n)\mathcal{O}(x^m) = \mathcal{O}(x^{m+n})\,. \tag{5}$$

Fourth, under addition,

$$\text{If } m \geq n\text{, then } \ \mathcal{O}(x^n) + \mathcal{O}(x^m) = \mathcal{O}(x^n)\,, \qquad \text{as } x \to 0\,. \tag{6}$$

This result is a consequence of the fact that as $x \to 0$, x^m approaches zero faster than x^n if $m > n$. Finally, we have the strange looking result

$$\text{If } m \geq n\text{, then } \ \mathcal{O}(x^m) = \mathcal{O}(x^n)\,, \qquad \text{as } x \to 0\,. \tag{7}$$

This is consistent with eq. (1) with $f(x) = x^m$ since $\lim_{x\to 0} |x^{m-n}| = 0$ (which is certainly a finite constant) if $m > n$. Eq. (7) seems to contradict our interpretation of $\mathcal{O}(x^{N+1})$ in eq. (2) as the power of the first neglected term of the Maclaurin series. Indeed, eq. (7) implies that $\mathcal{O}(x^{N+1}) = \mathcal{O}(x^p)$ for any non-negative integer $p < N+1$. Thus, it would be mathematically correct to replace $\mathcal{O}(x^{N+1})$ with $\mathcal{O}(x^p)$ in eq. (2). However, the most useful form of such an equation is obtained by choosing p to be the *largest* possible power, which for the case of eq. (2) is $p = N + 1$.

2. The behavior of $f(x)$ as $x \to 0$

Suppose that one can expand $f(x)$ in an asymptotic series (which may or may not be a Taylor series) as $x \to 0$,

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots\,.$$

Then, assuming that the coefficients a_1 and a_2 are non-vanishing, the *behavior* of $f(x)$ as $x \to 0$ is given by:

$$f(x) = f(0) + a_1 x + \mathcal{O}(x^2)\,.$$

The term $a_1 x$ indicates that the deviation of $f(x)$ from $f(0)$ as $x \to 0$ is linear, and the $\mathcal{O}(x^2)$ indicates the size of the first neglected term of the expansion. Of course, other possible behaviors are possible. More generally, the behavior of $f(x)$ as $x \to 0$ is given by:

$$f(x) = f(0) + a_M x^M + \mathcal{O}(x^N)\,,$$

where M is the smallest positive integer for which $a_M \neq 0$ and $N > M$ indicates the index of the first nonzero coefficient in the series after a_M.

A few examples will illustrate the above concepts.

(a) Find the behavior of $f(x) = \cos^2 x$ as $x \to 0$.

Starting with the Taylor series for $\cos x$,

$$\cos x = 1 - \tfrac{1}{2}x^2 + \mathcal{O}(x^4)\,,$$

we square the above result to obtain:

$$\cos^2 x = \left[1 - \tfrac{1}{2}x^2 + \mathcal{O}(x^4)\right]^2 = 1 - x^2 + \mathcal{O}(x^4)\,.$$

Note that $(1 - \frac{1}{2}x^2)^2 = 1 - x^2 - \frac{1}{4}x^4$, but we do not have to explicitly exhibit the $\frac{1}{4}x^4$ term above since according to eq. (6),

$$\tfrac{1}{4}x^4 + \mathcal{O}(x^4) = \mathcal{O}(x^4)\,.$$

Similarly, eqs. (5) and (6) imply that*

$$x^2\mathcal{O}(x^4) = \mathcal{O}(x^6) = \mathcal{O}(x^4)\,.$$

We conclude that the behavior of $\cos^2 x$ as $x \to 0$ is $\cos^2 x = 1 - x^2 + \mathcal{O}(x^4)$.

(b) Find the behavior of $f(x) = \dfrac{1}{\cos^2 x}$ as $x \to 0$.

In this case, we can first use the results of (a) above to obtain:

$$\frac{1}{\cos^2 x} = \frac{1}{1 - x^2 + \mathcal{O}(x^4)}\,. \qquad (8)$$

To complete the problem, we make use of the well known geometric series

$$\frac{1}{1-y} = \sum_{n=0}^{\infty} y^n = 1 + y + \mathcal{O}(y^2)\,. \qquad (9)$$

We can evaluate eq. (8) by taking $y \equiv x^2 - \mathcal{O}(x^4)$ in eq. (9). Hence,

$$\frac{1}{\cos^2 x} = 1 + x^2 - \mathcal{O}(x^4) + \mathcal{O}([x^2 - \mathcal{O}(x^4)]^2) = 1 + x^2 + \mathcal{O}(x^4)\,.$$

In obtaining this result, we used the properties of the big-Oh symbol given at the end of Section 1. In particular, $[x^2 - \mathcal{O}(x^4)]^2 = \mathcal{O}(x^4)$ and $\mathcal{O}(x^4) = -\mathcal{O}(x^4)$ [the latter follows from eq. (3) with $c = -1$]. We conclude that the behavior of $1/\cos^2 x$ as $x \to 0$ is $\cos^2 x = 1 + x^2 + \mathcal{O}(x^4)$.

*Equivalently, one can cay that as $x \to 0$, any $\mathcal{O}(x^6)$ term is negligible as compared to an $\mathcal{O}(x^4)$ term and can simply be neglected.

(c) Find the behavior of $\dfrac{1}{x} - \dfrac{1}{e^x - 1}$, as $x \to 0$.

In order to find the behavior, we must make sure that we keep enough explicit terms in our expansions. First, we write:

$$\frac{1}{x} - \frac{1}{e^x - 1} = \frac{1}{x}\left[1 - \frac{x}{e^x - 1}\right].$$

Next, we note that

$$e^x - 1 = \sum_{n=1}^{\infty} \frac{x^n}{n!} = x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \mathcal{O}(x^4) = x\left[1 + \tfrac{1}{2}x + \tfrac{1}{6}x^2 + \mathcal{O}(x^3)\right].$$

It then follows that

$$\frac{x}{e^x - 1} = \frac{1}{1 + \frac{1}{2}x + \frac{1}{6}x^2 + \mathcal{O}(x^3)}.$$

To evaluate this expression, we make use of the geometric series,

$$\frac{1}{1+y} = \sum_{n=0}^{\infty} (-1)^n y^n = 1 - y + y^2 + \mathcal{O}(y^3).$$

By choosing $y = \frac{1}{2}x + \frac{1}{6}x^2 + \mathcal{O}(x^3)$, and making use of the properties of the big-Oh symbol, it follows that $\mathcal{O}(y^3) = \mathcal{O}(x^3)$, and

$$\begin{aligned}
\frac{x}{e^x - 1} &= 1 - \tfrac{1}{2}x - \tfrac{1}{6}x^2 - \mathcal{O}(x^3) + [\tfrac{1}{2}x + \tfrac{1}{6}x^2 + \mathcal{O}(x^3)]^2 + \mathcal{O}(x^3). \\
&= 1 - \tfrac{1}{2}x + \left(\tfrac{1}{4} - \tfrac{1}{6}\right)x^2 + \mathcal{O}(x^3) \\
&= 1 - \tfrac{1}{2}x + \tfrac{1}{12}x^2 + \mathcal{O}(x^3).
\end{aligned}$$

Thus,

$$\frac{1}{x} - \frac{1}{e^x - 1} = \frac{1}{x}\left[\tfrac{1}{2}x - \tfrac{1}{12}x^2 + \mathcal{O}(x^3)\right] = \tfrac{1}{2} - \tfrac{1}{12}x + \mathcal{O}(x^2).$$

3. The behavior of $f(x)$ as $x \to \infty$

One can also consider Taylor series or asymptotic series about the point of infinity. In this case, we simply replace x with $1/x$ and 0 with ∞ in eq. (1). That is,

$$f(x) = \mathcal{O}\left(\frac{1}{x^n}\right), \quad \text{as } x \to \infty \quad \Longleftrightarrow \quad \lim_{x\to\infty} |x^n f(x)| = K, \tag{10}$$

where K is a non-negative finite constant. As a simple example, the behavior of e^{-1/x^2} as $x \to \infty$ is given by

$$e^{-1/x^2} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\frac{1}{x^2}\right)^n = 1 - \frac{1}{2x^2} + \mathcal{O}\left(\frac{1}{x^4}\right).$$