1. Consider the function $F(x) = \dfrac{x}{1-x-2x^2}\,.$

(a) Use the method of partial fractions to express $F(x)$ in terms of a sum (or difference) of two simpler terms. [*HINT:* factor the denominator.]

(b) Express $F(x)$ as a power series about $x = 0$,

$$F(x) = \sum_{n=0}^{\infty} a_n x^n\,. \tag{1}$$

This is most easily done by separately expanding the two terms obtained in part (a) and then combining the two sums. Determine a closed-form expression for a_n as a function of n. Write out the first seven values of a_n (for $n = 0, 1, 2, \ldots, 6$).

(c) What is the radius of convergence of the series obtained in eq. (1)?

2. Evaluate the following integral:

$$\int_{-\infty}^{\infty} e^{-t^2}\cos(2xt)\,dt \tag{2}$$

in two different ways.

(a) Before we attempt to integrate eq. (2), consider a related integral,

$$\int_{-\infty}^{\infty} e^{-t^2} e^{2ct}\,dt = e^{c^2}\int_{-\infty}^{\infty} e^{-(t-c)^2}\,dt\,. \tag{3}$$

By a change of variables, $u = t - c$, evaluate the integral on the right-hand side of eq. (3). Use this result to evaluate the integral given eq. (2) by first writing $\cos(2xt) = \text{Re}\; e^{2ixt}$. Then, you may choose $c = ix$, assuming that your result for eq. (3) is still valid for a purely imaginary c.

(b) Expand $\cos(2xt)$ in a Taylor series about $x = 0$. Integrate term by term, and sum the resulting series. Can you reproduce the answer obtained in part (a)?

HINT: Using the duplication formula for the gamma function, given on p. 545 of Boas, show that

$$\frac{\Gamma(n+\frac{1}{2})}{\Gamma(2n+1)} = \frac{\sqrt{\pi}}{2^{2n}n!}\,.$$

Use this result to simplify the series obtained at the end of part (b). You should then be able to sum the series in closed form.

3. A complex number $x + iy$ can be represented by the 2×2 matrix

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix}, \tag{4}$$

where x and y are real numbers. Verify that this is a sensible representation by answering the following questions.

(a) Show that the matrix representation of $(x + iy)(a + ib)$ is equal to

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix} \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

To show this, you should express the product $(x + iy)(a + ib)$ in the form of $X + iY$ and show that the matrix product above, when evaluated, is consistent with the form given by eq. (4).

(b) Show that the matrix representation of the complex number $\dfrac{1}{x + iy}$ is correctly given by the inverse of eq. (4).

(c) How is the determinant of the matrix given in eq. (4) related to the corresponding complex number, $x + iy$?

4. Consider the following interesting series of numbers:

$$0, 1, 1, 3, 5, 11, 21, \ldots \tag{5}$$

This series has been generated by the following rules. First, we define

$$x_0 = 0 \quad \text{and} \quad x_1 = 1\,. \tag{6}$$

Then for all positive integers $n = 1, 2, 3, \ldots,$

$$x_{n+1} = x_n + 2x_{n-1}\,. \tag{7}$$

Starting with $x_0 = 0$ and $x_1 = 1$, we can derive the values for x_2, x_3, $x_4, \ldots$ sequentially. For example, setting $n = 1$ in eq. (7) yields $x_2 = x_1 + 2x_0 = 1$. Next we can determine $x_3 = x_2 + 2x_1 = 3$ followed by $x_4 = x_3 + 2x_2 = 5$, etc. However, this is a very inefficient way of computing x_n for some large value of n (as it would take n separate computations).

Matrix methods can help us derive a simple rule for directly determining an arbitrary term x_n in the series. Consider the matrix equation:

$$\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix}. \tag{8}$$

(a) Show that this matrix equation is equivalent to the rule given in eq. (7).

(b) Defining the matrix:

$$M \equiv \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix},$$

which appears in eq. (8), prove that for any non-negative integer n:

$$\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = M^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{9}$$

HINT: Verify eq. (9) for $n = 0$ and $n = 1$. Then iterate the process using eq. (8).

(c) Compute M^n (for arbitrary n) by first diagonalizing the matrix M and raising the resulting diagonal matrix to the nth power. Once you have obtained an expression for M^n, use eq. (9) to write an explicit formula for x_n as a function of n. Check that your formula reproduces the series given in eq. (5).

5. We have learned two methods in this class for computing the inverse of a matrix. One method involves row reduction and the second method involves the transpose of the cofactor matrix. Consider the matrix

$$M = \begin{pmatrix} 4 & 0 & -1 \\ -2 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}. \tag{10}$$

(a) Using one of the two methods mentioned above, compute M^{-1}. Check your result by computing MM^{-1}.

(b) Here is a third method for computing M^{-1}. Diagonalize M and take the inverse of the diagonalizing equation. Then solve for M^{-1} (your formula should involve the inverse of a diagonal matrix, which can be obtained by inspection). Apply this technique to the matrix M given by eq. (10). Verify that the result obtained for M^{-1} by this method is correct.

(c) Here is a fourth method for computing M^{-1}. By the Cayley-Hamilton theorem, M solves its own characteristic equation. Compute the characteristic equation for the matrix M given by eq. (10). Multiply this equation by M^{-1}, and show that M^{-1} can be expressed in terms of M^2, M and the identity matrix. Use this result to evaluate M^{-1}, and compare with the results of parts (a) and (b).

6. A totally antisymmetric third-rank Cartesian tensor B_{ijk} is defined by the property that B_{ijk} changes sign if any two of its indices are interchanged.

(a) If i, j, and k can assume the values 1, 2 or 3, determine the number of non-zero components of B_{ijk}. How many components of B_{ijk} vanish? You may assume that the component B_{123} is nonzero.

(b) Show that B_{ijk} is proportional to the Levi-Civita tensor ϵ_{ijk}. What is the constant of proportionality?