INSTRUCTIONS: This is a three-hour exam. During the exam, you may refer to the textbook, the class handouts, or your own personal notes. Collaboration with your neighbor is strictly forbidden. In answering the questions, it is not sufficient to simply write the final result. You must provide the intermediate steps needed to arrive at the solution in order to get full credit. You may quote results that have been derived in the textbook, homework solutions and class handouts (but if you do so, please cite explicitly the source of any such quotations).

The exam consists of six problems with a total of 20 parts. Each part is worth ten points, for a total of 200 points.

1. The inverse hyperbolic tangent can be defined via the following integral:

$$\frac{\operatorname{arctanh} x}{x} = \int_0^1 \frac{dt}{1 - x^2 t^2}\,.$$

(a) Derive the power series expansion for $\operatorname{arctanh} x$ about $x = 0$. Determine the general form for the nth term of this power series, and write the power series using summation notation.

(b) What is the radius of convergence, R, of this power series? Does this power series converges if $x = R$ or if $x = -R$?

(c) Evaluate the integral:

$$\int_0^1 \frac{\operatorname{arctanh} t}{t}\, dt\,, \tag{1}$$

and express the result as an infinite series.

HINT: Using the power series expansion of part (a), you can integrate term by term.

(d) Evaluate the infinite series obtained in part (c) explicitly by relating it to one of the special functions discussed in class. Then obtain the value of the integral, eq. (1), in terms of known mathematical constants.

2. The Gamma function $\Gamma(z)$ can be defined for complex values of z. $\Gamma(z)$ is a single-valued complex function that satisfies $\overline{\Gamma(z)} = \Gamma(\overline{z})$, where $\overline{z}$ is the complex conjugate of z.

(a) Using the reflection formula for the Gamma function, express $\Gamma(1+z)\Gamma(1-z)$ in terms of elementary functions.

(b) Evaluate $|\Gamma(1+iy)|^2$ in terms of elementary functions, where y is a real number.

HINT: Use the result of part (a) with $z = iy$, keeping in mind that $\overline{\Gamma(z)} = \Gamma(\overline{z})$.

3. Consider the following system of equations:

$$\begin{aligned} 2x + 2y + 3z &= 0\,, \\ 4x + 8y + 12z &= -4\,, \\ 6x + 2y + cz &= 4\,. \end{aligned}$$

(a) Determine all values of c for which this system of equations is consistent.

(b) Determine all values of c for which there is a unique solution, and compute the solution for these cases.

(c) Determine all values of c for which there are an infinite number of solutions, and give the general solution for these cases.

4. Consider the matrix:

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\,. \tag{2}$$

(a) Show that $A^n = 0$ for all integer $n \geq 3$.

(b) Evaluate $(\mathbf{I} - A)^{-1}$, where $\mathbf{I}$ is the 3×3 identity matrix. Choose any technique for computing the inverse that you prefer.

(c) Check your answer to part (b) by the following sophisticated method. Use the geometric series to define:

$$(\mathbf{I} - A)^{-1} = \sum_{k=0}^{\infty} A^k\,.$$

where $A^0 = \mathbf{I}$ (assuming that the sum converges). Using the result of part (a), sum the series when A is given by eq. (2), and verify that you reproduce the result of part (b).

5. Consider the matrices

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix},$$

(a) Compute the eigenvalues and eigenvectors of A. Find the diagonalizing matrix P such that $D \equiv P^{-1}AP$ is a diagonal matrix.

(b) Compute e^D. Then, using the result of part (a), compute e^A.

(c) Using the Cayley-Hamilton theorem, find a general expression for A^n in terms of A. Then compute e^A directly from its series expansion. Verify that the result agrees with the one obtained in part (b).

(d) Compute e^B and e^{A+B} using one of the two methods above (your choice!) for evaluating the matrix exponential.

(e) Check to see whether $e^A e^B = e^B e^A$ is valid. Explain why $e^{A+B} \neq e^A e^B$ in this problem.

6. Let F_{jk} be an antisymmetric tensor, where the indices j and k can take on the values 1, 2 or 3. The vector B_i that is dual to F_{jk} is defined as

$$B_i \equiv \tfrac{1}{2}\epsilon_{ijk}F_{jk}\,. \tag{3}$$

(a) Solve eq. (3) for F_{jk}. That is, express F_{jk} in terms of the vector B_i.

HINT: Multiply eq. (3) by $\epsilon_{i\ell m}$ and make use of one of the Levi-Civita tensor identities. Be careful to keep track of your dummy indices!

(b) Suppose that F_{jk} can be written in terms of two vectors, $\vec{\boldsymbol{D}}$ and $\vec{\boldsymbol{A}}$ as follows:

$$F_{jk} = D_j A_k - D_k A_j\,.$$

Find a vector relation that expresses $\vec{\boldsymbol{B}}$ in terms of $\vec{\boldsymbol{D}}$ and $\vec{\boldsymbol{A}}$.

(c) Assuming that F_{jk} is a proper second-rank antisymmetric tensor, prove that $\vec{\boldsymbol{B}}$ transforms as a pseudovector under an orthogonal coordinate transformation.