The generalized Stirling series

In these notes, a generalization of the asymptotic Stirling series for the logarithm of the Gamma function is derived. This generalization is then used to examine the behavior of $\Gamma(x+iy)$ as $|y|\to\infty$.

1. Derivation of the generalized Stirling series

We begin with the full asymptotic Stirling series for the logarithm of the Gamma function,[1]

$$\operatorname{Ln}\Gamma(z)=(z-\tfrac{1}{2})\operatorname{Ln}z-z+\tfrac{1}{2}\operatorname{Ln}(2\pi)+\sum_{k=1}^{N}\frac{B_{2k}}{2k(2k+1)}\frac{1}{z^{2k-1}}+\mathcal{O}\left(\frac{1}{|z|^{2N+1}}\right)\,,\qquad(1)$$

where the B_{2k} are the Bernoulli numbers, which are defined in the class handout entitled, *Taylor Series Expansions*, and N is any positive integer. Eq. (1) is valid for complex numbers z such that $|\operatorname{Arg}z|<\pi$ (i.e. we exclude the origin of the complex plane and the negative real axis) as $|z|\to\infty$. Under the same assumptions, we shall derive the corresponding asymptotic expansion for $\Gamma(z+x)$, where x is any fixed real finite number. First we note that for any real number x,[2]

$$\operatorname{Ln}(z+x)=\operatorname{Ln}z+\operatorname{Ln}\left(1+\frac{x}{z}\right)\,.\qquad(2)$$

Hence, it follows that

$$\operatorname{Ln}\Gamma(z+x)=(z+x-\tfrac{1}{2})\operatorname{Ln}z-z+(z+x+\tfrac{1}{2})\operatorname{Ln}\left(1+\frac{x}{z}\right)-x+\tfrac{1}{2}\operatorname{Ln}(2\pi)$$
$$+\sum_{k=1}^{N}\frac{B_{2k}}{2k(2k+1)}\frac{1}{(z+x)^{2k-1}}+\mathcal{O}\left(\frac{1}{|z|^{2N+1}}\right)\,.\qquad(3)$$

Our strategy now is to expand about $x=0$. Note that:

$$\operatorname{Ln}\left(1+\frac{x}{z}\right)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\frac{x^n}{z^n}\,,\qquad(4)$$

[1] The logarithms on the right hand side of eq. (1) correspond to their principal values. Consequently, $\operatorname{Ln}\Gamma(z)$ is a single-valued function of the complex variable z, which is defined for all complex z excluding the non-positive integers (where $\Gamma(z)$ is singular). However, $\arg\operatorname{Ln}\Gamma(z)$ as defined by eq. (1) does not in general lie within the principal interval (between $-\pi$ and π). Hence, $\operatorname{Ln}\Gamma(z)$ can differ from the principal value of the logarithm of the complex Gamma function by an integer multiple of $2\pi i$ that depends on the value of z. For more details, see Chapter 9, Section 6 of Henri Cohen, *Number Theory, Volume II: Analytic and Modern Tools* (Springer Science, New York, 2007); in particular, problem 35 of Chapter 9, op. cit., is illuminating.

[2] If x is real (either positive, negative or zero) and $|\operatorname{Arg}z|<\pi$, then it is easy to verify that $|\operatorname{Arg}z+\operatorname{Arg}(1+\frac{x}{z})|<\pi$, in which case eq. (2) holds with no correction term [cf. eq. (54) of the class handout entitled, *The complex logarithm, exponential and power functions*].

and

$$(z+x)^{1-2k} = \frac{1}{z^{2k-1}}\left(1+\frac{x}{z}\right)^{1-2k} = \sum_{n=0}^{\infty}\binom{1-2k}{n}\frac{x^n}{z^{2k+n-1}} = \sum_{n=0}^{\infty}(-1)^n\binom{2k+n-2}{n}\frac{x^n}{z^{2k+n-1}}\,. \tag{5}$$

In obtaining the above result, we used the definition of the binomial coefficient. Namely,

$$\begin{aligned}\binom{-p}{n} &= \frac{-p(-p-1)(-p-2)\cdots(-p-n+1)}{n!} \\ &= \frac{(-1)^n p(p+1)(p+2)\cdots(p+n-1)}{n!} = (-1)^n\frac{(p+n-1)!}{n!(p-1)!} \\ &= (-1)^n\binom{p+n-1}{n}\,.\end{aligned}$$

Inserting eqs. (4) and (5) into eq. (3) yields:

$$\begin{aligned}\operatorname{Ln}\Gamma(z+x) = (z+x-\tfrac{1}{2})\operatorname{Ln}z - z + \tfrac{1}{2}\operatorname{Ln}(2\pi) + \sum_{n=2}^{2N}\frac{(-1)^{n+1}}{n}\frac{x^n}{z^{n-1}} + (x-\tfrac{1}{2})\sum_{n=1}^{2N-1}\frac{(-1)^{n+1}}{n}\frac{x^n}{z^n} \\ +\sum_{k=1}^{N}\sum_{n=0}^{2N-2k}\frac{(-1)^n B_{2k}}{2k(2k+1)}\binom{2k+n-2}{n}\frac{x^n}{z^{2k+n-1}} + \mathcal{O}\left(\frac{1}{|z|^{2N}}\right)\,,\end{aligned} \tag{6}$$

where all terms of $\mathcal{O}(1/|z|^{2N})$ or higher are neglected. After an appropriate relabeling of the summation index, one can rewrite the first two sums above as:

$$\sum_{p=1}^{2N-1}(-1)^{p+1}\left\{\frac{x-\frac{1}{2}}{p} - \frac{x}{p+1}\right\}\frac{x^p}{z^p} = \sum_{p=1}^{2N-1}(-1)^{p+1}\left\{\frac{x}{p(p+1)} - \frac{1}{2p}\right\}\frac{x^p}{z^p}\,. \tag{7}$$

In the double sum of eq. (6), we define a new summation variable, $p = 2k+n-1$. Then,

$$\sum_{k=1}^{N}\sum_{n=0}^{2N-2k}\frac{(-1)^n B_{2k}}{2k(2k+1)}\binom{2k+n-2}{n}\frac{x^n}{z^{2k+n-1}} = \sum_{k=1}^{N}\sum_{p=2k-1}^{2N-1}\frac{B_{2k}x^{p+1-2k}}{2k(2k+1)}\binom{p-1}{2k-2}\frac{(-1)^{p+1}}{z^p}\,, \tag{8}$$

after using the well known property of the binomial coefficient,

$$\binom{q}{r} = \binom{q}{q-r}\,.$$

We now interchange the order of the summation in eq. (8) to obtain:

$$\sum_{k=1}^{N}\sum_{n=0}^{2N-2k}\frac{(-1)^n B_{2k}}{2k(2k+1)}\binom{2k+n-2}{n}\frac{x^n}{z^{2k+n-1}} = \sum_{p=1}^{2N-1}\frac{(-1)^{p+1}}{z^p}\sum_{k=1}^{[\frac{1}{2}(p+1)]}\frac{B_{2k}x^{p+1-2k}}{2k(2k+1)}\binom{p-1}{2k-2}\,,$$

where $[\frac{1}{2}(p+1)]$ denotes the the largest integer less than or equal to $\frac{1}{2}(p+1)$. Since $B_{2p+1}=0$ for $p=1,2,3,\ldots$, one can re-express the sum above as:

$$\sum_{p=1}^{2N-1}\frac{(-1)^{p+1}}{z^p}\sum_{k=1}^{[\frac{1}{2}(p+1)]}\frac{B_{2k}x^{p+1-2k}}{2k(2k+1)}\binom{p-1}{2k-2}=\sum_{p=1}^{2N-1}\frac{(-1)^{p+1}}{z^p}\sum_{k=2}^{p+1}\frac{B_k x^{p+1-k}}{k(k+1)}\binom{p-1}{k-2}\,,$$

after replacing $2k\to k$ and summing over the new summation variable k. One can massage the above sum further by noting that:

$$\frac{1}{k(k+1)}\binom{p-1}{k-2}=\frac{(p-1)!}{k!(p-k+1)!}=\frac{1}{p(p+1)}\,\frac{(p+1)!}{k!(p-k+1)!}=\frac{1}{p(p+1)}\binom{p+1}{k}$$

Hence, it follows that:

$$\sum_{k=1}^{N}\sum_{n=0}^{2N-2k}\frac{(-1)^n B_{2k}}{2k(2k+1)}\binom{2k+n-2}{n}\frac{x^n}{z^{2k+n-1}}=\sum_{p=1}^{2N-1}\frac{(-1)^{p+1}}{p(p+1)z^p}\sum_{k=2}^{p+1}B_k x^{p+1-k}\binom{p+1}{k}\,. \tag{9}$$

Using the results of eqs. (7) and (9), we can rewrite eq. (6) as:

$$\begin{aligned}\operatorname{Ln}\Gamma(z+x)&=(z+x-\tfrac{1}{2})\operatorname{Ln}z-z+\tfrac{1}{2}\operatorname{Ln}(2\pi)\\&\quad+\sum_{p=1}^{2N-1}\frac{(-1)^{p+1}}{p(p+1)z^p}\left\{x^{p+1}-\tfrac{1}{2}(p+1)x^p+\sum_{k=2}^{p+1}B_k x^{p+1-k}\binom{p+1}{k}\right\}+\mathcal{O}\left(\frac{1}{|z|^{2N}}\right)\,.\end{aligned} \tag{10}$$

The expression within the braces above can be written in a more compact form by recalling that $B_0=1$ and $B_1=-\frac{1}{2}$. Then, eq. (10) yields

$$\operatorname{Ln}\Gamma(z+x)=(z+x-\tfrac{1}{2})\operatorname{Ln}z-z+\tfrac{1}{2}\operatorname{Ln}(2\pi)+\sum_{p=1}^{2N-1}\frac{(-1)^{p+1}}{p(p+1)z^p}\sum_{k=0}^{p+1}B_k x^{p+1-k}\binom{p+1}{k}+\mathcal{O}\left(\frac{1}{|z|^{2N}}\right)\,. \tag{11}$$

The Bernoulli polynomials $B_n(x)$ [for $n=0,1,2,3,\ldots$] are nth-order polynomials in x defined by

$$B_n(x)=\sum_{k=0}^{n}\binom{n}{k}B_k x^{n-k}\,.$$

For example, $B_0(x)=1$, $B_1(x)=x-\frac{1}{2}$, $B_2(x)=x^2-x+\frac{1}{6}$, etc. Three notable properties of the Bernoulli polynomials are:

$$\text{(i)}\;\; B_n(0)=B_n\,,\qquad \text{(ii)}\;\; B_n(1)=(-1)^n B_n\,,\qquad \text{(iii)}\;\; B_n(\tfrac{1}{2})=\left(\frac{1}{2^{n-1}}-1\right)B_n\,. \tag{12}$$

Thus, eq. (11) can be written more compactly in the following form, sometimes called the *generalized Stirling series*,[3]

$$\boxed{\operatorname{Ln}\Gamma(z+x)=(z+x-\tfrac{1}{2})\operatorname{Ln}z-z+\tfrac{1}{2}\operatorname{Ln}(2\pi)+\sum_{p=1}^{2N-1}\frac{(-1)^{p+1}B_{p+1}(x)}{p(p+1)z^p}+\mathcal{O}\left(\frac{1}{|z|^{2N}}\right)} \tag{13}$$

which is valid as $|z|\to\infty$ for any fixed real number x and $|\operatorname{Arg}z|<\pi$.

As a check, by setting $x=0$ in eq. (13) and using eq. (12) and $B_{2n+1}=0$ for $n=1,2,3,\ldots$, we recover eq. (1). Likewise, if we set $x=1$ in eq. (13), we obtain

$$\operatorname{Ln}\Gamma(z+1)=(z+\tfrac{1}{2})\operatorname{Ln}z-z+\tfrac{1}{2}\operatorname{Ln}(2\pi)+\sum_{k=1}^{N}\frac{B_{2k}}{2k(2k+1)}\,\frac{1}{z^{2k-1}}+\mathcal{O}\left(\frac{1}{|z|^{2N+1}}\right)\,, \tag{14}$$

which is the expected result in light of $\Gamma(z+1)=z\Gamma(z)$. That is, taking the logarithm of the latter expression,

$$\operatorname{Ln}\Gamma(z+1)=\operatorname{Ln}\Gamma(z)+\operatorname{Ln}z\,, \tag{15}$$

where $\operatorname{Ln}z$ is the principal value of the logarithm of z,[4] in which case eq. (14) immediately follows from eq. (1). Finally, if we set $x=-1$ in eq. (13) and use the relation, $B_n(-1)=B_n+(-1)^n\,n$, we end up with:

$$\operatorname{Ln}\Gamma(z-1)=(z-\tfrac{3}{2})\operatorname{Ln}z-z+\tfrac{1}{2}\operatorname{Ln}(2\pi)+\sum_{k=1}^{N}\frac{B_{2k}}{2k(2k+1)}\,\frac{1}{z^{2k-1}}+\sum_{p=1}^{2N-1}\frac{1}{pz^p}+\mathcal{O}\left(\frac{1}{|z|^{2N}}\right)\,. \tag{16}$$

Indeed, eq. (16) matches the result obtained from eq. (1) after taking the logarithm of $\Gamma(z-1)=\dfrac{\Gamma(z)}{z-1}$ and identifying $\operatorname{Ln}\left(1-\dfrac{1}{z}\right)=-\displaystyle\sum_{p=1}^{\infty}\frac{1}{pz^p}$.

2. Application to the asymptotic expansion of $\Gamma(x+iy)$ as $|y|\to\infty$

In the solutions to the final exam, the following asymptotic expressions were derived:

$$|\Gamma(1+iy)|^2=\frac{\pi y}{\sinh(\pi y)}=2\pi|y|e^{-\pi|y|}\left[1+\mathcal{O}\left(e^{-2\pi|y|}\right)\right]\,,\qquad \text{as } |y|\to\infty\,. \tag{17}$$

$$|\Gamma(iy)|^2=\frac{\pi y}{\sinh(\pi y)}=\frac{2\pi}{|y|}e^{-\pi|y|}\left[1+\mathcal{O}\left(e^{-2\pi|y|}\right)\right]\,,\qquad \text{as } |y|\to\infty\,, \tag{18}$$

$$|\Gamma(\tfrac{1}{2}+iy)|^2=\frac{2\pi\sinh(\pi y)}{\sinh(2\pi y)}=2\pi e^{-\pi|y|}\left[1+\mathcal{O}\left(e^{-2\pi|y|}\right)\right]\,,\qquad \text{as } |y|\to\infty\,, \tag{19}$$

[3]The asymptotic expansion given by eq. (13) was first obtained by E.W. Barnes, *The theory of the gamma function*, *Messenger Math.* **29** (1899) pp. 64–128. A more modern derivation is given by Masanori Katsurada, *Proc. Japan Acad.* **74**, Ser. A (1998) pp. 167–170, where the real number x is restricted to be positive. However, this restriction is not required in the derivation presented in these notes. Indeed, the generalized Stirling series given by eq. (13) can be shown to be valid for any *complex* number x [cf. eq. (12) on p. 48 of *Higher Transcendental Functions*, Volume I, edited by A. Erdélyi (McGraw-Hill Book Company, Inc., New York, 1953)].

[4]In light of footnote 1, one can show that there is *no* correction term to eq. (15), corresponding to some nonzero integer multiple of $2\pi i$.

where y is a real number. In particular, note that none of the three asymptotic results above possess correction terms of $\mathcal{O}(1/|y|^p)$ for any positive power p. We can understand this phenomenon as follows. Starting from eq. (13), we choose $z = iy$ where y is real. Recall that the principal value of the complex logarithm is given by $\operatorname{Ln} z = \operatorname{Ln}|z| + i \operatorname{Arg} z$, from which it follows that $\operatorname{Re} \operatorname{Ln} z = \operatorname{Ln}|z|$. Hence,

$$\operatorname{Ln}|\Gamma(x+iy)| = \operatorname{Re} \operatorname{Ln}\Gamma(x+iy)\,.$$

Thus, the asymptotic expansion of $\operatorname{Ln}|\Gamma(x+iy)|$ as $|y| \to \infty$ is given by

$$\operatorname{Ln}|\Gamma(x+iy)| = \operatorname{Re}\left\{(x+iy-\tfrac{1}{2})\operatorname{Ln}(iy) - iy + \tfrac{1}{2}\operatorname{Ln}(2\pi) + \sum_{p=1}^{2N-1}\frac{(-1)^{p+1}B_{p+1}(x)}{p(p+1)(iy)^p} + \mathcal{O}\left(\frac{1}{y^{2N}}\right)\right\}. \tag{20}$$

Note that the terms in the sum with p odd are all pure imaginary. Moreover,

$$\operatorname{Arg}(iy) = \frac{\pi}{2}\,\frac{|y|}{y} = \tfrac{1}{2}\pi \operatorname{sgn}(y) = \begin{cases} \tfrac{1}{2}\pi\,, & \text{if } y > 0\,, \\ -\tfrac{1}{2}\pi\,, & \text{if } y < 0\,, \end{cases} \tag{21}$$

where $\operatorname{sgn}(y)$ is the sign of y. It then follows that

$$\operatorname{Ln}(iy) = \operatorname{Ln}|y| + \tfrac{1}{2}i\pi \operatorname{sgn}(y)\,.$$

Hence, we can drop the odd p terms of the sum in eq. (20) and set $p = 2k$, which yields

$$\boxed{\operatorname{Ln}|\Gamma(x+iy)| = (x-\tfrac{1}{2})\operatorname{Ln}|y| - \tfrac{1}{2}\pi|y| + \tfrac{1}{2}\operatorname{Ln}(2\pi) + \sum_{k=1}^{N}\frac{(-1)^{k+1}B_{2k+1}(x)}{2k(2k+1)y^{2k}} + \mathcal{O}\left(\frac{1}{y^{2N+2}}\right)} \tag{22}$$

after using $|y| = y\operatorname{sgn}(y)$.

Using eq. (22), one can now understand why there are no $\mathcal{O}(1/|y|^p)$ corrections to eqs. (17)–(19) for any positive integer power p. This is a result of the following property of the Bernoulli polynomials:

$$B_{2n+1}(0) = B_{2n+1}(\tfrac{1}{2}) = B_{2n+1}(1) = 0\,, \qquad \text{for } n = 1, 2, 3, \ldots,$$

which is a consequence of $B_{2n+1} = 0$ for $n = 1, 2, 3, \ldots$, and the properties listed in eq. (12). Hence, it follows that

$$B_{2n+1}(x) = x(x-1)(x-\tfrac{1}{2})p_n(x)\,, \qquad n = 1, 2, 3, \ldots\,,$$

for some polynomial $p_n(x)$ which is of order $2n-2$. For example, we have $p_1(x) = 1$, $p_2(x) = -\frac{1}{10}(1+3x-3x^2)$, etc. Thus, we conclude that for $x = 0$, $x = \frac{1}{2}$ and $x = 1$,

$$\operatorname{Ln}|\Gamma(x+iy)| = (x-\tfrac{1}{2})\operatorname{Ln}|y| - \tfrac{1}{2}\pi|y| + \tfrac{1}{2}\operatorname{Ln}(2\pi)\,, \qquad \text{as } |y| \to \infty,$$

up to exponentially small corrections that are not visible to the asymptotic expansion of $\operatorname{Ln}|\Gamma(x+iy)|$. Exponentiating the above result then yields

$$|\Gamma(x+iy)| = (2\pi)^{1/2}|y|^{x-\frac{1}{2}}\,e^{-\frac{1}{2}\pi|y|}\,, \qquad \text{as } |y| \to \infty, \text{ for } x = 0,\ x = \tfrac{1}{2} \text{ and } x = 1, \tag{23}$$

up to exponentially small corrections, which confirms the results of eqs. (17)–(19). In general, we have

$$\boxed{|\Gamma(x+iy)| = (2\pi)^{1/2}|y|^{x-\frac{1}{2}}\, e^{-\frac{1}{2}\pi|y|}\left[1 + x(x-\tfrac{1}{2})(x-1)\mathcal{O}\left(\frac{1}{y^2}\right)\right]\,, \qquad \text{as } |y| \to \infty} \tag{24}$$

Note that $x = 0$, $x = \frac{1}{2}$ and $x = 1$ are the only real values of x for which $B_{2k+1}(x) = 0$ for *all* values of $k = 1, 2, 3, \ldots$. Thus, for any other real value of x, we expect $\mathcal{O}(1/y^{2p})$ corrections to eq. (23) for some positive integer power p, as indicated in eq. (24).